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COMMON FIXED POINT THEOREM FOR WEAKLY COMPATIBLE MAPPINGS WITHOUT CONTINUITY IN FUZZY METRIC SPACES

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ABSTRACT The aim of this paper is to prove a common fixed point theorem from the class of compatible maps to larger class of weakly compatible maps without appeal to continuity in fuzzy metric spaces

1. Introduction and preliminaries

In 1965, the concept of fuzzy sets was introduced by Zadeh [26]. Since then, many authors have expansively developed the theory of fuzzy sets and applications. Especially, Deng [6], Erceg [7], Kaleva and Seikkala [18] Kramosil and Michalek [20] have introduced the concepts of fuzzy metric spaces in different ways. However, it appears that the study of Kramosil and Michalek [20] of fuzzy metric spaces paves the way for developing a smoothing machinery in the field of fixed point theorems, in particular for the study of contractive maps.

Recently, many authors have also studied the fixed point theory in these fuzzy metric spaces ([1]-[4], [8], [10]-[14], [21], [24], [25]).

Grabiec [10] followed Kramosil and Michalek [20] and obtained the fuzzy version of Banach's fixed point theorem. Banach's fixed point theorem has many applications, but suffers from one drawback- the definition requires that the mapping be continuous throughout the space

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In 1976, Jungck [15] proved a common fixed point theorem for commuting maps generalizing the Banach's fixed point theorem. Sessa [23] defined a generalization of commutativity, which is called weak commutativity. Further Jungck [16] introduced more generalized commutativity, so called compatibility. Mishra et al [21] introduced the concept of compatibility in fuzzy metric spaces. Various fixed point theorems, for compatible mappings satisfying contractive type conditions and assuming continuity of at least one of the mappings in the compatible pairs in metric spaces and fuzzy metric spaces, have been obtained by many authors. It may be observed in this context that it is known since the paper of Kannan [19] in 1968 that there exist maps that have a discontinuity in their domain but which have fixed points, however, in all the known cases the maps involved were continuous at the fixed point.

In 1998 Jungck and Rhoades [17] introduced the notion of weakly compatible maps and showed that compatible maps are weakly compatible but converse need not true.

Recently, Chugh and Kumar [5] proved an interesting result in metric spaces for weakly compatible maps without assuming any mapping continuous.

In this paper, we improve the result of Mishra et al.[21] by relaxing the compatibility to weak compatibility, removing the assumption of continuity and commutativity in fuzzy metric spaces.

DEFINITION 1 [22]. A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is called a continuous t-norm if ([0,1], *) is an Abelian topological monoid with the unit 1 such that $a*b\le c*d$ whenever $a\le c$ and $b\le d$ for all $a,b,c,d\in[0,1]$.

Examples of t-norm are $a*b=a.b$ and $a*b=min\{a,b\}$.

DEFINITION 2 [20]. The 3-tuple $(X,M,*)$ is called a fuzzy metric space (shortly FM-space) if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set in $X^2\times[0,\infty)$ satisfying the following conditions for all $x,y,z\in X$ and $t,s>0$,

(FM-1) $M(x,y,0)=0$,

(FM-2) $M(x,y,t)=1$ for all $t>0$ if and only if $x=y$,

(FM-3) $M(x, y, t) = M(y, x, t)$,
(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(FM-5) $M(x, y, .) : [0, 1] \to [0, 1]$ is left continuous.

In what follows, $(X, M, *)$ will denote a fuzzy metric space. Note that $M(x, y, t)$ can be thought as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$ and $M(x, y, t) = 0$ with ∞ and we can find some topological properties and examples of fuzzy metric spaces in [9].

In the following example, we know that every metric induces a fuzzy metric.

EXAMPLE 1[9]. Let (X, d) be a metric space. Define $a * b = a.b$ (*or* $a * b = min\{a, b\}$) and for all $x, y \in X$ and $t > 0$,

$$\text{(i)} \qquad M(x, y, t) = \frac{t}{t + d(x, y)}$$

Then $(X, M, *)$ is a fuzzy metric space. We call this fuzzy metric M induced by the metric d the standard fuzzy metric. On the other hand, note that there exists no metric on X satisfying (i).

LEMMA 1 [10]. *For all* $x, y \in X$, $M(x, y, \ )$ *is nondecreasing.*

DEFINITION 3 [10] Let $(X, M, *)$ be a FM-space.

(1) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ (denoted by $\lim_{n\to\infty} x_n = x$) if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.
(2) A sequence $\{x_n\}$ in X is called Cauchy sequence if $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ for all $t > 0$ and $p > 0$.
(3) An FM-space in which every Cauchy sequence is convergent is said to be complete.

REMARK 1. Since * is continuous, it follows from (FM-4) that limit of sequence uniquely determined.

Let $(X, M, *)$ be an FM-space with the following condition:

(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$.

LEMMA 2 [4]. *Let $\{y_n\}$ be a sequence in a FM-space $(X, M, *)$ with the condition (FM-6). If there exists a number $k \in (0,1)$ such that*

$$M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t)$$

for all $t > 0$ and $n = 1, 2, \ldots$ then $\{y_n\}$ is a Cauchy sequence in X.

LEMMA 3 [21]. *If for all $x, y \in X$, $t > 0$ and for a number $k \in (0,1)$*

$$M(x, y, kt) \geq M(x, y, t),$$

then $x = y$.

DEFINITION 4 [21]. Let A and B be mappings from a FM-space $(X, M, *)$ into itself. The mappings A and B are said to be compatible if

$$\lim_{n \to \infty} M(ABx_n, BAx_n, t) = 1,$$

for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = z \quad for \ some \ z \in X.$$

DEFINITION 5 [17]. Two maps A and B are said to be weakly compatible if they commute at coincidence points

EXAMPLE 2. Let $X = [0, 2]$ with the metric d defined by $d(x, y) = |x - y|$. For each $t \in (0, \infty)$ define

$$M(x, y, t) = \frac{t}{t + d(x, y)}, \quad x, y \in X$$

$$M(x, y, 0) = 0, \quad x, y \in X.$$

Clearly $M(x, y, *)$ is a fuzzy metric space on X where * is defined by $a * b = ab$ or $a * b = min\{a, b\}$. Define $A, B : X \to X$ by

$$Ax = \begin{cases} 0.4 & \text{if } x \in [0, 0.3] \\ 1 & \text{if } x = 1 \\ 0.7 + x & \text{if } x > 0.3 \end{cases}$$

$$Bx = \begin{cases} 0.6 & \text{if } x \in [0, 0.3] \\ x & \text{if } x = 1 \\ \dfrac{x + 0.2}{0.5} & \text{if } x > 0.3 \end{cases}$$

Consider the sequence $\{x_n = 0.3+1/n : n \geq 1\}$ in X. Then $\lim_{n\to\infty} Ax_n = 1$, $\lim_{n\to\infty} Bx_n = 1$. But $\lim_{n\to\infty} M(ABx_n, BAx_n, t) = \frac{t}{t+|1.7-2.4|} = \frac{t}{t+0.7} \neq 1$.

Thus A and B are noncompatible. But A and B are weakly compatible since they commute at coincidence point $x = 1$. Hence weakly compatible maps need not be compatible.

2. Main results

THEOREM 1. *Let $(X, M, *)$ be a complete FM-space with $t*t \geq t$ for all $t \in [0, 1]$ and the condition (FM-6). Let A, B, S and T be mappings from X into itself such that*

(1) *$A(X) \subset T(X)$, $B(X) \subset S(X)$,*
(2) *the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible,*
(3) *there exists a constant $k \in (0, 1)$ such that $M(Ax, By, kt) \geq M(Ty, By, t) * M(Sx, Ax, t) * M(Sx, By, at) * M(Ty, Ax, (2-a)t) * M(Ty, Sx, t)$ for all $x, y \in X$, $a \in (0, 2)$ and $t > 0$.*

Then A, B, S and T have a common fixed point in X.

PROOF. By (1.1), since $A(X) \subset T(X)$, for any point $x_0 \in X$, there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$. Since $B(X) \subset S(X)$, for this point x_1 we can choose a point $x_2 \in X$ such that $Bx_1 = Sx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that

$$\begin{aligned} y_{2n} &= Ax_{2n} = Tx_{2n+1} \text{ and} \\ y_{2n+1} &= Bx_{2n+1} = Sx_{2n+2} \quad for \ n = 0, 1, \ldots \end{aligned} \tag{1.4}$$

By (1.3), for all $t > 0$ and $a = 1 - q$ with $q \in (0, 1)$, we have

$$
\begin{aligned}
& M(y_{2n+1}, y_{2n+2}, kt) \\
& = M(Bx_{2n+1}, Ax_{2n+2}, kt) \\
& = M(Ax_{2n+2}, Bx_{2n+1}, kt) \\
& \geq M(Tx_{2n+1}, Bx_{2n+1}, t) * M(Sx_{2n+2}, Ax_{2n+2}, t) \\
& * M(Sx_{2n+2}, Bx_{2n+1}, at) \\
& * M(Tx_{2n+1}, Ax_{2n+2}, (1+q)t) \\
& * M(Tx_{2n+1}, Sx_{2n+2}, t) \\
(1.5) \quad & = M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t) \\
& * M(y_{2n+1}, y_{2n+1}, at) * M(y_{2n}, y_{2n+2}, (1+q)t) \\
& * M(y_{2n}, y_{2n+1}, t) \\
& \geq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t) \\
& * 1 * M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, qt) \\
& * M(y_{2n}, y_{2n+1}, t) \\
& \geq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t) \\
& * M(y_{2n+1}, y_{2n+2}, qt).
\end{aligned}
$$

Since the t-norm * is continuous and $M(x, y, .)$ is left continuous letting $q \to 1$, we have

$$(1.6) \quad M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t).$$

Similarly, we have also

$$(1.7) \quad M(y_{2n+2}, y_{2n+3}, kt) \geq M(y_{2n+1}, y_{2n+2}, t) * M(y_{2n+2}, y_{2n+3}, t).$$

Thus by (1.6) and (1.7), it follows that

$$M(y_{n+1}, y_{n+2}, kt) \geq M(y_n, y_{n+1}, t) * M(y_{n+1}, y_{n+2}, t)$$

for $n = 1, 2, \ldots$ and so for positive integers n, p

$$M(y_{n+1}, y_{n+2}, kt) \geq M(y_n, y_{n+1}, t) * M(y_{n+1}, y_{n+2}, t/k^p).$$

Thus since $M(y_{n+1}, y_{n+2}, t/k^p) \to 1$ as $p \to \infty$, we have

$$M(y_{n+1}, y_{n+2}, kt) \geq M(y_n, y_{n+1}, t).$$

By Lemma 2, $\{y_n\}$ is a Cauchy sequence in X and so, since X is complete, $\{y_n\}$ converges to a point z (say) in X. So the subsequences $\{y_{2n}\}$ and $\{y_{2n+1}\}$ of $\{y_n\}$ also converge to the same limit z. Hence there exist two points u, v in X such that $Su = z$ and $Tv = z$, respectively.

By (1.3), with $a = 1$, we have

$$\begin{aligned} M(Au, y_{2n+1}, kt) &= M(Au, Bx_{2n+1}, kt) \\ &\geq M(Tx_{2n+1}, Bx_{2n+1}, t) * M(Su, Au, t) \\ &* M(Su, Bx_{2n+1}, t) * M(Tx_{2n+1}, Au, t) \\ &* M(Tx_{2n+1}, Su, t) \\ &= M(y_{2n}, y_{2n+1}, t) * M(Su, Au, t) \\ &* M(Su, y_{2n+1}, t) * M(y_{2n}, Au, t) \\ &* M(y_{2n}, Su, t), \end{aligned}$$

which implies that as $n \to \infty$

$$\begin{aligned} M(Au, z, kt) &\geq 1 * M(z, Au, t) * 1 * M(z, Au, t) * 1 \\ &\geq M(Au, z, t) \end{aligned}$$

Therefore by Lemma 3, we have $Au = z$. Since $Su = z$ thus $Au = z = Su$, i.e. u is a coincidence point of A and S. Similarly, we can show that v is a coincidence point of B and T.

Since the pair $\{A, S\}$ is weakly compatible therefore A and S commute at their coincidence point i.e. $ASu = SAu$ or $Az = Sz$. Similarly $BTv = TBv$ or $Bz = Tz$.

Now, we prove that $Az = z$. By (1.3) with $a = 1$ we have,

$$\begin{aligned} M(Az, y_{2n+1}, kt) &= M(Az, Bx_{2n+1}, kt) \\ &\geq M(Tx_{2n+1}, Bx_{2n+1}, t) \\ &* M(Sz, Az, t) * M(Sz, Bx_{2n+1}, t) \\ &* M(Tx_{2n+1}, Az, t) * M(Tx_{2n+1}, Sz, t) \\ &= M(y_{2n}, y_{2n+1}, t) * 1 * M(Az, y_{2n+1}, t) \\ &* M(y_{2n}, Az, t) * M(y_{2n}, Az, t). \end{aligned}$$

Proceeding limit as $n \to \infty$, we have

$$\begin{aligned} M(Az, z, kt) &\geq 1 * 1 * M(Az, z, t) * M(z, Az, t) * M(z, Az, t) \\ &\geq M(Az, z, t). \end{aligned}$$

Therefore, by Lemma 3, we have $Az = z = Sz$. Similarly, we have $Bz = z = Tz$. This means that z is a common fixed point of A, B, S and T.

For uniqueness of common fixed point let $w \neq z$ be another common fixed point of A, B, S and T. Then by (1.3) with $a = 1$ we have

$$\begin{aligned} M(z, w, kt) &= M(Az, Bw, kt) \\ &\geq M(Tw, Bw, t) * M(Sz, Az, t) \\ &* M(Sz, Bw, t) * M(Tw, Az, t) * M(Tw, Sz, t) \\ &\geq 1 * 1 * M(z, w, t) * M(w, z, t) * M(w, z, t) \\ &\geq M(z, w, t). \end{aligned}$$

Therefore, by Lemma 3, we have $z = w$. This completes the proof.

We now give an example to illustrate the above theorem.

EXAMPLE 3. Let $X = [0, 2]$ with the metric d defined by $d(x, y) = |x - y|$. For each $t \in (0, \infty)$ define

$$M(x, y, t) = \frac{t}{t + d(x, y)} \quad x, y \in X$$

$$M(x, y, 0) = 0 \quad x, y \in X$$

Then $(X, M, *)$ is a fuzzy metric space, where * is defined by $a*b = a.b$. Clearly $(X, M, *)$ is a complete fuzzy metric space. Define $A, B, S, T : X \to X$ by

$$Ax = \begin{cases} 0 & \text{if } x = 0 \\ 0.15 & \text{if } x > 0 \end{cases}$$

$$Bx = \begin{cases} 0 & \text{if } x = 0 \\ 0.35 & \text{if } x > 0 \end{cases}$$

$$Sx = \begin{cases} 0 & \text{if } x = 0 \\ 0.3 & \text{if } 0 < x \leq 0.5 \\ x - 0.35 & \text{if } x > 0.5 \end{cases}$$

$$Tx = \begin{cases} 0 & \text{if } x = 0 \\ 0.15 & \text{if } 0 < x \leq 0.5 \\ x - 0.15 & \text{if } x > 0.5 \end{cases}$$

If we take $k = 0.5$, $t = 1$ and $a = 1$ we see that A, B, S, T satisfy all the conditions of the above theorem and have a unique common fixed point $0 \in X$. It may be noted in this example that the mappings A and S commute at coincidence point $0 \in X$. So A and S are weakly compatible maps. Similarly B and T are weakly compatible maps. To see that the pairs $\{A, S\}$ and $\{B, T\}$ are non compatible, let us consider a decreasing sequence $\{x_n\}$ such that $x_n \to 0.5$. Then $Ax_n \to 0.15, Sx_n \to 0.15$, but $\lim_{n\to\infty} M(ASx_n, SAx_n, t) = \frac{t}{t+|0.15-0.3|} \neq 1$. Thus the pair $\{A, S\}$ is noncompatible. Also $Bx_n \to 0.35, Tx_n \to 0.35$ but $\lim_{n\to\infty} M(BTx_n, TBx_n, t) = \frac{t}{t+|0.35-0.6|} \neq 1$. So the pair $\{B, T\}$ is noncompatible. All the mappings involved in this example are discontinuous at the common fixed point.

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