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ON PAIRWISE FUZZY BASICALLY DISCONNECTED SPACES

G BALASUBRAMANIAN AND V. CHANDRASEKAR

ABSTRACT The concept of basic disconnectedness is introduced in the fuzzy bitopological spaces. Besides giving examples and some interesting properties of these spaces, we also establish several characterizations of these spaces

1. Introduction

The fundamental concept of fuzzy set introduced by ZADEH [23] provided a natural foundation for building new branches. Fuzzy sets have applications in many fields such as information [18] and control [20]. In 1968, CHANG [1] introduced the concept of fuzzy topological spaces and thereafter many fuzzy topologists [2, 5, 8, 10-16, 22] have contributed to the theory of fuzzy topological spaces In 1989, KANDIL [7] introduced the concept of fuzzy bitopological spaces and since then various notions in classical topology have been extended (see for example [17]) to fuzzy bitopological spaces. Extremally disconnected spaces were defined and studied in [6] where it is pointed out that the concept arose earlier in a paper of STONE [19]. The importance of these spaces lies in their connection with the completeness of $C(X)$ as a lattice. Basically disconnected spaces introduced in [5] also arose in this connection [9, 19]. The purpose of this paper is to

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introduce the concept of basic disconnectedness in fuzzy bitopological spaces. Besides giving some interesting properties of these spaces and examples, we also give several characterizations of these spaces.

2. Preliminaries

Let (X,T) be any fuzzy topological space [1] and let λ be any fuzzy set in X. Then we define $Cl_T(\lambda) = \wedge\{\mu|\lambda \leq \mu, \mu \text{ is } T - \text{fuzzy closed}\}$. $\text{Int}_T(\lambda) = \vee\{\mu|\mu \leq \lambda, \mu \text{ is } T- \text{fuzzy open}\}$. The relation between the interior and the closure operator is given by $1-\text{Int}_T(\lambda) = \text{Cl}_T(1-\lambda)$; $1-\text{Cl}_T(\lambda) = \text{Int}_T(1-\lambda)$. A fuzzy set λ in X is called *a $G_\delta-$ fuzzy set* if $\lambda = \wedge_{j=1}^{\infty}\lambda_j$ where $\lambda_j \in T$. And λ is called *a $F_\sigma-$ fuzzy set* if $\lambda = \vee_{j=1}^{\infty}\mu_j$ where $\mu_j^c = (1-\mu_j) \in T$.

By a fuzzy bitopological space [7] we mean an ordered triple (X, T_1, T_2) where T_1 and T_2 are fuzzy topologies on the non-empty set X. For any non-empty subset A of X we shall write $T_1/A = \{\lambda/A|\lambda \in T_1\}$ and $T_2/A = \{\mu/A|\mu \in T_2\}$. Clearly T_1/A and T_2/A are fuzzy topologies on A and the fuzzy bitopological space $(A, T_1/A, T_2/A)$ is called *a pairwise fuzzy subspace of* (X, T_1, T_2).

3. Pairwise fuzzy basically disconnected spaces

Based on the classical notion of basic disconnectedness [3, 5] we define the corresponding notion for fuzzy bitopological spaces as follows:

DEFINITION 3.1. A fuzzy bitopological space (X, T_1, T_2) is said to *be pairwise fuzzy basically disconnected* if the T_1- closure of each T_2- fuzzy open, $T_2 - F_\sigma$ fuzzy set is T_2- fuzzy open and T_2- closure of each T_1- fuzzy open, $T_1 - F_\sigma$ fuzzy set is T_1- fuzzy open.

EXAMPLE 3.2. Let $X = \{a,b,c,d\}$, $T_1 = \{0,1,\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$ and $T_2 = \{0,1,\mu_1,\mu_2,\mu_3,\mu_4\}$ where $\lambda_i : X \to I$, $i = 1,2,3,4$ and $\mu_j : X \to I$, $j = 1,2,3,4$ are defined as follows:

$$\lambda_1(x) = \begin{cases} 1 & x = b, \\ 0 & x = a,c,d \end{cases} \qquad \lambda_2(x) = \begin{cases} 1 & x = a,b, \\ 0 & x = c,d \end{cases}$$

$$\lambda_3(x) = \begin{cases} 1 & x = b,d \\ 0 & x = a,c \end{cases} \quad \text{and} \quad \lambda_4(x) = \begin{cases} 1 & x = a,b,d \\ 0 & x = c \end{cases}$$

$$\mu_1(x) = \begin{cases} 1 & x = c \\ 0 & x = a, b, d \end{cases} \qquad \mu_2(x) = \begin{cases} 1 & x = a, c \\ 0 & x = b, d \end{cases}$$

$$\mu_3(x) = \begin{cases} 1 & x = c, d \\ 0 & x = a, b \end{cases} \quad \text{and} \quad \mu_4(x) = \begin{cases} 1 & x = a, c, d \\ 0 & x = b \end{cases}$$

Then clearly (X, T_1) and (X, T_2) are fuzzy topological spaces. Also we can easily see that they are both fuzzy connected [4] spaces (since both (X, T_1) and (X, T_2) has no proper fuzzy clopen sets).

Also in fuzzy topological space (X, T_1), there is no such T_1- fuzzy open, $T_1 - F_\sigma$ fuzzy set and also there is no such T_2- fuzzy open, $T_2 - F_\sigma$ fuzzy set in (X, T_2).

Therefore, the fuzzy bitopological space (X, T_1, T_2) is pairwise fuzzy basically disconnected even though both (X, T_1) and (X, T_2) are fuzzy connected

EXAMPLE 3.3. Let $X = \{a, b, c\}$. Suppose $T_1 = \{0, 1, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\}$ and $T_2 = \{0, 1\}$ where $\lambda_i : X \to I$, $i = 1$ to 6 are defined as follows.

$$\lambda_1(x) = \begin{cases} 1 & x = a, b \\ 0 & x = c \end{cases} \qquad \lambda_2(x) = \begin{cases} 1 & x = b, c \\ 0 & x = a \end{cases}$$

$$\lambda_3(x) = \begin{cases} 1 & x = a, c \\ 0 & x = b \end{cases} \quad \text{and} \quad \lambda_4(x) = \begin{cases} 1 & x = a, \\ 0 & x = b, c \end{cases}$$

$$\lambda_5(x) = \begin{cases} 1 & x = b \\ 0 & x = a, c \end{cases} \qquad \lambda_6(x) = \begin{cases} 1 & x = c \\ 0 & x = a, b \end{cases}$$

Then clearly (X, T_1) is a fuzzy topological space and (X, T_2) is the indiscrete fuzzy topological space. Clearly (X, T_1) is a fuzzy disconnected space and (X, T_2) is a fuzzy connected space.

We claim the fuzzy bitopological space (X, T_1, T_2) is pairwise fuzzy basically disconnected space.

Let λ be any non-zero T_1- fuzzy open, $T_1 - F_\sigma$ fuzzy set. Then $Cl_{T_2}(\lambda) = 1$ which is clearly T_1- fuzzy open. Similarly, we can see that $Cl_{T_1}(\mu) = 1$ whenever μ is a non-zero T_2- fuzzy open, $T_2 - F_\sigma$ fuzzy set and clearly $Cl_{T_1}(\mu)$ is T_2- fuzzy open. Therefore, the fuzzy bitopological space (X, T_1, T_2) is pairwise fuzzy basically disconnected space.

REMARK. It is easy to see that the following are equivalent.

(1) (X, T_1, T_2) is pairwise fuzzy basically disconnected.
(2) Given T_2- fuzzy open set μ and T_1- fuzzy open set λ in (X, T_1, T_2) with $\lambda + \mu \leq 1$ and λ being $T_1 - F_\sigma$ fuzzy set or μ being a $T_2 - F_\sigma$ fuzzy set we have $Cl_{T_2}(\lambda) + Cl_{T_1}(\mu) \leq 1$.

4. Characterizations

THEOREM 4.1. *For any fuzzy bitopological space (X, T_1, T_2) the following are equivalent.*

(1) *(X, T_1, T_2) is pairwise fuzzy basically disconnected.*
(2) *Whenever λ is a T_1- fuzzy open and $T_1 - F_\sigma$ fuzzy set, $Int_{T_1}[Cl_{T_2}(\lambda)]$ is T_2- fuzzy closed. Similar statement holds when λ is a T_2- fuzzy open and $T_2 - F_\sigma$ fuzzy set.*
(3) *Whenever λ is T_1-fuzzy open and $T_1 - F_\sigma$ fuzzy set, we have*

$$Cl_{T_2}(\lambda) \leq 1 - Cl_{T_1}[1 - Cl_{T_2}(\lambda)].$$

Similar statement holds when λ is a T_2- fuzzy open and $T_2 - F_\sigma$ fuzzy set.
(4) *Whenever λ is a T_1- fuzzy open set and μ is a T_2-fuzzy open set such that $\lambda + \mu \leq 1$ and λ being $T_1 - F_\sigma$ fuzzy set or μ being a $T_2 - F_\sigma$ fuzzy set, we have*

$$Cl_{T_2}(\lambda) + Cl_{T_1}(\mu) \leq 1.$$

PROOF. (1) $\Rightarrow$ (2). Let λ be a T_1-fuzzy open and $T_1 - F_\sigma$ fuzzy set. Now

$$Int_{T_1}[Cl_{T_2}(\lambda)] = 1 - Cl_{T_1}[1 - Cl_{T_2}(\lambda)] \qquad (I)$$

By (1), $Cl_{T_2}(\lambda)$ is T_1- fuzzy open and therefore from (I) it follows that $Int_{T_1}[Cl_{T_2}(\lambda)]$ is T_2- fuzzy closed. Similar argument holds when λ is a T_2- fuzzy open and $T_2 - F_\sigma$ fuzzy set.

(2) $\Rightarrow$ (3). Let λ be a T_1- fuzzy open and $T_1 - F_\sigma$ fuzzy set and suppose that $Cl_{T_2}(\lambda) \nleq 1 - Cl_{T_1}[1 - Cl_{T_2}(\lambda)]$. Then there exists an

$x \in X$ such that $\{Cl_{T_2}(\lambda)\}(x) \nleq \{1 - Cl_{T_1}[1 - Cl_{T_2}(\lambda)]\}(x)$. Now by (2), $Int_{T_1}[Cl_{T_2}(\lambda)]$ is T_2- fuzzy closed. Also $Cl_{T_1}[1 - Cl_{T_2}(\lambda)] = 1 - Int_{T_1}[Cl_{T_2}(\lambda)]$. Hence it follows that

$$\begin{aligned}\{Cl_{T_2}(\lambda)\}(x) &\nleq 1 - \{1 - Int_{T_1}[Cl_{T_2}(\lambda)]\}(x) \\ &\nleq \{Int_{T_1}[Cl_{T_2}(\lambda)]\}(x)\end{aligned}$$

which is not possible; For by (2), $Int_{T_1}[Cl_{T_2}(\lambda)]$ is T_2- fuzzy closed containing λ. Hence $Cl_{T_2}(\lambda) \leq 1 - Cl_{T_1}[1 - Cl_{T_2}(\lambda)]$. Similar proof holds when λ is T_2- fuzzy open and $T_2 - F_\sigma$ fuzzy set.

(3) $\Rightarrow$ (4). Let λ be a T_1- fuzzy open, $T_1 - F_\sigma$ fuzzy set and μ be a T_2- fuzzy open such that $\lambda + \mu \leq 1$.

We know that $\mu \leq 1 - Cl_{T_2}(\lambda)$ and $\lambda \leq 1 - Cl_{T_1}(\mu)$. But by hypothesis $Cl_{T_2}(\lambda) \leq 1 - Cl_{T_1}[1 - Cl_{T_2}(\lambda)]$ and therefore $\mu \leq 1 - Cl_{T_2}(\lambda)$. Since $Cl_{T_1}(\mu)$ is the smallest T_1- fuzzy closed set containing μ, we have

$$Cl_{T_1}(\mu) \leq Cl_{T_1}[1 - Cl_{T_2}(\lambda)]. \tag{II}$$

Also since $Cl_{T_2}(\lambda) + Cl_{T_1}[1 - Cl_{T_2}(\lambda)] \leq 1$, it follows that

$$Cl_{T_2}(\lambda) + Cl_{T_1}(\mu) \leq 1 \qquad [\text{from } (II)].$$

(4) $\Rightarrow$ (1) Let λ be any T_1- fuzzy open and $T_1 - F_\sigma$ fuzzy set. We shall show that $Cl_{T_2}(\lambda)$ is T_1- fuzzy open. Let $\mu = 1 - Cl_{T_2}(\lambda)$. Clearly μ is T_2- fuzzy open and $\mu + \lambda \leq 1$. Hence, by (4), we have $Cl_{T_2}(\lambda) + Cl_{T_1}(\mu) \leq 1$ and therefore by construction of μ, we have $1 - Cl_{T_1}(\mu) = Cl_{T_2}(\lambda)$. This shows $Cl_{T_2}(\lambda)$ is T_1- fuzzy open. Similarly, we can show for any T_2- fuzzy open and $T_2 - F_\sigma$ fuzzy set λ, $Cl_{T_1}(\lambda)$ is T_2- fuzzy open.

5. Properties

In this section we shall establish some interesting properties of pairwise fuzzy basically disconnected spaces.

PROPOSITION 5.1. *Let (X, T_1, T_2) be a pairwise fuzzy basically disconnected space and let $(Y, T_1/Y, T_2/Y)$ be any pairwise fuzzy subspace of (X, T_1, T_2). Then $(Y, T_1/Y, T_2/Y)$ is pairwise fuzzy basically disconnected.*

PROOF. Let λ_1 and λ_2 be $T_1/Y-$ fuzzy open set and $T_2/Y-$ fuzzy open set in Y respectively and suppose that λ_1 is $T_1/Y - F_\sigma$ fuzzy set. Define λ_1^1 and λ_2^2 on X as follows:

$$\lambda_1^1 : X \to [0,1] \quad \text{where}$$

$$\lambda_1^1(x) = \begin{cases} \lambda_1(x) & \text{if } x \in Y, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\lambda_2^2 : X \to [0,1] \quad \text{where}$$

$$\lambda_2^2(x) = \begin{cases} \lambda_2(x) & \text{if } x \in Y, \\ 0 & \text{otherwise} \end{cases}$$

From [21], we know that λ_1^1 and λ_2^2 are T_1- fuzzy open set and T_2- fuzzy open set respectively and that λ_1^1 is $T_1 - F_\sigma$ fuzzy set. Since (X, T_1, T_2) is pairwise fuzzy basically disconnected, it follows that

$$Cl_{T_2}(\lambda_1^1) + Cl_{T_1}(\lambda_2^2) \leq 1$$

and this in turn implies

$$Cl_{T_2/Y}(\lambda_1) + Cl_{T_1/Y}(\lambda_2) \leq 1.$$

We arrive at the same conclusion when we assume λ_2 is $T_2/Y - F_\sigma$ fuzzy set. Hence the proposition holds.

DEFINITION 5.2. Let $\{(X_\alpha, T_\alpha, T_\alpha^*) | \alpha \in \Delta\}$ be any family of the disjoint fuzzy bitopological spaces. Let $X = \cup_{\alpha \in \Delta} X_\alpha$. Define $\oplus_{\alpha \in \Delta} T_\alpha = \{\lambda : X \to I | \lambda/_{X_\alpha} \in T_\alpha\}$ and $\oplus_{\alpha \in \Delta} T_\alpha^* = \{\lambda : X \to I | \lambda/_{X_\alpha} \in T_\alpha^*\}$. Then $(X, \oplus_{\alpha \in \Delta} T_\alpha, \oplus_{\alpha \in \Delta} T_\alpha^*)$ is a fuzzy bitopological space called *the fuzzy bitopological sum of* $\{(X_\alpha, T_\alpha, T_\alpha^*) | \alpha \in \Delta\}$.

PROPOSITION 5.3. *The fuzzy bitopological sum of a family of disjoint pairwise fuzzy basically disconnected spaces is pairwise fuzzy basically disconnected.*

PROOF. Let $\{(X_\alpha, T_\alpha, T^*_\alpha) | \alpha \in \Delta\}$ be a family of disjoint pairwise fuzzy basically disconnected spaces. Let $(X, \oplus_{\alpha\in\Delta} T_\alpha, \oplus_{\alpha\in\Delta} T^*_\alpha)$ be the fuzzy bitopological sum of these spaces. Let λ_1 and λ_2 be $\oplus_{\alpha\in\Delta} T_\alpha-$ fuzzy open and $\oplus_{\alpha\in\Delta} T^*_\alpha-$ fuzzy open sets in X respectively such that $\lambda_1 + \lambda_2 \leq 1$. Also we shall assume that λ_1 is $\oplus_{\alpha\in\Delta} T_\alpha - F_\sigma$ fuzzy set.

Now, from the assumptions, it is clear that $\lambda_1/_{X_\alpha}$ and $\lambda_2/_{X_\alpha}$ are $T_\alpha-$ fuzzy open and $T^*_\alpha-$ fuzzy open sets in X_α respectively for each $\alpha \in \Delta$. Also $\lambda_1/_{X_\alpha} + \lambda_2/_{X_\alpha} \leq 1$ and $\lambda_1/_{X_\alpha}$ is $T_\alpha - F_\sigma$ fuzzy set in X_α. Since $(X_\alpha, T_\alpha, T^*_\alpha)$ is pairwise fuzzy basically disconnected, we have

$$Cl_{T^*_\alpha}(\lambda_1/_{X_\alpha}) + Cl_{T_\alpha}(\lambda_2/_{X_\alpha}) \leq 1 \ , \ \alpha \in \Delta.$$

Hence

$$Cl_{\oplus_{\alpha\in\Delta} T^*_\alpha}(\lambda_1) + Cl_{\oplus_{\alpha\in\Delta} T_\alpha}(\lambda_2) \leq 1.$$

This proves that the fuzzy bitopological sum is a pairwise fuzzy basically disconnected space.

DEFINITION 5.4. A function $f : (X, T_1, T_2) \to (Y, T^*_1, T^*_2)$ is called *pairwise fuzzy continuous (resp., pairwise fuzzy open)* $\Leftrightarrow$ The induced function $f : (X, T_1) \to (Y, T^*_1)$ and $f : (X, T_2) \to (Y, T^*_2)$ are fuzzy continuous (resp., fuzzy open).

PROPOSITION 5.5. *Suppose (X, T_1, T_2) is a pairwise fuzzy basically disconnected space and $f : (X, T_1, T_2) \to (Y, T^*_1, T^*_2)$ is a pairwise fuzzy continuous and pairwise fuzzy open mapping. Then (Y, T^*_1, T^*_2) is pairwise fuzzy basically disconnected.*

PROOF. Let λ be an arbitrary T^*_2- fuzzy open and $T^*_2 - F_\sigma$ fuzzy set in Y. Since f is pairwise fuzzy continuous, $f^{-1}(\lambda)$ is T_2- fuzzy open and $T_2 - F_\sigma$ fuzzy set in X and since X is pairwise fuzzy basically disconnected $Cl_{T_1}[f^{-1}(\lambda)]$ is T_2- fuzzy open. Again by pairwise fuzzy continuity of f it follows that

$$f^{-1}[Cl_{T^*_1}(\lambda)] \leq Cl_{T_1}[f^{-1}(\lambda)].$$

Hence

$$\begin{aligned} f\{f^{-1}[Cl_{T_1^*}(\lambda)]\} &= Cl_{T_1^*}(\lambda) \leq f\{Cl_{T_1}[f^{-1}(\lambda)]\} \\ &\leq Cl_{T_1^*}\{f[f^{-1}(\lambda)]\} \\ &= Cl_{T_1^*}(\lambda) \text{ (by hypothesis on } f) \end{aligned}$$

That is $f\{Cl_{T_1}[f^{-1}(\lambda)]\} = Cl_{T_1^*}(\lambda)$. Since f is pairwise fuzzy open and $f\{Cl_{T_1}[f^{-1}(\lambda)]\} = Cl_{T_1^*}(\lambda)$ it follows that $Cl_{T_1^*}(\lambda)$ is T_2^*- fuzzy open. Similarly, we can show that $Cl_{T_2^*}(\lambda)$ is T_1^*- fuzzy open whenever λ is T_1^*- fuzzy open and $T_1^* - F_\sigma$ fuzzy set.

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Department of Mathematics
Periyar University
Salem 636011
Tamil Nadu, India
E-mail: vcsekar_5@yahoo.co.in