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# ALMOST PRETOPOLOGICAL CONVERGENCE SPACES

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Abstract In this paper, we introduce the notion of almost topological convergence spaces and almost pretopological convergence spaces, and prove that these are product invariant.

## 1. Introduction

A convergence structure defined by Kent [4] is a correspondence between the filters on a given set $X$ and the subsets of $X$ which specifies which filters converge to which points of $X$. This concept is defined to include types of convergence which are more general than that defined by specifying a topology on $X$. Thus, a convergence structure may be regarded as a generalization of a topology.

With a given convergence structure $q$ on a set $X$, Kent [4] introduced associated convergence structures which are called a topological modification, a pretopological modification and a pseudotopological modification. Also, Kent [2] introduced product convergence structures.

In this paper, we obtained two inequalities (Proposition 5) which hold between modifications of the product convergence structure and the product convergence structure of modifications associated with the factor convergence structures. Also, we introduce the notion of almost

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## 2. Preliminaries

A *convergence structure* $q$ on a set $X$ is defined to be a function from the set $F(X)$ of all filters on $X$ into the set $P(X)$ of all subsets of $X$, satisfying the following conditions:

(1) $x \in q(\dot{x})$ for all $x \in X$;

(2) $\Phi \subset \Psi$ implies $q(\Phi) \subset q(\Psi)$;

(3) $x \in q(\Phi)$ implies $x \in q(\Phi \cap \dot{x})$,

where $\dot{x}$ denotes the principal ultrafilter containing $\{x\}$; $\Phi$ and $\Psi$ are in $F(X)$. Then the pair $(X, q)$ is called a *convergence space.* If $x \in q(\Phi)$, then we say that $\Phi$ *q-converges* to $x$. The filter $V_q(x)$ obtained by intersecting all filters which $q$-converge to $x$ is called the *q-neighborhood filter* at $x$ If $V_q(x)$ $q$-converges to $x$ for each $x \in X$, then $q$ is said to be *pretopological* and the pair $(X, q)$ is called a *pretopological convergence space.* A convergence structure $q$ on $X$ is said to be *pseudotopological* if $\Phi$ $q$-converges to $x$ whenever each ultrafilter finer than $\Phi$ $q$-converges to $x$, and the pair $(X, q)$ is called a *pseudotopological convergence space.*

A convergence structure $q$ on $X$ is said to be *topological* if $q$ is pretopological and for each $x \in X$, the filter $V_q(x)$ has a filter base $B_q(x)$ with the following property:

$$y \in G(x) \in B_q(x) \ \text{ implies } \ G(x) \in B_q(y).$$

Then $(X, q)$ is called a *topological convergence space.*

Let $C(X)$ be the set of all convergence structures on $X$, partially orded as follows:

$$q_1 \leq q_2 \ \text{ iff } \ q_2(\Phi) \subset q_1(\Phi) \ \text{ for all } \Phi \in F(X).$$

If $q_1 \leq q_2$, then we say that $q_1$ is coarser than $q_2$, and $q_2$ is finer than $q_1$.

For any $q \in C(X)$, we define the following related convergence structures, $\rho(q), \pi(q)$ and $\lambda(q)$:

(1) $x \in \rho(q)(\Phi)$ iff $x \in q(\Phi')$ for each ultrafilter $\Phi'$ finer than $\Phi$.

(2) $x \in \pi(q)(\Phi)$ iff $V_q(x) \subset \Phi$.

(3) $x \in \lambda(q)(\Phi)$ iff $U_q(x) \subset \Phi$, where $U_q(x)$ is the filter generated by the sets $U \in V_q(x)$ which have the property: $y \in U$ implies $U \in V_q(y)$. In this case, $\rho(q), \pi(q)$ and $\lambda(q)$ are called the *pseudotopological modification*, the *pretopological modification* and the *topological modification* of $q$, and the pairs $(X, \rho(q)), (X, \pi(q))$ and $(X, \lambda(q))$ are called the *pseudotopological modification*, the *pretopological modification* and the *topological modification* of $(X, q)$, respectively.

PROPOSITION 1([4]). *(1) $\rho(q)$ is the finest pseudotopology coarser than $q$*

*(2) $\pi(q)$ is the finest pretopology coarser than $q$.*

*(3) $\lambda(q)$ is the finest topology coarser than $q$.*

*(4) $\lambda(q) \leq \pi(q) \leq \rho(q) \leq q$.*

Let $f$ be a map from $X$ into $Y$ and $\Phi$ a filter on $X$. Then $f(\Phi)$ means the filter generated by $\{f(F) \mid F \in \Phi\}$.

Let $f$ be a map from a convergence space $(X, q)$ to a convergence space $(Y, p)$. Then $f$ is said to be *continuous* at a point $x \in X$, if the filter $f(\Phi)$ on $Y$ $p$-converges to $f(x)$ for every filter $\Phi$ on $X$ $q$-converging to $x$ If $f$ is continuous at every point $x \in X$, then $f$ is said to be continuous. Also, $f$ is said to be *neighborhood preserving*, if $V_p(f(x)) = f(V_q(x))$.

Let $(X_\lambda, q_\lambda)$ be a convergence space, $X = \prod_{\lambda \in \Lambda} X_\lambda$ the product of sets and $P_\lambda \colon X \to (X_\lambda, q_\lambda)$ the $\lambda$-th projection for each $\lambda \in \Lambda$ The *product convergence structure* $q$ on $X$ is defined by specifying that for any $x \in X$ and $\Phi \in F(X)$,

$$x \in q(\Phi) \quad \text{iff} \quad P_\lambda(x) \in q_\lambda(P_\lambda(\Phi)) \text{ for each } \lambda \in \Lambda$$

In this case, the product convergence structure $q$ is denoted by $\prod_{\lambda \in \Lambda} q_\lambda$ and the pair $(X, q)$ is called the product convergence space of $\{(X_\lambda, q_\lambda) \mid \lambda \in \Lambda\}$. The product convergence structure $q$ is the coarsest convergence structure on $X$ with respect to which all projections $P_\lambda . X \to (X_\lambda, q_\lambda)$ are continuous. We know that, given a filter $\Phi_\lambda$ on $X_\lambda$ for each

$\lambda \in \Lambda$, the family $\{P_\lambda^{-1}(B_\lambda) \mid B_\lambda \in \Phi_\lambda, \lambda \in \Lambda\}$ has the finite intersection property. The *product filter* of $\{\Phi_\lambda \mid \lambda \in \Lambda\}$ means the filter on $X$ which has a base the set of subsets of $X$ of the form $\cap_{\lambda \in \Lambda'} P_\lambda^{-1}(B_\lambda)$, where $B_\lambda \in \Phi_\lambda$ for each $\lambda \in \Lambda$ and $\Lambda'$ is a finite subset of $\Lambda$. The product filter of $\{\Phi_\lambda \mid \lambda \in \Lambda\}$ is denoted by $\prod_{\lambda \in \Lambda} \Phi_\lambda$ and this product filter $\Phi = \prod_{\lambda \in \Lambda} \Phi_\lambda$ is the coarsest filter on $X$ such that $P_\lambda(\Phi) = \Phi_\lambda$ for each $\lambda \in \Lambda$.(See p64, [1])

PROPOSITION 2 ([7],[3]). *Let $(X_\lambda, q_\lambda)$ be a convergence space for each $\lambda \in \Lambda$ and $(X, q)$ the product convergence space of $\{(X_\lambda, q_\lambda) \mid \lambda \in \Lambda\}$. Then the following hold:*

*(1) $q_\lambda$ is pretopological for each $\lambda \in \Lambda$ iff $q$ is pretopological.*

*(2) $q_\lambda$ is pseudotopological for each $\lambda \in \Lambda$ iff $q$ is pseudotopological.*

*(3) $\rho(q) = \prod_{\lambda \in \Lambda} \rho(q_\lambda)$, where $\rho(q)$ and $\rho(q_\lambda)$ are the pseudotopological modifications of $q$ and $q_\lambda$, respectively.*

PROPOSITION 3 ([6]). *Let $(X_\lambda, \pi(q_\lambda))$ be the pretopological modification of convergence space $(X_\lambda, q_\lambda)$ for each $\lambda \in \Lambda$. Then the following are equivalent:*

*(a) $\prod_{\lambda \in \Lambda} \pi(q_\lambda) = \pi(\prod_{\lambda \in \Lambda} q_\lambda)$*

*(b) $V_{\prod_{\lambda \in \Lambda} q_\lambda}(x) = \prod_{\lambda \in \Lambda} V_{q_\lambda}(P_\lambda(x))$ for each $x \in X = \prod_{\lambda \in \Lambda} X_\lambda$.*

PROPOSITION 4. *Let $(X_\lambda, q_\lambda)$ be a convergence space for each $\lambda \in \Lambda$ and $(X, q)$ the product convergence space of $\{(X_\lambda, q_\lambda) \mid \lambda \in \Lambda\}$.*

*If $q_\lambda$ is topological for each $\lambda \in \Lambda$, then $q$ is topological.*

PROOF. Suppose that $q_\lambda$ is topological for each $\lambda \in \Lambda$. Let $x \in X$ and $P_\lambda(x) = x_\lambda$. Then the filter $V_{q_\lambda}(x_\lambda)$ has a filter base $B_{q_\lambda}(x_\lambda)$ with the following property: $y_\lambda \in G_\lambda \in B_{q_\lambda}(x_\lambda)$ implies $G_\lambda \in B_{q_\lambda}(y_\lambda)$. Let $B_q(x)$ be the family of subsets of $X$ of the form $\bigcap_{\lambda \in \Lambda'} P_\lambda^{-1}(B_\lambda)$, where $B_\lambda \in B_{q_\lambda}(x_\lambda)$ and $\Lambda'$ is a finite subset of $\Lambda$ Then $B_q(x)$ is the filter base for the product filter $\prod_{\lambda \in \Lambda} V_{q_\lambda}(x_\lambda)$. Since $q_\lambda$ is pretopological, by Proposition 3, we obtain $\prod_{\lambda \in \Lambda} V_{q_\lambda}(x_\lambda) = V_q(x)$. Therefore, $B_q(x)$ is the filter base for $V_q(x)$. Let $y \in G \in B_q(x)$ and $G = \bigcap_{\lambda \in \Lambda'} P_\lambda^{-1}(B_\lambda)$, where $B_\lambda \in B_{q_\lambda}(x_\lambda)$ and $\Lambda'$ is a finite subset of $\Lambda$. Then $y \in P_\lambda^{-1}(B_\lambda)$ and hence $P_\lambda(y) \in B_\lambda$ for each $\lambda \in \Lambda'$. Thus, $B_\lambda \in B_{q_\lambda}(P_\lambda(y))$ and $G \in B_q(y)$. Consequently, $q$ is topological.

It is clear that given convergences $q_\lambda$ and $p_\lambda$ on $X_\lambda$ for each $\lambda \in \Lambda$, if $q_\lambda \leq p_\lambda$ for each $\lambda \in \Lambda$, then, $\prod_{\lambda\in\Lambda} q_\lambda \leq \prod_{\lambda\in\Lambda} p_\lambda$.

The following Proposition 5 is an immediate result of Proposition 1, Proposition 2 and Proposition 4.

PROPOSITION 5. *Let $(X_\lambda, \lambda(q_\lambda))$, $(X_\lambda, \pi(q_\lambda))$ and $(X_\lambda, \rho(q_\lambda))$ be the topological, the pretopological and the pseudotopological modification of convergence space $(X_\lambda, q_\lambda)$ for each $\lambda \in \Lambda$, respectively. Then the following hold.*

$$(1) \prod_{\lambda\in\Lambda} \lambda(q_\lambda) \leq \lambda(\prod_{\lambda\in\Lambda} q_\lambda) \leq \pi(\prod_{\lambda\in\Lambda} q_\lambda).$$

$$(2) \prod_{\lambda\in\Lambda} \lambda(q_\lambda) \leq \prod_{\lambda\in\Lambda} \pi(q_\lambda) \leq \pi(\prod_{\lambda\in\Lambda} q_\lambda)$$

$$\leq \rho(\prod_{\lambda\in\Lambda} q_\lambda) = \prod_{\lambda\in\Lambda} \rho(q_\lambda) \leq \prod_{\lambda\in\Lambda} q_\lambda.$$

## 3. Main Results

Given convergence space $(X, q)$, we know that $q(\mathcal{M}) = \rho(q)(\mathcal{M})$ for each ultrafilter $\mathcal{M}$ on $X$, that is , every convergence structure is *almost pseudotopological.*

Let $\{(X_\lambda, q_\lambda) \mid \lambda \in \Lambda\}$ be a family of covergence spaces. Then we know that $\rho(\prod_{\lambda\in\Lambda} q_\lambda) = \prod_{\lambda\in\Lambda} \rho(q_\lambda)$, that is, every family of convergence structures is *pseudotopologically coherent.*

In this chapter, we will study characterizations of almost pretopological convergence structure and almost toplogical convergence structure. Also, we show that almost pretopological space and almost topological space are product invariant.

DEFINITION 6 ([6]). *Let $(X, q)$ be a convergence space. Then $q$ is said to be an almost pretopological (resp. almost topological ) convergence structure provided that*

$$q(\mathcal{M}) = \pi(q)(\mathcal{M}) \quad (resp.\ \ q(\mathcal{M}) = \lambda(q)(\mathcal{M}))$$

*for all ultrafilters $\mathcal{M}$ on $X$*

THEOREM 7. *Let $(X, q)$ be a convergence space. Then $q$ is almost pretopological iff $\rho(q) = \pi(q)$.*

PROOF. It is clear that $\rho(q) \geq \pi(q)$. Suppose that $q$ is almost pretopological. We show that $\rho(q) \leq \pi(q)$. Let $\Phi$ be a filter on X and $x \in \pi(q)(\Phi)$. Also, let $\mathcal{M}$ be an ultrafilter on X with $\Phi \subset \mathcal{M}$. Then $x \in \pi(q)(\Phi) \subset \pi(q)(\mathcal{M}) = q(\mathcal{M})$. Therefore $x \in \rho(q)(\Phi)$, and so $\rho(q) \leq \pi(q)$ Consequently, $\rho(q) = \pi(q)$.

Conversely, suppose that $\rho(q) = \pi(q)$ Let $\mathcal{M}$ be an ultrafilter on $X$. Then, $\pi(q)(\mathcal{M}) = \rho(q)(\mathcal{M}) = q(\mathcal{M})$. Thus $q$ is almost pretopological. The proof is complete.

THEOREM 8. *Let $(X, q)$ be a convergence space. Then $q$ is almost topological iff $\rho(q) = \lambda(q)$.*

PROOF. It is clear that $\rho(q) \geq \lambda(q)$. Suppose that $q$ is almost topological. We show that $\rho(q) \leq \lambda(q)$. Let $\Phi$ be a filter on$X$, $x \in \lambda(q)(\Phi)$ and $\Phi$ be an ultrafilter on $X$ with $\Phi \subset \mathcal{M}$. Then $x \in \lambda(q)(\Phi) \subset \lambda(q)(\mathcal{M}) = q(\mathcal{M})$. Thus $x \in \rho(q)(\Phi)$, and so $\lambda(q)(\Phi) \subset \rho(q)(\Phi)$. Consequently, $\lambda(q) \geq \rho(q)$, that is, $\rho(q) = \lambda(q)$.

Conversely, suppose that $\rho(q) = \lambda(q)$. Let $\mathcal{M}$ be an ultrafilter on X. Then $\lambda(q)(\mathcal{M}) = \rho(q)(\mathcal{M}) = q(\mathcal{M})$. Therefore $q$ is almost topological. The proof is complete.

COROLLARY 9. *Let $(X, q)$ be a convergence space on $X$. Then the following hold:*

*(1) If $q$ is pretopological, then $q$ is almost pretopological .*

*(2) If $q$ is topological, then $q$ is almost topological.*

*(3)If $q$ is almost topological, then $q$ is almost pretopological.*

PROOF. The proof is clear from Proposition 1 (4), Theorem 7 and Theorem 8.

THEOREM 10. *Let $(X, q)$ be a convergence space. Then the following hold ;*

*(1) $q$ is pretopological iff $q$ is almost pretopological and pseudotopological.*

*(2) $q$ is topological iff $q$ is almost topological and pseudotopological.*

PROOF. (1) Suppose that $q$ is almost pretopological and pseudotopological. Then $\rho(q) = \pi(q)$ and $q = \rho(q)$. Thus $q = \pi(q)$ and so $q$ is pretopological. The converse is clear.

(2) The proof is similar to (1).

THEOREM 11. *Let $(X_\lambda, q_\lambda)$ be a convergence space for each $\lambda \in \Lambda$ and $(X, q)$ the product convergence space of $\{(X_\lambda, q_\lambda) \mid \lambda \in \Lambda\}$. Then the following hold:*

*(1) If $q_\lambda$ is almost pretopological for each $\lambda \in \Lambda$, then $q$ is almost pretopological.*

*(2) If $q_\lambda$ is almost topological for each $\lambda \in \Lambda$, then $q$ is almost topological.*

PROOF (1) Suppose that $q_\lambda$ is almost pretopological for each $\lambda \in \Lambda$. Then $\rho(q_\lambda) = \pi(q_\lambda)$. By Proposition 5, we obtain $\rho(q) = \pi(q)$ Consequently, $q$ is almost pretopological.

(2) The proof is similar to (1)

DEFINITION 12 ([2], [8]). *Let $\{(X_\lambda, q_\lambda) \mid \lambda \in \Lambda\}$ be a family of covergence spaces . The family of convergence structures $\{q_\lambda \mid \lambda \in \Lambda\}$ is said to be pretopologically (resp topologically) coherent provided that*

$$\pi(\prod_{\lambda\in\Lambda} q_\lambda) = \prod_{\lambda\in\Lambda} \pi(q_\lambda) \quad (\textit{resp.} \quad \lambda(\prod_{\lambda\in\Lambda} q_\lambda) = \prod_{\lambda\in\Lambda} \lambda(q_\lambda)).$$

THEOREM 13 *Let $(X_\lambda, q_\lambda)$ be a convergence space for each $\lambda \in \Lambda$ and $(X, q)$ the product convergence space of $\{(X_\lambda, q_\lambda) \mid \lambda \in \Lambda\}$. Then the following hold:*

*(1) If $q_\lambda$ is almost pretopological for each $\lambda \in \Lambda$, then $\{q_\lambda \mid \lambda \in \Lambda\}$ is pretopologically coherent and $V_{\prod_{\lambda\in\Lambda} q_\lambda}(x) = \prod_{\lambda\in\Lambda} V_{q_\lambda}(P_\lambda(x))$ for each $x \in X$.*

*(2) If $q_\lambda$ is almost topological for each $\lambda \in \Lambda$. then $\{q_\lambda \mid \lambda \in \Lambda\}$ is topologically coherent.*

PROOF. (1) Since $q_\lambda$ is almost pretopological, $\rho(q_\lambda) = \pi(q_\lambda)$ for each $\lambda \in \Lambda$. Thus $\prod_{\lambda\in\Lambda} \rho(q_\lambda) = \prod_{\lambda\in\Lambda} \pi(q_\lambda)$. By Proposition 5, we obtain $\prod_{\lambda\in\Lambda} \pi(q_\lambda) = \pi(\prod_{\lambda\in\Lambda} q_\lambda)$. Therefore, $\{q_\lambda \mid \lambda \in \Lambda\}$ is pretopologically coherent. The remain is clear from Proposition 4, and this result is equal to Kent [6].

(2) The proof is similar to (1)

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