East Asian Math J 18(2002), No 2, pp 175-182

COMMON FIXED POINT, MULTIMAPS IN FUZZY METRIC SPACE

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ABSTRACT The purpose of this paper is to obtain some common fixed point theorems for multivalued mappings in fuzzy metric space Of course this is a new result on this line

1. Introduction and Preliminaries

In 1965, the concept of fuzzy sets was introduced initally by Zadeh [10]. Since then many authors have expansively developed the theory of fuzzy sets and applications Especially, Deng [24], Erceg [12], Kaleva and Seikkala [15], Kramosil and Michalek [5] have introduced the concept of fuzzy metric space in different ways.

Recently, many authors have also studied the fixed point theory in these fuzzy metric spaces are Badard [16], Chang, Cho, Lee, Jung, and Kang [18], Fang [7], Grabiec [11], Hadzic [13], [14], Jung, Cho and Kim [8], Jung, Cho, Chang, and Kang [9], Sharma [22], [23], Mishra, Sharma, and Singh [21] and for fuzzy mappings are Bose and Sahani [2], Butnariu [4], Chang [17] , Chang, Cho, Lee and Lee [19], Heilpern [20]. In this note we extend result of Grabiec [11] and others for multivalued mappings introduced by Kubiaczyk and Sharma [6].

Now we begin with some definitions:

Received March 11, 2002 Revised November 10, 2002.
2000 Mathematics Subject Classification: 47H10, 54H25
Key words and phrases fuzzy metric space, sequence, common fixed point.

DEFINITION 1 [3]. A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if $([0,1], *)$ is an abelian topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Examples of t-norm are $a * b = ab$ and $a * b = min\{a, b\}$.

DEFINITION 2 [5]. The 3-tuple $(X, M, *)$ is called a fuzzy metric space (shortly, FM-space) if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions: for all $x, y, z \in X$ and $s, t > 0$,

(FM-1) $M(x, y, 0) = 0$,

(FM-2) $M(x, y, t) = 1$, for all $t > 0$ if and only if $x = y$,

(FM-3) $M(x, y, t) = M(y, x, t)$,

(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

(FM-5) $M(x, y, \cdot) : [0, 1) \to [0, 1]$ is left continuous.

In what follows, $(X, M, *)$ will denote a fuzzy metric space. Note that $M(x, y, t)$ can be thought of as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$ and $M(x, y, t) = 0$ with ∞, and we can find some topological properties and examples of fuzzy metric spaces in paper of George and Veeramani [1].

In the following example, we know that every metric induces a fuzzy metric.

EXAMPLE 1 [1]. Let (X, d) be a metric space. Define $a * b = ab$ (or $a * b = min\{a, b\}$) and for all $x, y \in X$ and $t > 0$,

$$M(x, y, t) = \frac{t}{t + d(x, y)}. \tag{1.a}$$

Then $(X, M, *)$ is a fuzzy metric space. We call this fuzzy metric M induced by the metric d the standard fuzzy metric. On the other hand, note that there exists no metric on X satisfying (1.a).

LEMMA 1 [11]. *For all $x, y \in X$, $M(x, y, \cdot)$ is nondecreasing.*

DEFINITION 3 [11]. Let $(X, M, *)$ be a fuzzy metric space:

(1) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$, (denoted by $\lim_{n\to\infty} x_n = x$), if

$$\lim_{n\to\infty} M(x_n, x, t) = 1$$

for all $t > 0$.

(2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if

$$\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$$

for all $t > 0$ and $p > 0$.

(3) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

REMARK 1. Since $*$ is continuous, it follows from (FM-4) that the limit of the sequence in FM-space is uniquely determined.

Let $(X, M, *)$ is a fuzzy metric space with the following condition:

(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$.

LEMMA 2 [21]. *Let $\{y_n\}$ be a sequence in a fuzzy metric space $(X, M, *)$ with $t * t \geq t$ for all $t \in [0, 1]$ and the condition (FM-6). If there exists a number $q \in (0, 1)$ such that*

$$M(y_{n+2}, y_{n+1}, qt) \geq M(y_{n+1}, y_n, t)$$

for all $t > 0$ and $n = 1, 2, ...$ then $\{y_n\}$ is a Cauchy sequence in X.

LEMMA 3 [21] *If, for all $x, y \in X$, $t > 0$ and for a number $q \in (0, 1)$,*

$$M(x, y, qt) \geq M(x, y, t)$$

then $x = y$.

Kubiaczyk and Sharma [6] introduced the following concept of multivalued mappings in the sense of Kramosil and Michalek [5].

We denote by $CB(X)$ the set of all non-empty, bounded and closed subsets of X. We have

$$M^{\nabla}(B,y,t) = max\{M(b,y,t) : b \in B\}$$

$$M_{\nabla}(A,B,t) = min\{min_{a\in A}\{M^{\nabla}(a,B,t)\}, min_{b\in B}\{M^{\nabla}(A,b,t)\}\}$$

for all A, B in X and $t > 0$.

2. Main Results

THEOREM 1. *Let $(X, M, *)$ be a complete fuzzy metric space with $t * t \geq t$ for all $t \in [0,1]$ and the condition (FM-6). Let $F_1, F_2 : X \to CB(X)$ satisfying:*

(1.1) there exists a number $q \in (0,1)$ such that

$$M_{\nabla}(F_1x, F_2y, qt) \geq min\{M(x,y,t), M^{\nabla}(x, F_1x, t), M^{\nabla}(y, F_2y, t), M^{\nabla}(x, F_2y, (2-\alpha)t), M^{\nabla}(y, F_1x, t)\}$$

for all $x, y \in X$ and all $\alpha \in (0,2)$, $t > 0$. Then F_1 and F_2 have a common fixed point.

PROOF. Let x_0 is an arbitrary point in X and $x_1 \in X$ is such that $x_1 \in F_1x_0$ and

$$M(x_0, x_1, qt) \geq M^{\nabla}(x_0, F_1x_0, qt) - \varepsilon,$$

$x_2 \in X$ is such that $x_2 \in F_2x_1$ and

$$M(x_1, x_2, qt) \geq M^{\nabla}(x_1, F_2x_1, qt) - \frac{\varepsilon}{2}.$$

Inductively $x_{2n+1} \in X$ is such that $x_{2n+1} \in F_1x_{2n}$ and

$$M(x_{2n}, x_{2n+1}, qt) \geq M^{\nabla}(x_{2n}, F_1x_{2n}, qt) - \frac{\varepsilon}{2^{2n}}.$$

$x_{2n+2} \in X$ is such that $x_{2n+2} \in F_2x_{2n+1}$ and

$$M(x_{2n+1}, x_{2n+2}, qt) \geq M^{\nabla}(x_{2n+1}, F_2x_{2n+1}, qt) - \frac{\varepsilon}{2^{2n+1}}.$$

Now we show that $\{y_n\}$ is a Cauchy sequence.

By (1.1) for all $t > 0$ and $\alpha = 1 - k$ with $k \in (0, 1)$, we write

$$M(x_{2n+1}, x_{2n+2}, qt) \geq M^{\nabla}(x_{2n+1}, F_2x_{2n+1}, qt) - \frac{\varepsilon}{2^{2n+1}}$$
$$\geq M_{\nabla}(F_1x_{2n}, F_2x_{2n+1}, qt) - \frac{\varepsilon}{2^{2n+1}}$$

$$\geq min\{M(x_{2n}, x_{2n+1}, t), M^{\nabla}(x_{2n}, F_1x_{2n}, t), M^{\nabla}(x_{2n+1}, F_2x_{2n+1}, t),$$
$$M^{\nabla}(x_{2n}, F_2x_{2n+1}, (2 - \alpha)t), M^{\nabla}(x_{2n+1}, F_1x_{2n}, t)\} - \frac{\varepsilon}{2^{2n+1}}$$

$$\geq min\{M(x_{2n}, x_{2n+1}, t), M(x_{2n}, x_{2n+1}, t), M(x_{2n+1}, x_{2n+2}, t),$$
$$M(x_{2n}, x_{2n+2}, (1 + k)t), M(x_{2n+1}, x_{2n+1}, t)\} - \frac{\varepsilon}{2^{2n+1}}$$

Now using (FM-4), we write

$$\geq min\{M(x_{2n}, x_{2n+1}, t), M(x_{2n}, x_{2n+1}, t), M(x_{2n+1}, x_{2n+2}, t),$$
$$M(x_{2n}, x_{2n+1}, t) * M(x_{2n+1}, x_{2n+2}, kt), 1\} - \frac{\varepsilon}{2^{2n+1}}. \tag{1.2}$$

Since t-norm $*$ is continuous and $M(x, y, \cdot)$ is left continuous, letting $k \to 1$ in (1 2), we have

(1.3)
$$M(x_{2n+1}, x_{2n+2}, qt) \geq min\{M(x_{2n}, x_{2n+1}, t), M(x_{2n+1}, x_{2n+2}, t)\} - \frac{\varepsilon}{2^{2n+1}}.$$

Similarly we have also

(1.4)
$$M(x_{2n+2}, x_{2n+3}, qt) \geq min\{M(x_{2n+1}, x_{2n+2}, t), M(x_{2n+2}, x_{2n+3}, t)\} - \frac{\varepsilon}{2^{2n+2}}$$

Thus from (1.3) and (1.4), it follows that

$$M(x_{n+1}, x_{n+2}, qt) \geq min\{M(x_n, x_{n+1}, t), M(x_{n+1}, x_{n+2}, t)\} - \frac{\varepsilon}{2^{n+1}}$$

for $n = 1, 2, \ldots$ and so, for positive integers n, p,

$$M(x_{n+1}, x_{n+2}, qt) \geq min\{M(x_n, x_{n+1}, t), M(x_{n+1}, x_{n+2}, \frac{t}{q^P})\} - \frac{\varepsilon}{2^{n+1}}.$$

Thus since $M(x_{n+1}, x_{n+2}, \frac{t}{q^P}) \to 1$ as $n \to \infty$, we have

$$M(x_{n+1}, x_{n+2}, qt) \geq M(x_n, x_{n+1}, t) - \frac{\varepsilon}{2^{n+1}}.$$

ε is arbitrary making $\varepsilon \to 0$, we obtain

$$M(x_{n+1}, x_{n+2}, qt) \geq M(x_n, x_{n+1}, t).$$

Therefore by Lemma 2, $\{x_n\}$ converges to a point $z \in X$.

Now by (1.1) with $\alpha = 1$, we have

$$M^{\nabla}(x_{2n+2}, F_1 z, qt) \geq M_{\nabla}(F_1 z, F_2 x_{2n+1}, qt)$$

$$\geq min\{M(z, x_{2n+1}, t), M^{\nabla}(z, F_1 z, t), M^{\nabla}(x_{2n+1}, F_2 x_{2n+1}, t),$$
$$M^{\nabla}(z, F_2 x_{2n+1}, t), M^{\nabla}(x_{2n+1}, F_1 z, t)\}$$

$$\geq min\{M(z, x_{2n+1}, t), M^{\nabla}(z, F_1 z, t), M(x_{2n+1}, x_{2n+2}, t),$$
$$M(z, x_{2n+2}, t), M^{\nabla}(x_{2n+1}, F_1 z, t)\}.$$

Letting $n \to \infty$, we obtain

$$M^{\nabla}(z, F_1 z, qt) \geq min\{1, M^{\nabla}(z, F_1 z, t), 1, 1, M^{\nabla}(z, F_1 z, t)\}.$$

This gives

$$M^{\nabla}(z, F_1 z, qt) \geq M^{\nabla}(z, F_1 z, t).$$

Therefore by Lemma 3, $z \in F_1 z$. Similarly we can prove that $z \in F_2 z$.

References

[1] A. George and P Veeramani, *On some results in fuzzy metric spaces*, Fuzzy Sets and Systems **64** (1994), 395-399

[2] B K. Bose and D Sahani, *Fuzzy Mappings and fixed point theorems*, Fuzzy Sets and Systems **21** (1987), 53-58.

[3] B Schweizer and A Sklar, *Statistical metric space*, Pacific Journal Math. **10** (1960), 313-334

[4] D. Butnariu, *Fixed point for fuzzy mappings*, Fuzzy Sets and Systems **7** (1982), 191-207

[5] I. Kramosil and J Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetica **11** (1975), 326-334

[6] I Kubiaczyk and Sushil Sharma, *Common coincidence point in fuzzy metric space*, J Fuzzy Maths. (accepted)

[7] J X Fang, *On fixed point theorems in fuzzy metric spaces*, Fuzzy Sets and Systems **46** (1992), 107-113

[8] J.S Jung, Y J Cho and J K Kim, *Minimization theorems for fixed point theorems in fuzzy metric spaces and applications*, Fuzzy Sets and Systems **61** (1994), 199-207

[9] J.S. Jung, Y J. Cho, S S Chang and S M Kang, *Concidence theorems for set valued mappings and Ekland's variational principal in fuzzy metric spaces*, Fuzzy Sets and Systems **79** (1996), 239-250

[10] L A Zadeh, *Fuzzy Sets*, Inform. Control **8** (1965), 338-353.

[11] M. Grabiec, *Fixed point in fuzzy metric spaces*, Fuzzy Sets and Systems **27** (1988), 385-389.

[12] M.A Erceg, *Metric spaces in fuzzy set theory*, J Math Anal. Appl. **69** (1979), 205-230

[13] O Hadzic, *Fixed point theorems for multi-valued mappings in some classes of fuzzy metric spaces*, Fuzzy Sets and Systems **29** (1989), 115-125

[14] O Hadzic, *Fixed point theorems in probabilistic metric spaces*, Serbian Academy of Sciences and Arts Institute of Mathematics, University of Novi Sad, Yugoslavia (1995)

[15] O. Kaleva and S Seikkala, *On fuzzy metric spaces*, Fuzzy Sets and Systems **12** (1984), 215-229

[16] R. Badard, *Fixed point theorems for fuzzy number*, Fuzzy Sets and Systems **13** (1984), 291-302

[17] S S. Chang, *Fixed point theorems for fuzzy mappings*, Fuzzy Sets and Systems **17** (1985), 181-187

[18] S S Chang, Y J. Cho, B S. Lee, J S Jung and S M. Kang, *Coincidence point and minimization theorems in fuzzy metric spaces*, Fuzzy Sets and Systems **88**(1) (1997), 119-128

[19] S.S Chang, Y J Cho, B S Lee and G M. Lee, *Fixed degree and fixed point theorems for fuzzy mappings*, Fuzzy Sets and Systems **87**(3) (1997), 325-334

[20] S. Heilpern, *Fuzzy mappings and fixed point theorems*, J. Math Anal Appl. **83** (1981), 566-569.

[21] S.N. Mishra, N. Sharma and S.L Singh, *Common fixed points of maps on fuzzy metric spaces*, Internet J. Math and Math Sci **17** (1994), 253-258.

[22] Sushil Sharma, *On fuzzy metric space*, Southeast Asian Bulletin Maths., Springer-verlag Vol 6, No.1 (2002), 145-157

[23] Sushil Sharma, *Common fixed point theorems in fuzzy metric space*, Fuzzy Sets and Systems **127** (2002), 345-352

[24] Z K. Deng, *Fuzzy pseudo-metric spaces*, J Math Anal. Appl. **86** (1982), 74-95.

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