An Analytical Solution of the Schrödinger Equation for a Rectangular Barrier with Time-Dependent Position

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An analytical solution for the Schrödinger equation with time-dependent potential has been investigated extensively over past decades. In addition to its own mathematical interest, this problem has wide applications in many areas of physics, such as laser-induced dynamics, the motion of Paul trap ions,[1] and semiconductor physics.[2] Only systems with time-dependent potentials that are constant, linear, and quadratic in x are known to be analytically solved.[3]

For these problems, the well known methods for analytical wave functions are the famous invariant operator approach,[4] the propagator method,[5] and the time-space transformation method.[6] In general, systems with potentials of $V(x, t) = f(t)x^2 + g(t)x + h(t)$ has been solved exactly by these methods[7]. Among these systems, rectangular potentials with time-dependent height or depth are quite simple to solve.[8] A rectangular barrier with time-dependent position is, however, much more complex and the Schrödinger equation has not yet been solved analytically, although Moiseyev[9] studied the problem approximately by averaging the potential in time and by treating it as a time-independent bound system.

In the present work, we obtain the exact solution for the rectangular barrier whose position is oscillating in time. We use the Kramers-Henneberger transformation[10] which is a particular form of time-space transformation technique.

The Hamiltonian for the rectangular barrier with oscillating position is chosen as[9]

$$H(x, t) = \frac{p^2}{2m} + V(x, t), \tag{1}$$

where

$$V(x, t) = \begin{cases} V_0, \text{ if } |x + \alpha_0 \cos \omega t| < \frac{a}{2} \\ 0, \text{ elsewhere} \end{cases} \tag{2}$$

The position of the barrier oscillates with the frequency $\omega = 2\pi/T$ so that at $t = 0$ the barrier is centered at $x = -\alpha_0$, and at $t = T/2$ its center is at $x = +\alpha_0$. The Hamiltonian with the potential $V(x, t)$ of eq. (2) is obtained from $H = p^2/2m + V(x) + E_0 x\cos\omega t$ by Kramers-Henneberger transformation,[10] where $\alpha_0 = E_0/m\omega^2$. This Hamiltonian represents the system under the field $E_0 x\cos\omega t$.

If we introduce a new variable, $\xi(x, t) = x + \alpha_0\cos\omega t$, following the Kramers-Henneberger transformation,[10] the time-dependent wave function of the system, $\Psi(x, t)$, can be written as follows[3]

$$\Psi(x, t) = e^{-\frac{iEt}{\hbar}}\phi(\xi, t)\chi(x, t), \tag{3}$$

where E is a constant parameter which could be the energy of the system. Inserting $\Psi(x, t)$ of eq. (3) into time-dependent Schrodinger equation and changing x to ξ, we have

$$-\frac{\hbar^2}{2m}\left[\chi\frac{\partial^2\phi}{\partial\xi^2} + 2\frac{\partial\phi}{\partial\xi}\frac{\partial\chi}{\partial x} + \phi\frac{\partial^2\chi}{\partial x^2}\right] + V(\xi)\phi\chi = i\hbar\left[\chi\left(\frac{\partial\phi}{\partial\xi}\frac{\partial\xi}{\partial t} + \frac{\partial\phi}{\partial t}\right) + \phi\left(-\frac{iE}{\hbar}\chi + \frac{\partial\chi}{\partial t}\right)\right]. \tag{4}$$

Since the potential $V(\xi)$ in eq. (4) does not depend on t explicitly, $\phi(\xi, t)$ would be a time-independent solution if the following relation is satisfied:

$$-\frac{\hbar}{2m}\left[2\frac{\partial\phi}{\partial\xi}\frac{\partial\chi}{\partial x} + \phi\frac{\partial^2\chi}{\partial x^2}\right] = i\left[\chi\frac{\partial\xi}{\partial t}\frac{\partial\phi}{\partial\xi} + \phi\frac{\partial\chi}{\partial t}\right]. \tag{5}$$

Eq. (4) then becomes

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial\xi^2} + V(\xi) - E\right)\phi(\xi, t) = i\hbar\frac{\partial\phi(\xi, t)}{\partial t}. \tag{6}$$

Solutions of eq. (6) would be $e^{\pm c_1\xi}$, where $c_1 = ik$ or χ $(k = \sqrt{2mE}/\hbar)$, $(\kappa = \sqrt{2m(V_0 - E)}/\hbar)$, depending on the region of x.

Substituting $\partial\xi/\partial t = -(p(t)/m)$ and $\phi(\xi) = e^{c_1\xi}$ into eq. (5) and then rearranging it, we have

$$\frac{\hbar}{2m}\frac{\partial^2\chi}{\partial x^2} + \frac{\hbar c_1}{m}\frac{\partial\chi}{\partial x} - i\frac{c_1 p(t)}{m}\chi = -i\frac{\partial\chi}{\partial t}. \tag{7}$$

To determine the solution, we factorize $\chi(x, t)$ as $\chi(x, t) = u(t)\upsilon(x)$ since the eq. (7) is not coupled in x and t. Inserting $\chi(x, t)$ into eq. (7) and then dividing both sides by $u(t)\upsilon(x)$, we obtain.

$$\frac{\hbar}{2m}\frac{1}{\upsilon}\frac{d^2\upsilon}{dx^2} + \frac{\hbar c_1}{m}\frac{1}{\upsilon}\frac{d\upsilon}{dx} = -i\left(\frac{1}{u}\frac{du}{dt} - \frac{c_1 p(t)}{m}\right). \tag{8}$$

Since the left-hand side is a function of x only, while the right-hand side is a function of t, we let both sides equal to c_2 which is a constant. Thus we have $u(t)$ as given below,

$$u(t) = e^{ic_2 t + \frac{c_1}{m}\int p(t')dt'} = e^{ic_2 t - c_1\alpha_0\cos\omega t}. \tag{9}$$

The left-hand side would be an ordinary second-order differential equation for $\upsilon(x)$ as.

$$\frac{\hbar}{2m}\frac{d^2\upsilon}{dx^2}+\frac{\hbar c_1}{m}\frac{d\upsilon}{dx}-c_2\upsilon = 0 . \qquad (10)$$

Inserting $\upsilon(x) = e^{\lambda(x)}$ into eq. (10), we obtain the equation for $\lambda(x)$ given as,

$$\frac{\hbar}{2m}\left[\frac{d^2\lambda}{dx^2}+\left(\frac{d\lambda}{dx}\right)^2\right]+\frac{\hbar c_1}{m}\frac{d\lambda}{dx}-c_2 = 0 . \qquad (11)$$

If we define $d\lambda/dx = w(x)$ and insert it into eq. (11), we finally have the first-order differential equation for $w(x)$ as given below,

$$\frac{\hbar}{2m}\frac{dw}{dx} = c_2-\frac{\hbar c_1}{m}w-\frac{\hbar}{2m}w^2 , \qquad (12)$$

which can be easily solved by integrating the equation given as,

$$\left(\frac{2m}{\hbar}c_2-2c_1w-w^2\right)^{-1}dw = dx . \qquad (13)$$

Integrating eq. (13), we would have

$$x = -\frac{2}{\sqrt{-\Delta}}\tanh^{-1}\left(-\frac{2(c_1+w)}{\sqrt{-\Delta}}\right), \quad \Delta < 0$$

$$= \frac{2}{\sqrt{\Delta}}\tan^{-1}\left(-\frac{2(c_1+w)}{\sqrt{\Delta}}\right), \quad \Delta > 0 , \qquad (14)$$

where $\Delta = -4(2m/\hbar\, c_2 + c_1^2)$. Determining $w(x)$ from eq. (14) and integrating it again, we have $\lambda(x)$ as given below.

$$\lambda(x) = \ln\left[\cosh\left(-\frac{\sqrt{-\Delta}}{2}x\right)\right]-c_1x , \quad \Delta < 0$$

$$= \ln\left[\cos\left(\frac{\sqrt{\Delta}}{2}x\right)\right]-c_1x , \quad \Delta > 0 . \qquad (15)$$

From $\upsilon(x) = e^{\lambda(x)}$, we get

$$\upsilon(x) = \cosh\left(-\frac{\sqrt{-\Delta}}{2}x\right)e^{-c_1x} , \quad \Delta < 0$$

$$= \cos\left(\frac{\sqrt{\Delta}}{2}x\right)e^{-c_1x} , \quad \Delta > 0 . \qquad (16)$$

Thus we have $\chi(x, t)$ as

$$\chi(x, t) = e^{ic_3t-\alpha_0c_1\cos\omega t}\cosh\left(-\frac{\sqrt{-\Delta}}{2}x\right)e^{-c_1x} , \quad \Delta < 0$$

$$= e^{ic_3t-\alpha_0c_1\cos\omega t}\cos\left(\frac{\sqrt{-\Delta}}{2}x\right)e^{-c_1x} , \quad \Delta > 0 . \qquad (17)$$

Inserting $\chi(x, t)$ from eq. (17) and $\phi(\xi, t)$ which is $e^{\mp c_1\xi}$ into eq. (3), we can exactly determine $\Psi(x, t)$ for the system of rectangular barrier with the oscillating position.

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