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NOTE ON THE MULTIPLE GAMMA FUNCTIONS

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ABSTRACT. Recently the theory of the multiple Gamma functions, which were studied by Barnes and others a century ago, has been revived in the study of determinants of Laplacians Here we are aiming at evaluating the values of the multiple Gamma functions $\Gamma_n(\frac{1}{2})$ in terms of the Hurwitz or Riemann Zeta functions

Recently the theory of multiple gamma functions, which were studied systematically by Barnes [1, 2] and others in about 1900, has been revived according to the study of determinants of Laplacians (see, *e.g.*[3], [6]). Barnes [2] introduced these functions through n-ple (or multiple) Hurwitz zeta functions.

Let $s = \sigma + it$, where σ, $t \in \mathbb{R}$. The n-ple Hurwitz zeta function is initially defined, when $\sigma > n$ and $a > 0$, by the series

$$\zeta_n(s, x|w_1, w_2, \ldots, w_n) = \sum_{k_1,k_2,\ldots,k_n=0}^{\infty} \frac{1}{(a+\Omega)^s} \tag{1}$$

where $\Omega = k_1w_1 + k_2w_2 + \cdots + k_nw_n$ Letting $w_k = 1$ $(k = 1, 2, \ldots, n)$ and $a > 0$ in (1) reduces to

$$\zeta_n(s, x) := \sum_{k_1,k_2,\ldots,k_n=0}^{\infty} (x + k_1 + k_2 + \cdots + k_n)^{-s}, \tag{2}$$

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which becomes, for $n = 1$, the generalized (or Hurwitz) Zeta function

$$\zeta_1(s,x) = \sum_{k=0}^{\infty}(x+k)^{-s} := \zeta(s,x). \tag{3}$$

The case $x = 1$ of (3) denoted by $\zeta(s)$ is the familiar Riemann Zeta function.

It is remarked in passing that the n-ple series in (2) can be shown to be analytic for $\Re(s) = \sigma > n$ by Eisenstein's Theorem and furthermore continued analytically to the whole s-plane with simple poles only at $s = k$ ($k = 1, 2, \ldots, n$) by the contour integral representation (see [5], [6]).

Vignéras [7] introduced the Weierstrass canonical product form of the n-ple Gamma functions by the following recurrence formula: She defined the n-ple Gamma functions $\Gamma_n(z)$ by

$$\Gamma_n(z) := G_n(z)^{(-1)^{n-1}} \quad (G_n(x+1) := \exp(f_n(x)),$$

where $f_n(x)$ satisfy

$$f_n(x) = -xA_n(1) + \sum_{h=1}^{n-1} \frac{p_h(x)}{h!}\left[f_{n-1}^{(h)}(0) - A_n^{(h)}(1)\right] + A_n(x),$$

with

$$A_n(x) = \sum_{m \in \mathbb{N}_0{}^{n-1}\times\mathbb{N}} \frac{1}{n}\left(\frac{x}{L(m)}\right)^n - \frac{1}{n-1}\left(\frac{x}{L(m)}\right)^{n-1} + \cdots$$
$$+ (-1)^{n-1}\frac{x}{L(m)} + (-1)^n \log\left(1 + \frac{x}{L(m)}\right),$$

$L(m) = m_1 + m_2 + \cdots + m_n$ if $m = (m_1, m_2, \cdots, m_n) \in \mathbb{N}_0{}^{n-1} \times \mathbb{N}$ and $p_h(x)$ is the unique polynomial of degree $n+1$ satisfying the equation $f(x+1) - f(x) = x^h, h \geq 1,\ x \geq 0$ and $p_h(0) = 0$, where $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

It is often useful to observe some properties of the multiple Gamma functions through Vignéras's Weierstrass canonical product form. Here, by using the expression Γ_n in terms of Hurwitz Zeta functions, we are aiming at evaluating the special values $\Gamma_n\left(\frac{1}{2}\right)$, which are sometimes very useful in various applications.

By simple combinatorial mind, it is easy to check that the number of elements of the following set

$$S_n := \{(k_1, k_2, \ldots, k_n) \in {\mathbb{N}_0}^n | k_1 + k_2 + \cdots + k_n = k,\ k_i \in \mathbb{N}_0,\\ i = 1, 2, \ldots, n\}$$

is equal to $\binom{k+n-1}{n-1}$. From this observation the n-ple series $\zeta_n(s,x)$ is written as a single series

$$\zeta_n(s,x) = \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} / (x+k)^s. \tag{4}$$

If we use the Stirling numbers of the first kind $s(n,k)$ in the binomial coefficients in the summation part of (4), we readily express (4) as follows (see Choi [4]):

$$\zeta_n(s,x) = \frac{1}{(n-1)!} \sum_{k=0}^{\infty} \left(\sum_{i=0}^{n-1} |s|(n, i+1) k^i \right) / (x+k)^s, \tag{5}$$

where $|s|(n,k) := |s(n,k)|$.

It is shown that $\zeta_n(s,x)$ can be expressed in the following form :

$$\zeta_n(s,x) = \sum_{i=0}^{n-1} P_{n,i}(x) \zeta(s-i, x), \tag{6}$$

where

$$P_{n,i}(x) = \frac{1}{(n-1)!} \sum_{j=i}^{n-1} (-1)^{n+1-i} \binom{j}{i} s(n, j+1) x^{j-i}$$

and so we observe $P_{n,i}(x)$ a polynomial in x of degree $n-1-i$ with rational coefficients, and we denote $P_{n,0}(x)$ by $P_n(x)$.

It is not difficult to show that, for $i = 0, 1, \ldots, n-1$, we have

$$P_{n,i}(x) = \frac{(-1)^i}{i!} P_n^{(i)}(x). \tag{7}$$

where $P_n^{(i)}(x)$ is the i-th derivative. Indeed, Differentiating $P_{n,i}(x)$, we find that

$$\begin{aligned} P'_{n,i}(x) &= \frac{1}{(n-1)!} \sum_{j=i+1}^{n-1} (-1)^{n+1-i} \binom{j}{i} (j-i) s(n, j+1) x^{j-i-1} \\ &= (-1)(i+1) \frac{1}{(n-1)!} \sum_{j=i+1}^{n-1} (-1)^{n-1} \binom{j}{i+1} s(n, j+1) x^{j-i-1} \\ &= (-1)(i+1) P_{n,i+1}(x). \end{aligned}$$

In [2], Barnes defines the multiple Gamma function by using the multiple Hurwitz Zeta function. Now define $G_n(x) = e^{\zeta'_n(0,x)}$ where $\zeta'_n(s,x) = \frac{\partial}{\partial s}\zeta_n(s,x)$. Then we get the relationship between multiple Gamma functions and multiple Hurwitz Zeta functions (see [5], [6]):

$$\Gamma_n(x) = \left[\prod_{m=1}^{n} R_{n-m+1}^{(-1)^m \binom{x}{m-1}} \right] G_n(x), \tag{8}$$

where

$$R_m = \exp\left[\sum_{k=1}^{m} \zeta'_k(0,1) \right] \quad (n \in \mathbb{N}).$$

From (6) and (7), (8) can be expressed in the following equivalent form:

$$\Gamma_n(x) = \left[\prod_{m=1}^{n} R_{n-m+1}^{(-1)^m \binom{x}{m-1}} \right] \exp\left[\sum_{i=0}^{n-1} P_{n,i}(x) \zeta'(-i, x) \right], \tag{9}$$

where

$$R_m = \exp\left[\frac{1}{(n-1)!}\sum_{i=1}^{m}(-1)^{m-i}s(m,i)\zeta'(1-i)\right]. \tag{10}$$

Setting $x = \frac{1}{2}$ in (9) yields

(11)

$$\Gamma_n\left(\tfrac{1}{2}\right) = \left[\prod_{m=1}^{n} R_{n-m+1}^{(-1)^m\binom{\frac{1}{2}}{m-1}}\right] \times \exp\left[\sum_{j=0}^{n-1} P_{n,j}\left(\tfrac{1}{2}\right)\left\{\frac{B_{j+1}\log 2}{2^j(j+1)} + \left(2^{-j}-1\right)\zeta'(-j)\right\}\right],$$

where R_m are given as in (10) and B_n Bernoulli numbers (see [5, p. 59]).

The special cases of (11) when $n = 2$ and $n = 3$ are recorded here.

$$\begin{aligned}\Gamma_2\left(\tfrac{1}{2}\right) &= 2^{-\frac{1}{24}}\cdot\pi^{\frac{1}{4}}\cdot\exp\left[-\tfrac{3}{2}\zeta'(-1)\right]\\ &= 2^{-\frac{1}{24}}\cdot\pi^{\frac{1}{4}}\cdot e^{-\frac{1}{8}}\quad A^{\frac{3}{2}},\end{aligned}$$

where A is the Glaisher-Kinkelin constant which has been shown to have the following relation (see [6, p. 506], see also [5, p. 87]):

$$\log A = -\zeta'(-1) + \tfrac{1}{12};$$

$$\Gamma_3\left(\tfrac{1}{2}\right) = 2^{\frac{1}{24}}\cdot\pi^{\frac{3}{16}}\cdot\exp\left[-\tfrac{3}{2}\zeta'(-1) - \tfrac{7}{8}\zeta'(-2)\right].$$

References

[1] E W Barnes, *The theory of the G-function*, Quart J Math **31** (1899), 264-314.

[2] E W. Barnes, *On the theory of the multiple Gamma function*, Trans Cambridge Philos Soc **19** (1904), 374-425

[3] J Choi, *Determinant of Laplacian on S^3*, Math. Japon **40** (1994), 155-166

[4] J. Choi, *Explicit formulas for Bernoulli polynomials of order n*, Indian J Pure Appl. Math **27** (1996), 667-674.

[5] H M Srivastava and J Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Series on Mathematics and Its Applications, Vol 531, Kluwer Academic Publishers, Dordrecht, Boston, and London, 2001

[6] I. Vardi, *Determinants of Laplacians and multiple Gamma functions*, SIAM J. Math Anal **19** (1988), 493-507.

[7] M -F Vignéras, *L'équation fonctionnelle de la fonction zêta de Selberg du groupe moudulaire $PSL(2, Z)$*, in "Journées Arithmétiques de Luminy" (Collq Internat CNRS, Centre Univ Luminy, Luminy, 1978), pp 235–249, Astérisque **61**, Soc Math France, Paris (1979)

[8] E.T Whittaker and G N Watson, *A Course of Modern Analysis* (4th Ed.), Cambridge University Press, 1963

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