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SOME RESULTS ON ENDOMORPHISMS OF PRIME RING WHICH ARE (σ,τ)-DERIVATION

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ABSTRACT. Let R be a prime ring with characteristic not two and U is a nonzero left ideal of R which contains no nonzero nilpotent right ideal as a ring For a $(\sigma,\ \tau)-$derivation d $R \to R$, we prove the following results. (1) If d is an endomorphism on R then $d = 0$. (2) If d is an anti-endomorphism on R then $d = 0$ (3) If $d(xy) = d(yx)$, for all $x, y \in R$ then R is commutative. (4) If d is an homomorphism or anti-homomorphism on U then $d = 0$

1. Introduction

The primary purpose of this paper is to investigate about a $(\sigma,\tau)-$ *derivation* d which is a ring endomorphism or anti-endomorphism on R. Bell and Kappe ([2]) proved that if d is a derivation of R which is either an endomorphism or anti-endomorphism in semi-prime ring R, then $d = 0$, and if d acts as a homomorphism or anti-homomorphism is a nonzero right ideal U of prime ring R, then $d = 0$ on R. It is our aim in this paper to extend the above mentioned results to a more general situation

In this paper, R will represent an associative ring Recall that a ring R is prime if $aRb = \{0\}$ implies that $a = 0$ or $b = 0$. Let R be a ring and σ, τ be two automorphisms of R . We write $[x, y]$, $[x, y]_{\sigma,\tau}$,

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for $xy - yx$ and $x\sigma(y) - \tau(y)x$ respectively and make extensive use of basic commutator identities: $[xy,z]_{\sigma,\tau} = x[y,z]_{\sigma,\tau} + [x,\tau(z)]y = x[y,\sigma(z)] + [x,z]_{\sigma,\tau}y$. We set $Z = \{c \in R | cx = xc$, for all $x \in R\}$ and call the *center of R*.

An additive mapping $d : R \to R$ is called a *derivation* if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. A derivation d is *inner* if there exits an $a \in R$ such that $d(x) = [a,x]$ holds for all $x \in R$ and d is called (σ,τ) – *derivation* if $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$ for all $x, y \in R$. On the other hand we said that d is an endomorphism or anti-endomorphism respectively $d(xy) = d(x)d(y)$ or $d(xy) = d(y)d(x)$ for all $x, y \in R$.

2. Results

THEOREM 1. *Let R be a prime ring. If d is a (σ,τ) – derivation of R which is an endomorphism on R, then $d = 0$.*

PROOF. Since d acts as a homomorphism on R, we have

$$(2.1) \qquad d(xy) = d(x)\sigma(y) + \tau(x)d(y) = d(x)d(y) \quad \text{for all} \quad x, y \in R.$$

Substituting xr for $x, r \in R$ in (2.1), we get

$$d(xr)\sigma(y) + \tau(xr)d(y) = d(xr)d(y).$$

Since d is an homomorphism on R and τ is an automorphism of R, we have

$$d(x)d(r)\sigma(y) + \tau(x)\tau(r)d(y) = d(x)d(r)d(y).$$

Expanding the last equation one obtains,

$$\begin{aligned} d(x)d(r)\sigma(y) + \tau(x)\tau(r)d(y) &= d(x)d(ry) \\ &= d(x)d(r)\sigma(y) + d(x)\tau(r)d(y) \end{aligned}$$

or equivalently,

$$\begin{aligned} 0 &= d(x)\tau(r)d(y) - \tau(x)\tau(r)d(y) \\ &= (d(x) - \tau(x))\tau(r)d(y). \end{aligned}$$

Since τ is an automorphism of R, we get

$$(d(x)-\tau(x))Rd(y)=0 \quad \text{for all} \quad x,y\in R.$$

Since R is a prime ring, we conclude that

$$d(x)=\tau(x) \quad \text{for all} \quad x\in R \text{ or } \quad d=0. \tag{2.2}$$

Assume $d(x)=\tau(x)$ for all $x\in R$. Replacing x by $xy, y\in R$ in this equation we have

$$d(xy)=\tau(xy)=\tau(x)\tau(y).$$

On the other hand, recalling d is a $(\sigma,\tau)-$ *derivation* and (2.2), it follows

$$d(x)\sigma(y)+\tau(x)d(y)=\tau(x)d(y)$$

and so,

$$d(x)\sigma(y)=0 \quad \text{for all} \quad x,y\in R.$$

Since R is a prime ring, we see that $d=0$ on R.

THEOREM 2. *Let R be a prime ring If d is a $(\sigma,\tau)-$derivation of R which is an anti-endomorphism on R, then $d=0$.*

PROOF. Since d acts as a anti-homomorphism on R, we get

$$d(xy)=d(x)\sigma(y)+\tau(x)d(y)=d(y)d(x) \quad \text{for all} \quad x,y\in R. \tag{2.3}$$

Replacing y by xy in (2.3), we have

$$d(x)\sigma(xy)+\tau(x)d(xy)=d(xy)d(x).$$

Recall that d is a $(\sigma,\tau)-$*derivation* of R which is an anti-endomorphism on R, we have

$$d(x)\sigma(x)\sigma(y)+\tau(x)d(y)d(x)=d(x)\sigma(y)d(x)+\tau(x)d(y)d(x)$$

Since the second terms on the both sides are equal, we conclude that

$$d(x)\sigma(y)d(x)-d(x)\sigma(x)\sigma(y)=0 \quad \text{for all} \quad x,y\in R. \tag{2.4}$$

Substituting $yr, r \in R$ for y in this equation, we get

$$\begin{aligned} 0 &= d(x)\sigma(yr)d(x) - d(x)\sigma(x)\sigma(yr) \\ &= d(x)\sigma(y)\sigma(r)d(x) - d(x)\sigma(x)\sigma(y)\sigma(r). \end{aligned}$$

Using (2.4), it gives

$$\begin{aligned} 0 &= d(x)\sigma(y)\sigma(r)d(x) - d(x)\sigma(y)d(x)\sigma(r) \\ &= d(x)\sigma(y)[\sigma(r), d(x)]. \end{aligned}$$

Since σ, τ are automorphisms of R, we obtain

$$d(x)R[\sigma(r), d(x)] = 0 \quad \text{for all} \quad x, r \in R.$$

Since R is a prime ring,

$$d(x) = 0 \text{ or } d(x) \in Z \text{ for all } x \in R.$$

If $d(x) = 0$ then $d(x) \in Z$. So, we can take $d(R) \subset Z$ which forces d to be an endomophism of R. It follows $d = 0$ from Theorem 1. This completes the proof of the Theorem 2.

THEOREM 3. *Let R be a prime ring of characteristic not two. If d is a nonzero (σ, τ) – derivation of R and $d(xy) = d(yx)$ for all $x, y \in R$, then R is a commutative ring.*

PROOF

For any element $c \in R$ such that $d(c) = 0$, for example $c = [x, y]$, we have

$$d(z)\sigma(c) = d(zc) = d(cz) = \tau(c)d(z)$$

for all $z \in R$.

Thus

$$[d(z), c]_{\sigma,\tau} = 0 \quad \text{for all} \quad z \in R. \tag{2.5}$$

This reduces $c \in Z$ for all $c \in R$ such that $d(c) = 0$ by [4, Theorem 1]. In view of (2.5), we obtain $[x, y] \in Z$ for all $x, y \in R$ because of $d([x, y]) = 0$. Thus R is commutative by [3, Lemma 1.5].

LEMMA 1. *Let R be a prime ring and U a nonzero left ideal of R which is semiprime as a ring If $Ua = 0$ ($aU = 0$) for $a \in R$ then $a = 0$.*

PROOF. Since R is a prime ring and U is a nonzero left ideal of R, if $aU = 0$ then $a = 0$. Now, let us show that $Ua = 0$ then $a = 0$. Assume that $a \neq 0$. Define L by

$$L = \{x \in R \mid Ux = 0\}.$$

Since $0 \neq a \in L$ it is clearly that L is a nonzero right ideal of R such that $UL = (0)$. On the other hand, $L \cap U$ is a right ideal of U and

$$(L \cap U)(L \cap U) \subset UL = (0),$$

that is,

$$(L \cap U)^2 = (0).$$

Since U is semiprime,we have $L \cap U = 0$ In this case, we have

$$LU \subset L \cap U = (0)$$

Since R is a prime ring and U is a nonzero left ideal of R, one obtains $L = (0)$ Thus we get $a = 0$.

LEMMA 2. *Let R be a prime ring and U a nonzero left ideal of R which is semiprime as a ring. If d is a $(\sigma,\tau) -$ derivation of R such that $d(U) = 0$ then $d = 0$.*

PROOF. By hypothesis for all $x \in R$, $m \in U$, we get

$$0 = d(xm) = d(x)\sigma(m) + \tau(x)d(m) = d(x)\sigma(m).$$

Since σ is an automorphism of R, it follows from Lemma 1 that $d(x) = 0$ for all $x \in R$.

THEOREM 4. *Let R be a prime ring, U a nonzero left ideal of R which is semiprime as a ring. If d is a nonzero (σ,τ)-derivation of R such that $d(U)a = 0(ad(U) = 0)$, then $a = 0$.*

PROOF. For all $u \in U$, $x \in R$ we have

$$0 = d(xu)a = d(x)\sigma(u)a + \tau(x)d(u)a.$$

From the hypothesis, we take

$$d(x)\sigma(u)a = 0 \quad \text{for all} \quad u \in U, \ x \in R.$$

That is $U\sigma^{-1}(a) = 0$ by [1, Lemma 1]. And so, $a = 0$ by Lemma 1. If $ad(U) = 0$, then for all $u, v \in U$,

$$0 = ad(uv) = ad(u)\sigma(v) + a\tau(u)d(v).$$

That is,

$$a\tau(u)d(v) = 0 \quad \text{for all} \quad u, v \in U.$$

We can take $\tau^{-1}(a)U\tau^{-1}(d(v)) = 0$ for all $u, v \in U$ since τ is an automorphism of R. $U\tau^{-1}(d(v))$ is a left ideal of R, we obtain $a = 0$ or $U\tau^{-1}(d(v)) = 0$ from Lemma 1. If $U\tau^{-1}(d(v)) = 0$ for all $v \in U$ then by Lemma 1 and Lemma 2 we get $d = 0$.

THEOREM 5. *Let R be a prime ring, U is a nonzero left ideal of R which is semiprime as a ring and d is a (σ, τ) – derivation of R. If d acts as a homomorphism on U, then $d = 0$.*

PROOF. Since d acts as a homomorphism on U, we have

$$d(vu) = d(v)d(u) = d(v)\sigma(u) + \tau(v)d(u) \quad \text{for all} \quad u, v \in U.$$

Substituting $ut, t \in U$ for u, we get

$$\begin{aligned} d(v)\sigma(u)d(t) + \tau(v)d(u)d(t) &= d(vu)d(t) = d(v)d(u)d(t) \\ &= d(v)d(ut) = d(v(ut)) \\ &= d(v)\sigma(u)\sigma(t) + \tau(v)d(ut) \\ &= d(v)\sigma(u)\sigma(t) + \tau(v)d(u)d(t) \end{aligned}$$

and so,

$$d(U)\sigma(u)(d(t) - \sigma(t)) = 0 \quad \text{for all} \quad u, t \in U.$$

Using Theorem 4, we get $d = 0$ by Lemma 2 or

$$U\sigma^{-1}(d(t) - \sigma(t)) = 0 \quad \text{for all} \quad t \in U.$$

If $U\sigma^{-1}(d(t) - \sigma(t)) = 0$, then by Lemma 1, one obtains,

$$d(t) = \sigma(t) \quad \text{for all} \quad t \in U. \tag{2.6}$$

Replacing t by $tu, t, u \in U$ in (2.6)

$$\begin{aligned} \sigma(t)\sigma(u) &= \sigma(tu) = d(tu) \\ &= d(t)\sigma(u) + \tau(t)d(u) \\ &= \sigma(t)\sigma(u) + \tau(t)d(u) \end{aligned}$$

that is,

$$\tau(t)d(u) = 0, \quad \text{for all} \quad t, u \in U.$$

By Theorem 4, we get $d = 0$.

THEOREM 6. *Let R be a prime ring, U is a nonzero left ideal of R which is semiprime as a ring and d is a (σ,τ) – derivation of R. If d acts an anti-homomorphism on U, then $d = 0$.*

PROOF. Since d acts as a anti-homomorphism on U, we have

$$d(uv) = d(v)d(u) = d(u)\sigma(v) + \tau(u)d(v) \quad \text{for all} \quad u, v \in U. \tag{2.7}$$

Substituting uv for v in (2.7), we get

$$\begin{aligned} d(u)\sigma(u)\sigma(v) + \tau(u)d(v)d(u) &= d(u)\sigma(uv) + \tau(u)d(uv) = d(uv)d(u) \\ &= d(u)\sigma(v)d(u) + \tau(u)d(v)d(u) \end{aligned}$$

or equivalently,

$$d(u)\sigma(v)d(u) = d(u)\sigma(u)\sigma(v) \quad \text{for all} \quad u, v \in U. \tag{2.8}$$

Replacing v by $vt, t \in U$ in (2.8) and using (2.8), we have,

$$d(u)\sigma(v)\sigma(t)d(u) = d(u)\sigma(u)\sigma(v)\sigma(t) = d(u)\sigma(v)d(u)\sigma(t)$$

and so,

$$d(u)\sigma(v)[\sigma(t), d(u)] = 0 \quad \text{for all} \quad u, v, t \in U.$$

That is,

$$\sigma^{-1}(d(u))U[t, \sigma^{-1}(d(u))] = 0, \quad \text{for all} \quad u, t \in U$$

Since $U[t, \sigma^{-1}(d(u))]$ is a left ideal and R is a prime ring it gives

$$d(u) = 0 \quad \text{or} \quad [\sigma(t), d(u)] = 0 \quad \text{for all} \quad u, t \in U.$$

Define for fixed $t \in R, K = \{u \in U | d(u) = 0\}$ and $L = \{u \in U | [\sigma(t), d(u)] = 0\}$. A group can not be the set theoretic union of two proper subgroups, hence $U = K$ or $U = L$. In the former case, $d(U) = (0)$. It gives that $d = 0$ by Lemma 2. So we have $[\sigma(t), d(u)] = 0$, for all $u, t \in U$. Replacing t by $rt, r \in R$ we have

$$[\sigma(r), d(u)]\sigma(t) = 0 \text{ for all } u, t \in U, r \in R$$

and so,

$$[R, \sigma^{-1}(d(u))]U = 0.$$

Since U is a left ideal of R and R is a prime ring, we get $d(U) \subset Z$ which forces d to be an endomophism of R. It follows $d = 0$ from Theorem 5 This completes the proof of the Theorem 6.

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