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ON B-ALGEBRAS AND GROUPS

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ABSTRACT In the paper the following propositions are proved. 1) If $(Q, \ , e)$ is a B-algebra, then there exists a group $(Q, A, ^{-1}, 1)$ such that the following equalities hold $e = 1$ and $\ = {}^{-1}A$, where ${}^{-1}A(x,y) = z \overset{def}{\Longleftrightarrow} A(z,y) = x$, and 2) If $(Q, A, ^{-1}, e)$ is a group, then $(Q, {}^{-1}A, e)$ is a B-algebra

1. Preliminaries

DEFINITION 1 1(CF. [3]). Let $(Q, \cdot)$ be a groupoid. Let also e be a (fixed) element of the set Q. $(Q, \cdot, e)$ is said to be a *B-algebra* iff the following laws hold:

(i) $x \cdot x = e$,
(ii) $x \ \ e = x$ and
(iii) $(x \cdot y) \cdot z = x \cdot (z \cdot (e \cdot y))$.

PROPOSITION 1.2 [3]. *If $(Q, \cdot, e)$ is a B-algebra, then $(Q, \cdot)$ is a quasigroup.*

PROPOSITION 1.3 [3]. *Let $(Q, \cdot, e)$ be a B-algebra. Then for all $x \in Q$ the following equality holds:*

(iv) $e \cdot (e \cdot x) = x$

2. Two auxiliary proposition

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PROPOSITION 2.1. *Let $(Q,\cdot,e)$ be a B-algebra. Let also*

$$f(x)\stackrel{def}{=}e\cdot x$$

for all $x\in Q$. Then:

(1) f is a permutation of the set Q; and

(2) $f\circ f=I$, where $I=\{(x,x)|x\in Q\}$.

PROOF. By Proposition 1.2 and Proposition 1.3.

PROPOSITION 2.2. *Let $(Q,\cdot,e)$ be a B-algebra. Let also*

$$f(x)\stackrel{def}{=}e\cdot x$$

for all $x\in Q$. Then, for each $x,y\in Q$ the following equality holds:

$$x\cdot f(y)=f(f(y)\cdot x)$$

PROOF. By Definition 1.1(iii), $x=e$ and Proposition 2.1.

3. Results

THEOREM 3.1. *Let $(Q,\cdot,e)$ be a B-algebra and let*

(v) $f(x)\stackrel{def}{=}e\cdot x$ *for all* $x\in Q$.

Let also

(vi) $a*b\stackrel{def}{=}a\cdot f(b)$ *for each* $a,b\in Q$.

Then the groupoid $(Q,*)$ is a group. Moreover, for each $x,y\in Q$ the following equalities hold:

(a) $x*e=x$,

(b) $x*f(x)=e$, *and*

(c) $x\cdot y={}^{-1}A(x,y)$, *where* $A=*$ *and* ${}^{-1}A(a,b)=c\stackrel{def}{\Longleftrightarrow}A(c,b)=a$ *for each* $a,b,c\in Q$.

PROOF. Firstly we observe that under the assumptions the following statements hold:

(1) The groupoid $(Q,*)$ is isotopic (in the sense of [1] and [2]) to the groupoid $(Q,\cdot)$ (by (v),(vi) and Proposition 2.1);

(2) The groupoid $(Q,*)$ is a quasigroup (by (1) and Proposition 1.2); and

(3) The groupoid $(Q,*)$ is a semigroup. Indeed, by Definition 1.1 (iii), (v), (vi), Proposition 2.1 and Proposition 2.2, we conclude that the following series of implications hold:

$(a\cdot b)\cdot c = a\cdot(c\cdot f(b)) \overset{2.2}{\Longrightarrow} (a\cdot b)\cdot c = a\cdot f(f(b)\cdot c) \overset{(vi),2.1}{\Longrightarrow}$
$(a*f(b))*f(c) = a*f(f(f(b)*f(c))) \overset{2.1(2)}{\Longrightarrow}$
$(a*f(b))*f(c) = a*(f(b)*f(c))$
for all $a,b,c\in Q$. Whence, by Proposition 2.1(1), we conclude that the groupoid $(Q,*)$ is a semigroup.

By (2) and (3), we conclude that the groupoid $(Q,*)$ is a group. The proof of (a):

$$x \overset{(ii)}{=} x\cdot e \overset{(vi),2.1}{=} x*f(e) \overset{1.3,2.1}{=} x*e.$$

The proof of (b):

$$e \overset{(i)}{=} x\cdot x \overset{(vi),2.1}{=} x*f(x).$$

The proof of (c):

$$\begin{aligned} {}^{-1}A(x,y) = z &\overset{def}{\Longleftrightarrow} A(z,y) = x \overset{A=*}{\Longleftrightarrow} z*y = x \\ &\Longleftrightarrow z = x*y^{-1} \overset{(b)}{\Longleftrightarrow} z = x*f(y) \overset{(vi),2.1}{\Longleftrightarrow} z = x\cdot y \end{aligned}$$

THEOREM 3.2 *Let $(Q,A,{}^{-1},e)$ be a group and let*

$$ {}^{-1}A(x,y) = z \overset{def}{\Longleftrightarrow} A(z,y) = x$$

for each $x,y,z\in Q$. Then the algebra $(Q,{}^{-1}A,e)$ is a B-algebra.

PROOF. At first observe that

$$^{-1}A(x,y) = z \stackrel{def}{\iff} A(z,y) = x \iff z = A(x,y^{-1}).$$

$$^{-1}A(x,e) = A(x,e^{-1}) = A(x,e) = x.$$

$$^{-1}A(x,x) = A(x,x^{-1}) = e.$$

$$x^{-1} = A(e,x^{-1}) =^{-1} A(e,x).$$

Now, we have

$$\begin{aligned} ^{-1}A(^{-1}A(x,y),z) &= A(A(x,y^{-1}),z^{-1}) = A(x,A(y^{-1},z^{-1})) \\ &= A(x,(A(z,y))^{-1}) =^{-1} A(x,A(z,y)) \\ &=^{-1} A(x,A(z,(y^{-1})^{-1})) =^{-1} A(x,^{-1}A(z,y^{-1})) \\ &=^{-1} A(x,^{-1}A(z,^{-1}A(e,y))). \end{aligned}$$

This completes the proof.

4. Remarks

4.1. In the [4] the following proposition is proved: A BCI-algebra $(Q,\cdot,0)$ is a BCI-quasigroup iff there exists a commutative group $(Q,+,0)$ such that $x \cdot y = x - y$. (In this case $x + y = x(0y)$.) In [5] the following propositions are proved:

(1) A BCI-algebra $(Q,\cdot,0)$ is right alternative, left alternative or flexible iff it is a group of the exponent 2; and

(2) A weak BCC-algebra is a Boolean group iff it satisfies (at least) one of the following identities $x \cdot (y \cdot x) = y$, $(x \cdot y) \cdot x = y$ See, also [6].

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4.2. The algebra $(Q,\cdot,f)$ satisfying the law

$$(x \cdot y) \cdot z = x \cdot f(f(y) \cdot z)$$

is described in [7]

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