East Asian Math J 18(2002), No 2, pp 225-233

FIXED POINT THEOREMS FOR FUZZY MAPPINGS SATISFYING AN IMPLICIT RELATION

SUSHIL SHARMA

ABSTRACT In this paper, we obtain the common fixed point for fuzzy mappings satisfying an implicit relation We improve earlier results of this line.

1. Introduction

In 1981, Heilpern [7] introduced the concept of fuzzy mappings. In 1987, Bose and Sahani [5] gave an improved version of Heilpern Fixed point theorems for fuzzy mappings have been studied by Butnariu [3], Chang [6], Chitra [1], Weiss [4], Lee and Cho [2] and Arora and Sharma [8] In the present paper we improve results of Arora and Sharma [8].

2. Terminology

The definitions and terminology for further discussions are taken from Heilpern [7]

Let (X, d) be metric linear space, $F(X)$ the collection of all fuzzy sets in X and $W(X)$ the collection of all those fuzzy sets A of $F(X)$ whose α-level sets.

$$A_\alpha = \{x \in X : A(x) \geq \alpha\}$$

Received October 2, 2002 Revised December 12, 2002

2000 Mathematics Subject Classification 47H10, 54H25.

Key words and phrases· complete metric linear space, fuzzy mapping, approximate quantities, fixed point theorem,.

for each $\alpha \in [0,1]$ and

$$A_0 = \overline{\{x \in X : A(x) > 0\}}$$

are compact and convex with $\sup_{x \in X} A(x) = 1$. $A(x)$ being the grade of membership of x in A. The members of $W(X)$ are called the approximate quantities. If $A, B \in W(X)$ then we say $A \subset B$ if and only if $A(x) \leq B(x)$ for each $x \in X$.

By a fuzzy map F on X, we mean a mapping $F : X \to W(X)$. A point $x \in X$ is a common fixed point of a family f of fuzzy maps if $\{x\} \subset F_i(x)$ for all $F_i \in f$.

If $A, B \in W(X)$ and $\alpha \in [0,1]$, then denote

$$p_\alpha(A,B) = \underset{\substack{x \in A \\ y \in B}}{Inf}\, d(x,y)$$

$$D_\alpha(A,B) = H(A_\alpha, B_\alpha),$$

where H denotes the Hausdorff distance.

Also

$$D(A,B) = \underset{a}{sup} D_\alpha(A,B),$$

$$p(A,B) = \underset{a}{sup}\, p_\alpha(A,B),$$

For the proof of our theorems we need following lemmas due to Heilpern [7].

LEMMA 1. *Let $x \in X$, $A \in W(X)$ and $\{x\}$ be a fuzzy set with membership function equal to the characteristic function of set $\{x\}$. If $\{x\} \subset A$, then $p_\alpha(x,A) = 0$ for each $\alpha \in [0,1]$.*

LEMMA 2. *$p_\alpha(x,A) \leq d(x,y) + p_\alpha(y,A)$ for any $x, y \in X$.*

LEMMA 3. *If $\{x_0\} \subset A$ then $p_\alpha(x_0,B) \leq D_\alpha(A,B)$ for each $B \in W(X)$.*

LEMMA 4. *[8] Let (X,d) be metric linear space $F : X \to W(X)$ be a fuzzy map and $x_0 \in X$, then there exists $x_1 \in X$ such that $\{x_1\} \subset F(x_0)$.*

Implicit Relation

Let ψ be the set of all continuous functions $\varphi : R_+^6 \to R$ satisfying the following conditions :

(ψ_1) $\varphi(t_1, \ldots, t_6)$ is decreasing in variables $t_2, \ldots, t_6$.

(ψ_2) there exists $h \in (0,1)$ such that the inequalities :

(i) $u \le t$ and $\varphi(t, v, v, u, u+v, 0) \le 0$ or

(ii) $u \le t$ and $\varphi(t, v, u, v, 0, u+v) \le 0$ implies $t \le hv$.

EXAMPLE 1. $\varphi(t_1, \ldots, t_6) = t_1 - \{at_2t_3 + bt_3t_4 + ct_5t_6\}^{\frac{1}{2}}$ where $a, b, c > 0$ and $a + b < 1$.

(ψ_1) : Obviously

(ψ_2) : Let $u > 0$, $u \le t$ and $\varphi(t, v, v, u, u+v, 0) = t - \{av^2 + bvu + 0\}^{\frac{1}{2}} \le 0$.

If $v \le u$ then $u \le t \le (a+b)^{\frac{1}{2}} u < u$, a contradiction. Thus $u < v$ and $t \le (a+b)^{\frac{1}{2}} v = hv$, where $h = (a+b)^{\frac{1}{2}}$ Similarly, $u \le t$ and $\varphi(t, v, u, v, 0, u+v) \le 0$ implies $t \le hv$. If $u = 0$, then $u \le v$ and $t \le (a+b)^{\frac{1}{2}} v = hv$

EXAMPLE 2

$\varphi(t_1, \ldots, t_6) = {t_1}^3 - m\,max\{t_2t_4^2, t_3^2t_4, t_5^2t_6, t_5t_6^2\}$, where $m \in (0,1)$.

(ψ_1) : Obviously.

(ψ_2) : Let $u > 0$, $u \le t$ and $\varphi(t, v, v, u, u+v, 0)$ $= t^3 - m\,max\{vu^2, v^2u, 0, 0\} \le 0$. If $v \le u$ then $u \le t \le m^{\frac{1}{3}} u < u$, a contradiction.

Thus $u < v$ and $t \le m^{\frac{1}{3}} v = hv$, where $h = m^{\frac{1}{3}}$ Similarly, $u \le t$ and $\varphi(t, v, u, v, 0, u+v,) \le 0$ implies $t \le hv$. If $u = 0$, then $u \le v$ and $t \le m^{\frac{1}{3}} v = hv$.

EXAMPLE 3. $\varphi(t_1, \ldots, t_6) = t_1 - m\,max\{t_2, \frac{1}{2}(t_3+t_4), \frac{1}{2}(t_5+t_6)\}$, where $m \in (0,1)$.

(ψ_1) : Obviously.

(ψ_2) : Let $u > 0$, $u \le t$ and $\varphi(t, v, v, u, u+v, 0) = t - m\, max\{v, \frac{1}{2}(v+u), \frac{1}{2}(u+v)\} \le 0$. If $v \le u$ then $u \le t \le mu < u$, a contradiction. Thus $u < v$ and $t \le mv = hv$, where $h = m \in (0,1)$. Similarly, $u \le t$ and $\varphi(t, v, u, v, 0, u+v) \le 0$ implies $t \le mv = hv$. If $u = 0$, then $u \le v$ and $t \le mv = hv$.

Main Results

THEOREM 1. *Let (X, d) be a complete metric linear space and $F_i : X \to W(X)$ be fuzzy mappings for $i = 1, 2$ such that for all $x, y \in X$*

(1.1)

$$\varphi(D(F_1x, F_2y), d(x,y), p(x, F_1x), p(y, F_2y), p(x, F_2y), p(y, F_1x)) \le 0$$

Then F_1 and F_2 have a common fixed point.

PROOF. Let $x_0 \in X$. Then by lemma 4 there exists $x_1 \in X$ such that $\{x_1\} \subset F_1(x_0)$. For $x_1 \in X$, by lemma 4 the 1-level set $F_2(x_1)_1$ of $F_2(x_1)$ is a compact nonempty subset of X. Thus, there exists $x_2 \in F_2(x_1)_1$ such that

$$d(x_1, x_2) = \inf_{x \in F_2(x_1)} d(x_1, x)$$

By Lemma 3 , we have

$$\begin{aligned} d(x_1, x_2) &= p_1(x_1, F_2x_1) \\ &\le D_1(F_1x_0, F_2x_1) \\ &\le D(F_1x_0, F_2x_1) \end{aligned}$$

Similarly, for $x_2 \in X$, there exists $x_3 \in F_1(x_2)_1$ such that

$$\begin{aligned} d(x_2, x_3) &= p_1(x_2, F_1x_2) \\ &\le D_1(F_2x_1, F_1x_2) \\ &\le D(F_2x_1, F_1x_2) \end{aligned}$$

Continuing this way , we can obtain a sequence $\{x_n\}$ of X such that

$$\begin{aligned}\{x_{2n+1}\} &\subset F_1x_{2n}\\ \{x_{2n+2}\} &\subset F_2x_{2n+1},\ \ n=1,2,\dots with\\ d(x_{2n},x_{2n+1}) &= p_1(x_{2n},F_1x_{2n})\\ &\le D_1(F_1x_{2n},F_2x_{2n-1})\\ &\le D(F_1x_{2n},F_2x_{2n-1})\end{aligned}$$

and

$$\begin{aligned}d(x_{2n+1},x_{2n+2}) &= p_1(x_{2n+1},F_2x_{2n+1})\\ &\le D_1(F_1x_{2n},F_2x_{2n+1})\\ &\le D(F_1x_{2n},F_2x_{2n+1})\end{aligned}$$

By (1.1), we write

$$\varphi(D(F_1x_{2n},F_2x_{2n+1}),d(x_{2n},x_{2n+1}),p(x_{2n},F_1x_{2n}),p(x_{2n+1},F_2x_{2n+1}),\\ p(x_{2n},F_2x_{2n+1}),p(x_{2n+1},F_1x_{2n}))\le 0$$

$$\varphi(D(F_1x_{2n},F_2x_{2n+1}),d(x_{2n},x_{2n+1}),d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2}),\\ d(x_{2n},x_{2n+2}),d(x_{2n+1},x_{2n+1}))\le 0$$

$$\varphi(d(x_{2n+1},x_{2n+2}),d(x_{2n},x_{2n+1}),d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2}),\\ d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2}),0)\le 0$$

By implicit relation (i), we have

(1.2) $$d(x_{2n+1},x_{2n+2})\le hd(x_{2n},x_{2n+1})$$

Similarly, by (1.1) and implicit relation (ii), we have

(1.3) $$d(x_{2n+1},x_{2n})\le hd(x_{2n},x_{2n-1})$$

and so

$$d(x_{2n+1}, x_{2n+2}) \le h^{2n+1} d(x_0, x_1) \tag{1.4}$$

Since $h \in (0,1)$ it follows from (1.4) that $\{x_n\}$ is a Cauchy sequence and hence convergent in X. Let $lim_{n\to\infty} x_n = z \in X$.

We claim that z is a fixed point of both F_1 and F_2.

Now by (1.1) we write

$$\varphi(D(F_1z, F_2x_{2n+1}), d(z, x_{2n+1}), p(z, F_1z), p(x_{2n+1}, F_2x_{2n+1}), \\ p(z, F_2x_{2n+1}), p(x_{2n+1}, F_1z)) \le 0$$

$$\varphi(p(F_1z, x_{2n+2}), d(z, x_{2n+1}), p(z, F_1z), d(x_{2n+1}, x_{2n+2}), \\ d(z, x_{2n+2}), p(x_{2n+1}, F_1z)) \le 0$$

Letting $n \to \infty$, we obtain

$$\varphi(p(F_1z, z), 0, p(z, F_1z), 0, 0, p(z, F_1z)) \le 0$$

By implicit relation (ii) we see that $\{z\} \subset F_1z$. Proceeding similarly, it can be verified that $p(z, F_2z) = 0$. Hence $\{z\} \subset F_2z$, i.e. z is a common fixed point of F_1 and F_2

This completes the proof of the theorem.

Let us replace F_1 by F_0 and F_2 by $F_n (n \ne 0)$ and as done in Theorem 1, choose the sequence $\{x_n\}$ as

$$x_0 \in X, \{x_1\} \subset F_0(x_0), \{x_2\} \subset F_n(x_1), \{x_3\} \subset F_0(x_2), \dots, \\ \{x_{2n-1}\} \subset F_0(x_{2n-2}), \{x_{2n}\} \subset F_n(x_{2n-1}), \dots .$$

Following the procedure of Theorem 1, we get a common fixed point for each pair (F_0, F_i), $i = 1, 2, \dots$. Thus we state

THEOREM 2. *Let* $\{F_n : n \in Z^+\}$ *be a collection of fuzzy maps from* $X \to W(X)$, X *being a complete metric linear space, and for all* $x, y \in X$, $n = 1, 2, \ldots$.

$$\varphi(D(F_0x, F_ny), d(x,y), p(x, F_0x), p(y, F_ny), p(x, F_ny), p(y, F_0x)) \leq 0$$

Then there exists a fixed point of the family $\{F_n : n \in Z^+\}$. *Letting* $F_1 = F_2 = F$ *in Theorem 1 and* $x_0 \in X$, *by Lemma 4 we can obtain a sequence* $\{x_n\}$ *of* X *such that for all* $n = 1, 2, \ldots$

$$\{x_n\} \subset F(x_{n-1})$$

and

$$d(x_n, x_{n+1}) \leq D(Fx_{n-1}, Fx_n).$$

Now as F *satisfies*

$$(1.5)\quad \varphi(D(Fx, Fy), d(x,y), p(x, Fx), p(y, Fy), p(x, Fy), p(y, Fx)) \leq 0$$

for every $x, y \in X$ *It can be easily proved that* $\{x_n\}$ *is a Cauchy sequence.*

THEOREM 3. *Let* (X, d) *be a metric linear space and* $F : X \to W(X)$ *be a fuzzy mapping satisfying (1.5). Then* F *has a fixed point in* X *if any one of the following conditions is true*

(i) X *is complete,*
(ii) $\{x_n\}$ *converges to* $z \in X$,
(iii) $\{x_n\}$ *has a convergent subsequence* $\{x_{n_k}\}$.

The following corollaries follow immediately from the Theorems.

COROLLARY 1. *Let* (X, d) *be a complete metric linear space and* $F_i : X \to W(X)$ *be fuzzy mappings for* $i = 1, 2$ *such that for all* $x, y \in X$ *and* $q \in (0, \frac{1}{2})$

$$D(F_1x, F_2y) \leq qMax\{d(x,y), p(x, F_1x), p(y, F_2y), p(x, F_2y), p(y, F_1x)\}$$

Then F_1 *and* F_2 *have a common fixed point.*

COROLLARY 2. *Let* $\{F_n : n \in Z^+\}$ *be a collection of fuzzy maps from* $X \to W(X)$, X *being a complete metric linear space, and for all* $x, y \in X$, *and* $q \in (0, \frac{1}{2})$, $n = 1, 2, \dots$.

$$D(F_0x, F_ny) \leq qMax\{d(x,y), p(x, F_0x), p(y, F_ny), p(x, F_ny), p(y, F_0x)\}$$

Then there exists a fixed point of the family $\{F_n : n \in Z^+\}$.

COROLLARY 3. *Let* (X, d) *be a metric linear space and* $F : X \to W(X)$ *be a fuzzy mapping satisfying :*

$$D(Fx, Fy) \leq qMax\{d(x,y), p(x, Fx), p(y, Fy), p(x, Fy), p(y, Fx)\}$$

for all $x, y \in X$ *and some* $q \in (0, \frac{1}{2})$. *Then* F *has a fixed point in* X *if any one of the following conditions is true*

(i) X *is complete,*
(ii) $\{x_n\}$ *converges to* $z \in X$,
(iii) $\{x_n\}$ *has a convergent subsequence* $\{x_{n_k}\}$.

Acknowledgement

Author extend thanks to Professor Ireneusz Kubiaczyk for this paper.

REFERENCES

[1] J A Chitra, *A note on fixed points of fuzzy maps on partially ordered topological spaces*, Fuzzy Sets and Systems **19** (1986), 305–308.
[2] B S Lee, S.J Cho, *A fixed point theorem for contractive type fuzzy mappings*, Fuzzy Sets and Systems **61** (1994), 309–312
[3] D. Butnariu, *Fixed point for fuzzy mappings*, Fuzzy Sets and Systems **7** (1982), 191–207.
[4] M D Weiss, *Fixed point and induced fuzzy topologies for fuzzy sets*, J. Math Anal Appl **50** (1975), 142–150
[5] R K Bose and D Sahani, *Fuzzy mappings and fixed point theorems*, Fuzzy Sets and Systems **21** (1987), 53–58
[6] S S. Chang, *Fixed point theorems for fuzzy mappings*, Fuzzy Sets and Systems **17** (1985), 181–187.
[7] S Heilpern, *Fuzzy mappings and fixed point theorem*, J Math Anal. Appl **83** (1981), 566–569.

[8] S.C Arora and V Sharma, *Fixed point theorems for fuzzy mappings*, Fuzzy Sets and Systems **110** (2000), 127–130

[9] V. Popa, *A general coincidence theorem for compatible multivalued mappings satisfying an implicit relation*, Demonstratio Mathematica **53**(1) (2000), 159-164.

114, A-Sector
Vivekanand Colony
Ujjain-456010,India
E-mail. sksharma2005@yahoo.com