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RADIATIVE TRANSFER IN A SCATTERING SPHERICAL ATMOSPHERE

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ABSTRACT

We have written a code called QDM_sca, which numerically solves the problem of radiative transfer in an anisotropically scattering, spherical atmosphere. First we formulate the problem as a second order differential equation of a quasi-diffusion type. We then apply a three-point finite differencing to the resulting differential equation and transform it to a tri-diagonal system of simultaneous linear equations. After boundary conditions are implemented in the tri-diagonal system, the QDM_sca radiative code fixes the field of specific intensity at every point in the atmosphere. As an application example, we used the code to calculate the brightness of atmospheric diffuse light(ADL) as a function of zenith distance, which plays a pivotal role in reducing the zodiacal light brightness from night sky observations. On the basis of this ADL calculation, frequent uses of effective extinction optical depth have been fully justified in correcting the atmospheric extinction for such extended sources as zodiacal light, integrated starlight and diffuse galactic light. The code will be available on request.

I. INTRODUCTION

A code called QDM_sca has been developed to solve the problem of radiative transfer in a spherical atmosphere, whose constituents are supposed to scatter light anisotropically and to absorb as well. This paper explains basic principles behind the code and gives details on its inner workings.

The QDM_sca code evolved from QDM_abs, which treats the same radiative transfer problem in an absorbing spherical atmosphere. When Park (1997) developed QDM_abs on the basis of work by Leung & Liszt (1976), we were interested in probing the excitation conditions of CO molecules in interstellar clouds (c.f. Park, Hong, & Minh 1996, Park & Hong 1998a, 1998b). Later, we realized a possible application of the code to the problem of atmospheric diffuse light (c.f. Hong et al. 1998, Kwon, Hong, & Weinberg 2000) and decided to add the capability of handling the problem of particle scattering. In its original version we used the usual Legendre expansion to describe the scattering phase function and applied the code to the analysis of night sky observations. However, the scattering phase functions of the atmospheric aerosols, the interplanetary dust particles (Hong 1985), and the interstellar grains in the Orion Nebula (Shin & Hong 1987; Park 1999) are all of very strongly forward throwing nature. This requires an excessively large number of terms in the Legendre expansion. We thus decided to numerically integrate the scattering phase function directly. In the current version of QDM_sca, optical properties of the atmospheric constituents may vary with radial distance. Aiming at a variety of problems in the Earth's atmosphere (e.g. Kwon 1989), we also allow the atmosphere to have a layer of airglow emitting sources.

This paper explains basic principles of the quasi-diffusion method. In the next section we will show how the standard equation of radiative transfer can be transformed, under spherical geometry, to an equation of quasi-diffusion type. In the third section we will introduce the ray tracing method and explain how the quasi-diffusion

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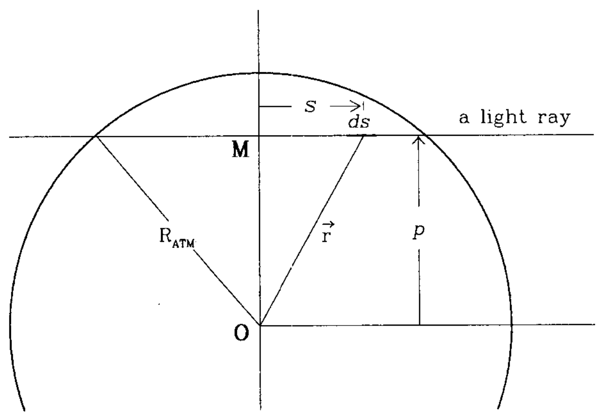

Fig. 1.— Coordinate system for the ray tracing scheme. The light-path distance, s, is measured from the mid-point, M, of each ray, and the projected distance, p, represents minimum separation between the origin, O, and the ray. Outer boundary of the atmosphere is placed at radial distance R_{ATM}. A simple geometrical relation $r^2 = s^2 + p^2$ holds true for the light-path element, ds, located at $\vec{r}$.

equation can be solved by the method. The fourth section explains how the boundary conditions are incorporated into the ray tracing technique. The fifth section will outline the iteration procedure of the code. In the sixth section we will briefly explain how this code was used in reducing the zodiacal light brightness from night sky observations. The last section will summarize the paper.

The code will be available on request. Since a single main program consists major part of the current version, readers will face a difficulty in tailoring the code to their own problems. To ease the difficulty we heavily commented the source code, and tried to make this document as friendly as possible. In a later version we will modularize the code by dividing it into a dozen or so sub-routines. The modularized version will also be available for general users, and it can be easily modified by the users.

II. BASIC FORMULATIONS OF THE RADIATIVE TRANSFER

We first describe the coordinate system. In most text books the (r, θ, ϕ) system of spherical coordinates is used to formulate the problem of radiative transfer in spherical geometry. Since azimuthal symmetry holds true for most problems of astronomical interest, the ϕ dependence is usually replaced by a 2π factor. Therefore, the space dependence of local intensity $I(r, \theta)$ can be fully described by the radial distance r from the system center and the polar angle θ, which is usually measured from the radial direction. Instead of using r and θ the QDM scheme adopts s and p as independent coordinates. As can be seen from Fig 1, the coordinate s is measured along a given ray of light, and the other coordinate p represents the minimum distance of the ray from the origin. We call s the *light-path distance* and p the *projected distance.* At a fixed position $\vec{r}$ the whole range of polar angles $-\pi/2 \leq \theta \leq \pi/2$ could be completely covered if one would change the projected distance p from zero to r. In other words, a light ray of given θ makes a unique projected distance from the center. Thus $I(r, p)$ recovers the full information content in $I(r, \theta)$.

a) Equation of Radiative Transfer

Let us examine how the specific intensity varies with the light-path distance along a ray of given projected distance. The change dI over line element ds is described by the following equation:

$$\frac{dI}{ds} = -(\kappa_{\rm abs} + \kappa_{\rm sca})I + \kappa_{\rm sca} \int_{\Omega'} I(\Omega')\Phi(\Omega,\Omega')\,d\Omega', \tag{1}$$

where $\kappa_{\rm abs}$ is the mean volume *absorption* coefficient, $\kappa_{\rm sca}$ represents the mean volume *scattering* coefficient, and Φ is the mean volume *scattering phase function* of the atmospheric constituents. These optical properties may vary with radial distance r. We have denoted the direction of light impinged upon a "mean" scatterer by the symbol Ω', and the direction into which the light is scattered by Ω. Given the identity

$$\begin{aligned}\frac{d}{ds} &= \frac{\partial r}{\partial s}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial s}\frac{\partial}{\partial \theta} \\ &= \mu\frac{\partial}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial}{\partial \mu}\end{aligned}$$

with $\mu = \cos\theta$, one may recast the radiative transfer equation into its more familiar form than the one in Eq. (1). If the line element is along the light path with fixed p, we may replace the total derivative of I with respect to s by a partial derivative and have

$$\frac{\partial I}{\partial s} = -\kappa I + \kappa_{\rm sca} \int_{\Omega'} I(\Omega')\Phi(\Omega,\Omega')\,d\Omega', \tag{2}$$

where $\kappa(= \kappa_{\rm abs} + \kappa_{\rm sca})$ represents the mean volume *extinction* coefficient of the atmospheric constituents. It is convenient to introduce the mean volume *emissivity* η defined by

$$\eta(r,\mu) = \kappa_{\rm sca}(r) \int_{\Omega'} I(\Omega')\Phi(\Omega,\Omega')\,d\Omega'. \tag{3}$$

Then the above equation takes a rather compact form:

$$\frac{\partial}{\partial s} I(r,\mu) = -\kappa(r) I(r,\mu) + \eta(r,\mu), \tag{4}$$

where the azimuthal symmetry has been assumed implicitly and Ω reduced to $\mu(= \cos\theta)$. To remind readers of the position dependence we have explicitly shown r within parentheses.

b) The Mean Volume Optical Properties

To avoid unnecessary confusion it may worth to clarify the concept of the mean volume optical properties. Since there are many types of constituents in the atmosphere, the optical properties should be the "proper" averages over all these constituents. Denoting each type by index k, we define the mean volume absorption coefficient as

$$\kappa_{\rm abs}(r) = \sum_{\rm k} \int \pi a^2\, Q_{\rm abs\,k}(a)\, n_{\rm k}(a;r)\, da, \tag{5}$$

where $Q_{\rm abs\,k}(a)$ represents the absorption efficiency factor of a k-type particle with radius a, and $n_{\rm k}(a;r)da$ is the number of the k-type particles with radius $a \sim a + da$ in a unit volume located at radial distance r from the center. This is what the proper average is all about. Similarly, the mean volume scattering coefficient is defined by

$$\kappa_{\rm sca}(r) = \sum_{\rm k} \int \pi a^2\, Q_{\rm sca\,k}(a)\, n_{\rm k}(a;r)\, da, \tag{6}$$

where $Q_{\rm sca\,k}(a)$ means the scattering efficiency factor. All the atmospheric constituents are treated here as spherical particles. For non-spherical particles and molecules, we simply replace $\pi a^2\, Q_{\rm abs}$ and $\pi a^2\, Q_{\rm sca}$ by the absorption and scattering cross-sections, respectively.

We first define the differential scattering phase function, $\varphi(\Omega; a)$, of a particle with radius a in the following way: Out of the photons that are scattered into the entire 4π radians, the fraction comprised of those photons that are scattered into $\Omega \sim \Omega + d\Omega$ is given by $\varphi(\Omega; a)d\Omega$. Then the mean volume scattering phase function $\Phi(\Omega; r)$ of the particles in unit volume at r can be obtained from

$$\Phi(\Omega; r) = \frac{1}{\kappa_{\rm sca}(r)} \sum_{\rm k} \int \varphi_{\rm k}(\Omega; a)\, \pi a^2\, Q_{\rm sca\,k}\, n_{\rm k}(a; r)\, da. \tag{7}$$

This should be a normalized function, because the differential scattering phase function itself satisfies the normalization condition

$$\int_{4\pi} \varphi_{\rm k}(\Omega; a)\, d\Omega = 1$$

for any type k and size a.

c) The Henyey-Greenstein Scattering Phase Function

As a good parametric representation of the particle scattering phase function $\varphi_{\rm k}$ or the mean volume scattering phase function Φ, we utilize the Henyey-Greenstein function,

$$\Phi_{\rm HG}(\Theta) = \frac{1}{4\pi} \frac{1 - g^2}{(1 + g^2 - 2g\cos\Theta)^{3/2}}, \tag{8a}$$

for its versatility. Here, azimuthal symmetry has been assumed for the scattering phase function, and scattering angle is simply denoted by Θ. g is the asymmetry factor. This single parameter function is well known for its flexibility in portraying gross properties of scattering phase functions (c.f. van de Hulst 1980). A linear sum of a few H-G functions with different asymmetry factors would faithfully describe such major characteristics of the scattering function as the rapid increase towards the forward direction, almost an isotropic nature in intermediate scattering angles, and the backward enhancement (Hong 1985).

If $\hat{n}$ and $\hat{n}'$ are unit vectors in the scattered and incident directions, respectively, cosine of the scattering angle is given by

$$\cos\Theta = \hat{n} \cdot \hat{n}' = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi').$$

Since azimuthal symmetry holds true for the phase function and the azimuthal angle ϕ can be counted from any direction, we take ϕ to be zero without any loss of generality. Replacing $\cos\theta$ by μ, we then have

$$\cos\Theta = \mu\mu' + (1 - \mu^2)^{1/2}(1 - \mu'^2)^{1/2}\cos\phi'.$$

With this relation we can write the scattering phase function in the form

$$\Phi_{\rm HG}(\mu, \mu', \phi'; g) = \frac{1 - g^2}{4\pi}\left[1 + g^2 - 2g\,\mu\mu' - 2g\,(1 - \mu^2)^{1/2}(1 - \mu'^2)^{1/2}\cos\phi'\right]^{-3/2}. \tag{8b}$$

In terms of function $PH(\mu, \mu'; g)$ defined as

$$PH(\mu, \mu'; g) \equiv \int_0^{2\pi} \Phi_{\rm HG}(\mu, \mu', \phi'; g)\, d\phi', \tag{8c}$$

the volume emissivity integral in Eq. (3) becomes

$$\eta(r, \mu) = \kappa_{sca}(r) \int_{-1}^{+1} d\mu'\, I(r, \mu') \sum_g w_g PH(\mu, \mu'; g), \tag{9}$$

where w_g means relative contribution of an H-G function with asymmetry factor g to the total volume scattering function. Of course the sum of w_g's is unity.

For most types of particulate components three different H-G functions are enough to fully characterize their scattering behaviors. When a model of H-G functions is fixed, one may tabulate $PH(\mu, \mu'; g)$'s for a chosen set of

μ and μ' combinations. If the relative weights, w_g's, are kept as functions of radial distance, position dependence of the scattering behaviors can also be treated.

d) The Quasi-Diffusion Equation

Because we take the derivative in Eq. (4) along the path with projected distance being fixed, we have

$$\left[\frac{\partial}{\partial s}\right]_p = \left[\mu\frac{\partial}{\partial r}\right]_p , \tag{10}$$

which in turn leads us to re-write Eq. (4) as

$$\mu\frac{\partial}{\partial r}I(r,\mu) = -\kappa(r)I(r,\mu) + \eta(r,\mu) \quad ; \quad -1 \le \mu \le +1. \tag{11a}$$

Let us divide the whole range of μ into positive, $0 \le \mu \le +1$, and negative, $-1 \le \mu < 0$, domains. In the two domains the radiative transfer equation takes exactly the same form as

$$\begin{aligned} \mu\frac{\partial}{\partial r}I(r,\mu) &= -\kappa(r)I(r,\mu) + \eta(r,\mu) \quad ; \quad 0 \le \mu \le +1, \\ \mu\frac{\partial}{\partial r}I(r,\mu) &= -\kappa(r)I(r,\mu) + \eta(r,\mu) \quad ; \quad -1 \le \mu < 0. \end{aligned} \tag{11b}$$

If we keep μ to be positive, the above equations assume

$$\begin{aligned} \mu\frac{\partial}{\partial r}I(r,\mu) &= -\kappa(r)I(r,\mu) + \eta(r,\mu) \quad ; \quad 0 \le \mu \le +1, \\ -\mu\frac{\partial}{\partial r}I(r,-\mu) &= -\kappa(r)I(r,-\mu) + \eta(r,-\mu) \quad ; \quad 0 \le \mu \le +1. \end{aligned} \tag{11c}$$

Although μ is kept within its positive domain, this set of equations describes the intensity field in the full domain of μ. This observation suggests us to name $I(r,\mu)$ as $I^+(r,\mu)$ with μ being positive and $I(r,-\mu)$ as $I^-(r,\mu)$ with μ again being positive.

Figure 2 will help illustrate geometrical meanings of this naming practice. With these new names we re-write Eq. (11c) as

$$\begin{aligned} \mu\frac{\partial}{\partial r}I^+(r,\mu) &= -\kappa(r)I^+(r,\mu) + \eta(r,\mu) \quad ; \quad 0 \le \mu \le +1, \\ \mu\frac{\partial}{\partial r}I^-(r,\mu) &= \kappa(r)I^-(r,\mu) - \eta(r,-\mu) \quad ; \quad 0 \le \mu \le +1. \end{aligned} \tag{11d}$$

To combine this set of two first order differential equations into a second order one we introduce two pairs of new functions. The first pair are defined by

$$\begin{aligned} u(r,\mu) &\equiv \tfrac{1}{2}\left[I^+(r,\mu) + I^-(r,\mu)\right], \\ v(r,\mu) &\equiv \tfrac{1}{2}\left[I^+(r,\mu) - I^-(r,\mu)\right], \end{aligned} \tag{12}$$

and the second by

$$\begin{aligned} \eta_{\rm evn}(r,\mu) &\equiv \tfrac{1}{2}\left[\eta(r,\mu) + \eta(r,-\mu)\right], \\ \eta_{\rm odd}(r,\mu) &\equiv \tfrac{1}{2}\left[\eta(r,\mu) - \eta(r,-\mu)\right]. \end{aligned} \tag{13}$$

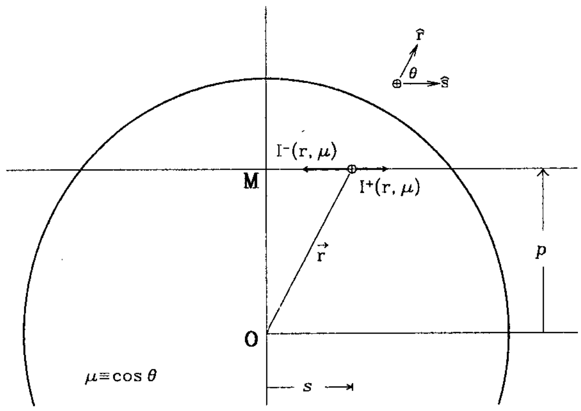

Fig. 2.— A two stream representation of specific intensity. The specific intensity along a given light path has two components: The intensity in the outward direction is denoted by $I^+(r,\mu)$ and the inward one by $I^-(r,\mu)$. Here, $\mu = \cos\theta = \hat{r}\cdot\hat{s}$ is kept positive.

By adding one to the other and subtracting from each other in Eq. (11d) and using the four new functions in Eq. (12) and Eq. (13), we finally have

$$\mu\frac{\partial}{\partial r}\,u(r,\mu) = -\kappa(r)\,v(r,\mu) + \eta_{\rm odd}(r,\mu), \tag{14a}$$

and

$$\mu\frac{\partial}{\partial r}\,v(r,\mu) = -\kappa(r)\,u(r,\mu) + \eta_{\rm evn}(r,\mu). \tag{14b}$$

From now on we will not specify the range for μ, since it will be always kept positive. From these two equations one may eliminate u in favor of v and construct a second order differential equation

$$\mu\frac{\partial}{\partial r}\left[\frac{\mu}{\kappa(r)}\frac{\partial}{\partial r}\,v(r,\mu)\right] = \kappa(r)\,v(r,\mu) - \eta_{\rm odd}(r,\mu) + \mu\frac{\partial}{\partial r}\left[\frac{1}{\kappa(r)}\,\eta_{\rm evn}(r,\mu)\right]. \tag{15a}$$

If v is eliminated, we similarly have

$$\mu\frac{\partial}{\partial r}\left[\frac{\mu}{\kappa(r)}\frac{\partial}{\partial r}\,u(r,\mu)\right] = \kappa(r)\,u(r,\mu) - \eta_{\rm evn}(r,\mu) + \mu\frac{\partial}{\partial r}\left[\frac{1}{\kappa(r)}\,\eta_{\rm odd}(r,\mu)\right]. \tag{15b}$$

Let us go back to the light-path distance, s, from the variable, r, and re-write the first order differential equations as

$$\frac{\partial u}{\partial s} = -\kappa\,v + \eta_{\rm odd} \quad ; \qquad \frac{\partial v}{\partial s} = -\kappa\,u + \eta_{\rm evn}. \tag{16a}$$

This makes the second order differential equations more compact:

$$\begin{aligned} \frac{\partial}{\partial s}\left[\frac{1}{\kappa}\frac{\partial}{\partial s}v\right] &= \kappa\,v - \eta_{\rm odd} + \tfrac{\partial}{\partial s}\left[\tfrac{1}{\kappa}\,\eta_{\rm evn}\right], \\ \frac{\partial}{\partial s}\left[\frac{1}{\kappa}\frac{\partial}{\partial s}u\right] &= \kappa\,u - \eta_{\rm evn} + \tfrac{\partial}{\partial s}\left[\tfrac{1}{\kappa}\,\eta_{\rm odd}\right]. \end{aligned} \tag{16b}$$

We have transformed the radiative transfer equation into the *quasi-diffusion* equation. Please keep in mind that $\kappa = \kappa(r)$, $u = u(r,\mu)$, $v = v(r,\mu)$, $\eta_{\rm odd} = \eta_{\rm odd}(r,\mu)$, and $\eta_{\rm evn} = \eta_{\rm evn}(r,\mu)$.

It is worthwhile to comment on the choice of r and s. We had used s all the way up to Eq. (4) and from Eq. (10) on we switched the variable to r. One might wonder why we went to r in Eq. (11a, b, c, and d) and then back to s in Eq. (16a and b). If we had kept s, we would have some difficulties in matching the sign of ds with the domain of μ. Because the sign of ds is dictated by that of μ, we could easily be confused when I was divided into I^+ and I^-.

III. A TRI-DIAGONAL SYSTEM OF FINITE DIFFERENCE EQUATIONS

A standard way of solving a second order differential equation is first to approximate differentials to finite differences and then to obtain solutions of the resulting simultaneous linear equations. If they comprise a tri-diagonal system, one can easily find the solutions. In this section we will construct a tri-diagonal system out of the second order differential equation. Upon applying boundary conditions to the system, one obtains solutions of u and v, which will eventually lead us to specify the radiation field in terms of intensity itself.

a) Three-Point Finite Differencing

Any combination of three points, (x_i, y_i) with i=1, 2, and 3, can be fitted to a parabola. If we take

$$y(x) = \alpha + \beta(x - x_{i-1}) + \gamma(x - x_{i-1})(x - x_i) \tag{17a}$$

as an equation of the parabola, the coefficients α, β and γ can be determined by substituting (x_{i-1}, y_{i-1}), (x_i, y_i) and (x_{i+1}, y_{i+1}) to the equation in sequence:

$$\alpha = y_{i-1} \quad ; \qquad \beta = \frac{1}{b}(y_i - y_{i-1}) \quad ; \qquad \gamma = \frac{1}{bf(b+f)}\left[f\, y_{i-1} - (b+f)\, y_i + b\, y_{i+1}\right]. \tag{18a}$$

Here we have denoted the *backward* step $x_i - x_{i-1}$ by b and the *forward* step $x_{i+1} - x_i$ by f. Since the derivative of the parabolic function is

$$\frac{d}{dx}y(x) = \beta + \gamma(x - x_{i-1}) + \gamma(x - x_i), \tag{19}$$

its value can be calculated at the left end, middle, and right end points, respectively, by

$$\begin{aligned}
\left[\frac{dy}{dx}\right]_{x_{i-1}} &= \frac{1}{bf(b+f)}\left[-f(2b+f)y_{i-1} + (b+f)^2 y_i - b^2 y_{i+1}\right], \\
\left[\frac{dy}{dx}\right]_{x_i} &= \frac{1}{bf(b+f)}\left[-f^2 y_{i-1} + (f^2 - b^2)y_i + b^2 y_{i+1}\right], \\
\left[\frac{dy}{dx}\right]_{x_{i+1}} &= \frac{1}{bf(b+f)}\left[f^2 y_{i-1} - (b+f)^2 y_i + b(b+2f)y_{i+1}\right].
\end{aligned} \tag{20}$$

Please note that the backward and forward steps, b and f, depend on index i. The first of these three equations is particularly useful at the left most point; while the last one is at the right most one.

Equation (16) has terms of the types $\frac{d}{dx}[K(x)\, y(x)]$ and $\frac{d}{dx}[K(x)\,\{dy(x)/dx\}]$. For the former Eq. (20) can be used exactly as it is:

$$\left[\frac{d}{dx}K(x)\, y(x)\right]_{x_i} = \frac{1}{bf(b+f)}\left[-f^2 (K\, y)_{i-1} + (f^2 - b^2)(K\, y)_i + b^2 (K\, y)_{i+1}\right]. \tag{21}$$

For the latter type, however, care should be exercised. Since our scheme of finite differencing is accurate only to the second order in step size Δx_i, the product of the functions $K(x)$ and $dy(x)/dx$ should also be kept to the same order. By using three (K_i, x_i) pairs we first fit $K(x)$ to a parabola

$$K(x) = A + B(x - x_{i-1}) + G(x - x_{i-1})(x - x_i), \tag{17b}$$

where the coefficients A, B and G are determined exactly as in Eq. (18b):

$$A = K_{i-1} \quad ; \quad B = \frac{1}{b}(K_i - K_{i-1}) \quad ; \quad G = \frac{1}{bf(b+f)}\left[f\,K_{i-1} - (b+f)\,K_i + b\,K_{i+1}\right]. \tag{18b}$$

If terms are kept only up to the second order, the product $K(x)\,\{dy(x)/dx\}$ becomes

$$K(x)\frac{d}{dx}\,y(x) = A\beta + (A\gamma + B\beta)\,(x - x_{i-1}) + A\gamma(x - x_i) + B\gamma\,(x - x_{i-1})^2 + (B\gamma + G\beta)\,(x - x_{i-1})(x - x_i),$$

whose derivative is

$$\frac{d}{dx}\left[K(x)\frac{d}{dx}y(x)\right] = (2A\gamma + B\beta) + (3B\gamma + G\beta)\,(x - x_{i-1}) + (B\gamma + G\beta)\,(x - x_i).$$

Consequently, the derivative at $x = x_i$ is calculated from

$$\left[\frac{d}{dx}\left(K(x)\frac{d}{dx}y(x)\right)\right]_{x_i} = (2A\gamma + B\beta) + (3B\gamma + G])b.$$

Into this equation we substitute Eq. (18a) for γ and β, and Eq. (18b) for A, B and G, and finally express the derivative in terms of K_i's and y_i's:

$$\begin{aligned}\left[\frac{d}{dx}\left(K(x)\frac{d}{dx}y(x)\right)\right]_{x_i} &= -\frac{1}{bf(b+f)}\;[-bK_{i-1} + 3bK_i]\,y_{i+1} \\ &+ \frac{1}{bf(b+f)}\cdot\left[\left(-\frac{f^2}{b} + b + f\right)K_{i-1} + \left(\frac{f^2}{b} - 3f - 4b\right)K_i + bK_{i+1}\right]y_i \\ &+ \frac{1}{bf(b+f)}\cdot\left[\left(\frac{f^2}{b} - f\right)K_{i-1} + \left(-\frac{f^2}{b} + b + 3f\right)K_i - bK_{i+1}\right]y_{i-1}.\end{aligned} \tag{22}$$

Please note that this expression is somewhat different from equation (2.13) of Cassinelli & Hummer (1971). If our expressions are re-written in their way, the quantity within the last pair of square brackets, for example, becomes

$$\left[\left(\frac{f^2}{b}K_{i-1} + (2f - \frac{f^2}{b})K_i\right) - \{fK_{i-1} - (b+f)K_i + bK_{i+1}\}\right]. \tag{23}$$

The term in the first pair of parentheses is exactly the same for both expressions, but the remaining term was not included in their scheme of finite differencing. If K_i varies almost linearly with x_i, the term in the curly brackets would become almost zero. This can be seen from the relation in Eq. (18b). In problems of our interest, function $K(x)$ corresponds to $1/\kappa(s)$, and in some part of the Earth's atmosphere, the extinction coefficient varies quite rapidly with s. However, in the current version of `QDM_sca` we use exactly the same finite differencing as the one by Cassinelli & Hummer (1971).

b) Consideration of Dimensions

In numerical calculations it is convenient to work with dimensionless variables. We therefore normalize all the variables to their characteristic values that are dictated by the problem and use *tilde* sign for the time being to signify their dimensionless nature:

$$\tilde{s} \equiv s/r_\mathrm{u} \quad ; \quad \tilde{u} \equiv u/u_\mathrm{u} \quad ; \quad \tilde{K} \equiv K/K_\mathrm{u}(= \kappa_\mathrm{u}/\kappa) \quad ; \quad \tilde{\eta} \equiv \eta/\eta_\mathrm{u} \equiv \eta[u_\mathrm{u}/K_u]^{-1}. \tag{24}$$

Since $K(r) = 1/\kappa(r)$ and the dimension of κ is an inverse of length, the dimension of K_u ought to be *length.* And if a notice is made to the dimensional relation $\eta \sim \kappa u$ or $\eta \sim K^{-1}u$, one can use $\eta_\mathrm{u} = u_\mathrm{u}/K_\mathrm{u}$. In terms of the dimensionless variables the quasi-diffusion equation takes the following form

$$\frac{\partial}{\partial\tilde{s}}\left[\tilde{K}\frac{\partial\tilde{u}}{\partial\tilde{s}}\right] = \frac{r_\mathrm{u}^2}{K_\mathrm{u}^2}\frac{\tilde{u}}{\tilde{K}} - \frac{r_\mathrm{u}^2}{K_\mathrm{u}^2}\tilde{\eta}_\mathrm{evn} + \frac{r_\mathrm{u}}{K_\mathrm{u}}\frac{\partial}{\partial\tilde{s}}\left[\tilde{K}\tilde{\eta}_\mathrm{odd}\right]. \tag{25}$$

Putting the last two terms of the above equation in $\mathcal{E}$,

$$\mathcal{E}(\tilde{K},\tilde{\eta}) = \frac{r_{\rm u}^2}{K_{\rm u}^2}\,\tilde{\eta}_{\rm evn} - \frac{r_{\rm u}}{K_{\rm u}}\,\frac{\partial}{\partial\tilde{s}}\left[\tilde{K}\tilde{\eta}_{\rm odd}\right], \tag{26}$$

we re-write the quasi-diffusion equation in terms of the normalized variables as

$$\frac{\partial}{\partial\tilde{s}}\left[\tilde{K}\frac{\partial\tilde{u}}{\partial\tilde{s}}\right] = \frac{r_{\rm u}^2}{K_{\rm u}^2}\,\frac{\tilde{u}}{\tilde{K}} - \mathcal{E}(\tilde{K},\tilde{\eta}). \tag{27}$$

It is this form of quasi-diffusion equation that will be transfigured to a tri-diagonal system of difference equations.

For the parameter $u_{\rm u}$ the intensity of the incident radiation is a natural choice of unit. For the scale parameter $r_{\rm u}$ we may take the geometrical or scale height of the given atmosphere or the Earth's radius, while the total extinction optical depth of the atmosphere will guide us to fix the parameter $K_{\rm u}$. Scattering and absorption characteristics of the atmospheric constituents and the inner surface introduce additional parameters to the system, which could be the particle albedo, the particle scattering asymmetry factor, and the surface albedo.

c) The Tri-diagonal System of Linear Equations

The finite difference given in Eq. (22) transforms the normalized quasi-diffusion equation into a tri-diagonal system

$$q_{i-1}\tilde{u}_{i-1} + q_i\tilde{u}_i + q_{i+1}\tilde{u}_{i+1} = bf(b+f)\left[\frac{r_{\rm u}^2}{K_{\rm u}^2}\frac{\tilde{u}_i}{\tilde{K}_i} - \mathcal{E}(\tilde{K}_i,\tilde{\eta}_i)\right],$$

where

$$\begin{aligned} q_{i-1} &= \left[\left(\frac{f^2}{b} - f\right)\tilde{K}_{i-1} + \left(-\frac{f^2}{b} + b + 3f\right)\tilde{K}_i - b\tilde{K}_{i+1}\right], \\ q_i &= \left[\left(-\frac{f^2}{b} + b + f\right)\tilde{K}_{i-1} + \left(\frac{f^2}{b} - 3f - 4b\right)\tilde{K}_i + b\tilde{K}_{i+1}\right], \\ q_{i+1} &= \left[-b\tilde{K}_{i-1} + 3b\tilde{K}_i\right]. \end{aligned} \tag{28a}$$

We re-arrange the terms of the above two equations and cast the tri-diagonal system into the following form:

$$\mathcal{A}_i\tilde{u}_{i-1} + \mathcal{B}_i\tilde{u}_i + \mathcal{C}_i\tilde{u}_{i+1} = \mathcal{D}_i; \quad i = 1,2,3, \quad \ldots \quad N,$$

where

$$\mathcal{A}_i = q_{i-1} \quad ; \quad \mathcal{B}_i = -\left\{q_i + \frac{bf(b+f)}{\tilde{K}_i}\frac{r_{\rm u}^2}{\tilde{K}_{\rm u}^2}\right\} \quad ; \quad \mathcal{C}_i = q_{i+1} \quad ; \quad \mathcal{D}_i = -bf(b+f)\,\mathcal{E}_i. \tag{28b}$$

In matrix form Eq. (28b) becomes

$$\begin{pmatrix} \mathcal{B}_1 & \mathcal{C}_1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ \mathcal{A}_2 & \mathcal{B}_2 & \mathcal{C}_2 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & \mathcal{A}_3 & \mathcal{B}_3 & \mathcal{C}_3 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{A}_4 & \mathcal{B}_4 & \mathcal{C}_4 & \ldots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & \mathcal{A}_{N-1} & \mathcal{B}_{N-1} & \mathcal{C}_{N-1} \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & \mathcal{A}_N & \mathcal{B}_N \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \\ u_{N-1} \\ u_N \end{pmatrix} = \begin{pmatrix} \mathcal{D}_1 \\ \mathcal{D}_2 \\ \mathcal{D}_3 \\ \mathcal{D}_4 \\ \vdots \\ \mathcal{D}_{N-1} \\ \mathcal{D}_N \end{pmatrix}. \tag{29}$$

The inner and outer boundary conditions will specify the first and last rows of the tri-diagonal system, respectively.

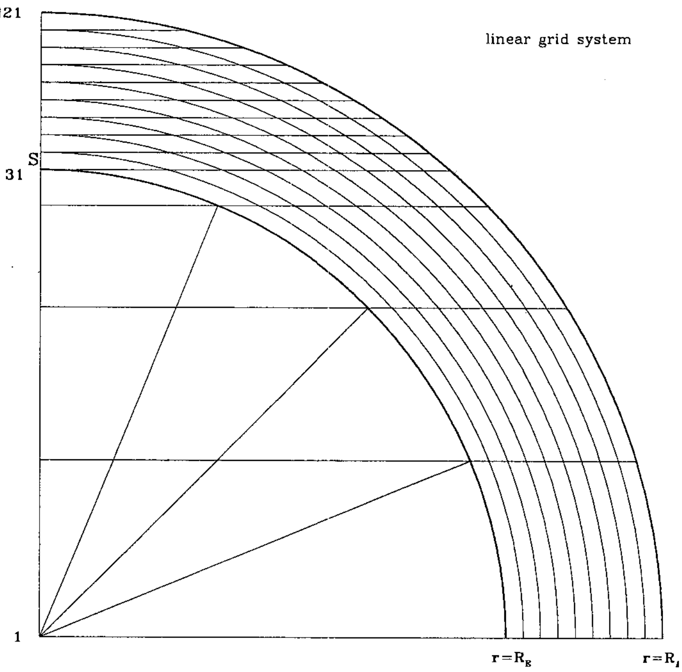

Fig. 3.— An example of grid system. Out of the 121 rays, in total, 30 are placed below the horizon of an observer at point S, and the remaining 91 are above and on the horizon, dividing the atmosphere into spherical shells of equal thickness. Each of the 31 light rays that are blocked off by or tangent to the Earth's surface has 90 light-path segments; while each of the remaining 90 rays consists of different numbers of segments.

IV. IMPLEMENTATION OF THE BOUNDARY CONDITIONS

Boundary conditions are dictated by the nature of the problem one is working on. Therefore, it is the users who are to specify the conditions. For example, in dark clouds one may not have any solid cores at the center, but there may be some radiation sources like proto-stars. On the other hand, for the Earth we have a solid surface at bottom of the atmosphere. In this paper we will derive the boundary conditions only for the case of Earth's atmosphere.

The (s, p) grid system is illustrated in Fig 3, where s increases outward along the light path and p upward from the horizontal line. Light rays with projected distance larger than the Earth's radius, R_E, do not meet the solid surface. From symmetry of the problem we may easily specify the conditions to be met by such rays at the mid-point where s=0. We call this mid-point boundary condition. The light rays with $p \leq R_E$ will hit the Earth's surface and then get partially absorbed and reflected as well. Depending upon the optical properties of the surface, the boundary condition there can take different characteristics. We call this the surface boundary condition. At the top of the atmosphere we have to take into account the ambient radiation field, from which photons get into the atmosphere.

To simplify the notation, from now on we omit all the tilde signs from the dimensionless variables.

a) Mid-point Boundary Condition

Near the mid-point of a light path the field u can be Taylor-expanded as

$$u(s \simeq 0) = u(0) + s \left[\frac{\partial u}{\partial s}\right]_{s=0} + \frac{s^2}{2}\left[\frac{\partial^2 u}{\partial s^2}\right]_{s=0} + \quad \ldots$$

The symmetry of u with respect to the mid-point requires the first derivative $\partial u/\partial s$ to be zero at s = 0, and the quasi-diffusion equation itself, Eq. (27), tells us that the second derivative is

$$\left[\frac{\partial^2 u}{\partial s^2}\right]_{s=0} = \frac{1}{K(s=0)}\left[\frac{r_{\rm u}^2}{K_{\rm u}^2}\frac{u}{K} - \mathcal{E}\right]_{s=0}.$$

Here we have made use of the fact that $\partial K/\partial s = 0$ at the mid-point. With these pieces of information, the Taylor expansion of u gives us

$$u_2 = u_1 + \frac{1}{2}\frac{s_2^2}{K_1}\left[\frac{r_{\rm u}^2}{K_{\rm u}^2}\frac{u_1}{K_1} - \mathcal{E}_1\right].$$

Please note that in our index convention subscript $i = 1$ *indicates the mid-point location* where $s = 0$, and also note that $f = s_2 - s_1 = s_2 - 0 = s_2$. Re-arranging terms of the above equation we have

$$\left(2 + \frac{s_2^2}{K_1^2}\frac{r_{\rm u}^2}{K_{\rm u}^2}\right)u_1 - 2\,u_2 = \frac{s_2^2}{K_1}\mathcal{E}_1.$$

Consequently, the mid-point boundary condition is described as

$$\mathcal{B}_1 = 2 + \frac{s_2^2}{K_1^2}\frac{r_{\rm u}^2}{K_{\rm u}^2} \quad ; \qquad \mathcal{C}_1 = -2 \quad ; \qquad \mathcal{D}_1 = \frac{s_2^2}{K_1}\mathcal{E}_1. \qquad (MPBC)$$

This can be applied to all the light rays whose projected distances are greater than $\mathrm{R_E}$.

b) Inner Surface Boundary Condition

For the rays with p less than $\mathrm{R_E}$ we have to take into account the reflection by the Earth's surface. Upon meeting the surface, a part of the incident intensity $I^-(0,\mu)$ will be absorbed and the rest will be reflected out again. Since we take the reflection angle equal to the incidence angle, as illustrated in Fig 4, we can express the intensity $I^+(0,\mu)$ of the reflected light as a product of the surface reflection coefficient SR and the incident intensity $I^-(0,\mu)$. At the surface $u(0) = [(1/2)(1+\mathrm{SR})]I^-(0,\mu)$ and $v(0) = [-(1/2)(1-\mathrm{SR})]I^-(0,\mu)$ hold true, and, consequently, we may have $v(0) = -\mathrm{F}\,u(0)$ with F being the ratio $(1-\mathrm{SR})/(1+\mathrm{SR})$. This is the condition we can impose upon the light rays hitting the Earth's surface. The reflection factor F could be a function of μ, because in general SR depends on the direction of incidence. Even so one may still directly relate v to u at the surface.

Having established $v(0) = -\mathrm{F}u(0)$, we can use Eq. (16a) to get the first derivative at the surface. In terms of the dimensionless variables the equation becomes

$$\left[\frac{\partial u}{\partial s}\right]_{s=0} = \frac{r_{\rm u}}{K_{\rm u}}\left[\mathrm{F}\frac{u}{K} + \eta_{\rm odd}\right]_{s=0}.$$

This will immediately fix the first derivative in the Taylor expansion. For the second derivative, however, a non-zero term $\partial K/\partial s$ has to be taken into account. From Eq. (27) we obtain

$$\begin{aligned}\left[\frac{\partial^2 u}{\partial s^2}\right]_{s=0} &= \left[\frac{1}{K}\left\{\frac{r_{\rm u}^2}{K_{\rm u}^2}\frac{u}{K} - \mathcal{E} - \frac{\partial K}{\partial s}\frac{\partial u}{\partial s}\right\}\right]_{s=0} \\ &= \left[\frac{1}{K}\left\{\frac{r_{\rm u}^2}{K_{\rm u}^2}\frac{u}{K} - \mathcal{E} - \frac{\partial K}{\partial s}\frac{r_{\rm u}}{K_{\rm u}}\left(F\frac{u}{K} + \eta_{\rm odd}\right)\right\}\right]_{s=0} \\ &= \left[\frac{1}{K}\left\{\frac{r_{\rm u}}{K_{\rm u}}\left(\frac{1}{K}\frac{r_{\rm u}}{K_{\rm u}} - \mathrm{F}\frac{\partial K}{\partial s}\right)u - \left(\mathcal{E} + \frac{r_{\rm u}}{K_{\rm u}}\frac{\partial K}{\partial s}\eta_{\rm odd}\right)\right\}\right]_{s=0}.\end{aligned}$$

Upon the above two equations being substituted for the first and second derivatives, the Taylor expansion becomes

$$u(s) = u(0) + s\,\frac{r_{\rm u}}{K_{\rm u}}\left[\mathrm{F}\frac{u}{K} + \eta_{\rm odd}\right]_{s=0} + \frac{s^2}{2}\left[\frac{1}{K}\left\{\frac{r_{\rm u}}{K_{\rm u}}\left(\frac{1}{K}\frac{r_{\rm u}}{K_{\rm u}} - \mathrm{F}\frac{\partial K}{\partial s}\right)u - \left(\mathcal{E} + \frac{r_{\rm u}}{K_{\rm u}}\frac{\partial K}{\partial s}\eta_{\rm odd}\right)\right\}\right]_{s=0} + \cdots.$$

After the terms involving $u(s)$ and $u(0)$ are sorted out, this equation takes in our index notation the following form:

$$\left[1 + \frac{s_2 r_{\rm u}}{K_1 K_{\rm u}}\left\{\left(1 - \frac{1}{2}\frac{s_2}{K_1}\frac{\partial K_1}{\partial s}\right)\mathrm{F} + \frac{1}{2}\frac{s_2 r_{\rm u}}{K_1 K_{\rm u}}\right\}\right]u_1 - u_2 = -s_2\left\{\frac{r_{\rm u}}{K_{\rm u}}\left(1 - \frac{1}{2}\frac{s_2}{K_1}\frac{\partial K_1}{\partial s}\right)\eta_{\rm odd}(0) - \frac{1}{2}\frac{s_2}{K_1}\mathcal{E}_1\right\},$$

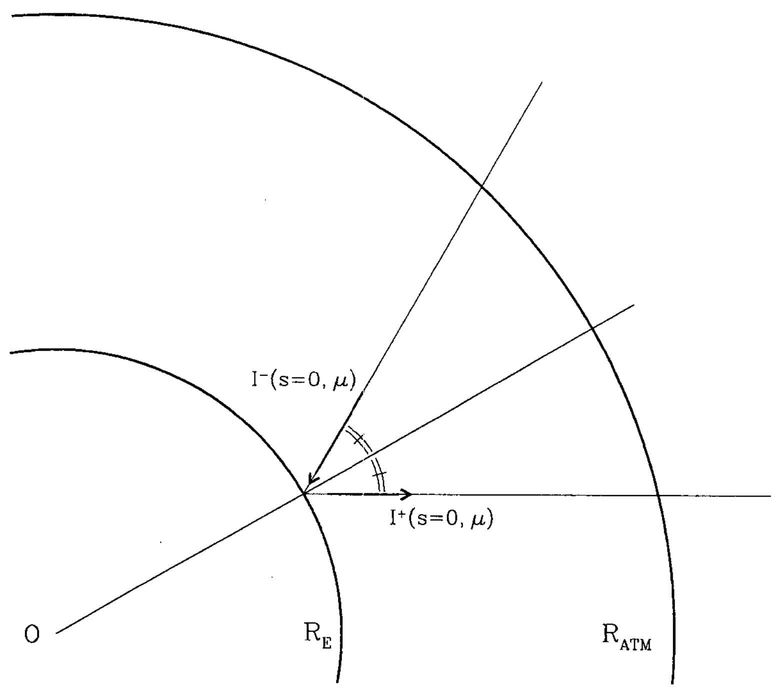

Fig. 4.— Reflection by the Earth's surface. For each ray a fraction SR of the inward intensity is reflected into the outward intensity, and the angle of reflection is kept equal to the incidence angle. In this figure it appears that light paths marked with I^+ and I^- reside in two different rays; however, they are actually in the same ray. This is because the two light paths make the same projected distance with the origin. If one would extend the two paths beyond the origin, one would easily see that they are at the same projected distance. Consequently, the two light paths are in the same ray. The next figure will help us better understand the geometry involving I^+ and I^-.

where $\partial K_1/\partial s$ means the value of derivative evaluated at $s = s_1 = 0$ and $\eta_{\rm odd}(0)$ is $\eta_{\rm odd\ 1}$ in our index notation. Consequently, the inner surface boundary condition is described as

$$\begin{aligned} \mathcal{B}_1 &= 1 + \frac{s_2 r_{\rm u}}{K_1 K_{\rm u}} \left\{ \left(1 - \frac{1}{2} \frac{s_2}{K_1} \frac{\partial K_1}{\partial s} \right) \mathrm{F} + \frac{1}{2} \frac{s_2 r_{\rm u}}{K_1 K_{\rm u}} \right\}, \\ \mathcal{C}_1 &= -1, \\ \mathcal{D}_1 &= -s_2 \left\{ \frac{r_{\rm u}}{K_{\rm u}} \left(1 - \frac{1}{2} \frac{s_2}{K_1} \frac{\partial K_1}{\partial s} \right) \eta_{\rm odd\ 1} - \frac{1}{2} \frac{s_2}{K_1} \mathcal{E}_1 \right\}. \end{aligned} \qquad (ISBC)$$

This should be applied to the rays whose projected distances are less than $\mathrm{R_E}$.

c) Outer Surface Boundary Condition

Near the outer boundary we expand u as

$$u_{N-1} = u_N + (s_{N-1} - s_N) \left[\frac{\partial u}{\partial x} \right]_N + \frac{1}{2} (s_{N-1} - s_N)^2 \left[\frac{\partial^2 u}{\partial s^2} \right]_N + \ \ldots \ .$$

Since $v = (1/2)(I^+ - I^-) = (1/2)(I^+ + I^-) - I^- = u - I^-$ holds true everywhere in the atmosphere and at the outer boundary I^- is nothing but the intensity $I_{\rm inc}$ incident upon the top of the Earth's atmosphere from the surrounding radiation field, one of the equations in Eq. (16a) becomes in the dimensionless variables

$$\left[\frac{\partial u}{\partial s}\right]_N = \frac{r_u}{K_u}\left[-\frac{u}{K} + \frac{I_{inc}}{K} + \eta_{odd}\right]_N .$$

Just as was done at the inner surface, we use the ray equation to derive the second derivative:

$$\begin{aligned}\left[\frac{\partial^2 u}{\partial s^2}\right]_N &= \left[\frac{1}{K}\left\{\frac{r_u^2}{K_u^2}\frac{u}{K} - \mathcal{E} - \frac{\partial K}{\partial s}\frac{\partial u}{\partial s}\right\}\right]_N \\ &= \left[\frac{1}{K}\left\{\frac{r_u^2}{K_u^2}\frac{u}{K} - \mathcal{E} + \frac{\partial K}{\partial s}\frac{r_u}{K_u}\left(\frac{u}{K} - \frac{I_{inc}}{K} - \eta_{odd}\right)\right\}\right]_N .\end{aligned}$$

With the first and second derivatives, we can completely describe the Taylor expansion and obtain

$$\begin{aligned}u_{N-1} &= u_N - b\frac{r_u}{K_u}\left[-\frac{u}{K} + \left(\frac{I_{inc}}{K} + \eta_{odd}\right)\right]_N \\ &\quad + \frac{b^2}{2}\left[\frac{1}{K}\left\{\frac{r_u^2}{K_u^2}\frac{u}{K} - \mathcal{E} + \frac{\partial K}{\partial s}\frac{r_u}{K_u}\frac{u}{K} - \frac{\partial K}{\partial s}\frac{r_u}{K_u}\left(\frac{I_{inc}}{K} + \eta_{odd}\right)\right\}\right]_N + \quad \ldots, \\ &= u_N + b\frac{r_u}{K_u}\left[\frac{u}{K}\right]_N - b\frac{r_u}{K_u}\left[\frac{I_{inc}}{K} + \eta_{odd}\right]_N \\ &\quad + \frac{b^2}{2}\frac{r_u}{K_u}\left[\frac{1}{K}\left(\frac{r_u}{K_u} + \frac{\partial K}{\partial s}\right)\frac{u}{K}\right]_N - \frac{b^2}{2}\left[\frac{1}{K}\left\{\mathcal{E} + \frac{\partial K}{\partial s}\frac{r_u}{K_u}\left(\frac{I_{inc}}{K} + \eta_{odd}\right)\right\}\right]_N + \quad \ldots .\end{aligned}$$

After terms are sorted out, this becomes

$$-u_{N-1} + \left[1 + \frac{b\,r_u}{K_u}\left\{1 + \frac{b}{2K}\left(\frac{r_u}{K_u} + \frac{\partial K}{\partial s}\right)\right\}\right]_N u_N = \frac{b\,r_u}{K_u}\left[\left(\frac{I_{inc}}{K} + \eta_{odd}\right)\left(1 + \frac{b}{2K}\frac{\partial K}{\partial s}\right)\right]_N + \frac{b^2}{2}\left[\frac{\mathcal{E}}{K}\right]_N .$$

In the code we implement this condition in the following form:

$$\begin{aligned}\mathcal{A}_N &= -1, \\ \mathcal{B}_N &= 1 + \frac{b\,r_u}{K_u}\left\{1 + \frac{b}{2K_N}\left(\frac{r_u}{K_u} + \frac{\partial K_N}{\partial s}\right)\right\}, \qquad\qquad (OSBC) \\ \mathcal{D}_N &= \frac{b\,r_u}{K_u}\left(\frac{I_{inc}}{K_N} + \eta_{odd\ N}\right)\left(1 + \frac{b}{2K_N}\frac{\partial K_N}{\partial s}\right) + \frac{b^2}{2}\frac{\mathcal{E}_N}{K_N}.\end{aligned}$$

V. GRID MESH, ATMOSPHERE MODEL, AND THE η-CONVERGENCY

Either MPBC or ISBC would give us specific "numbers" for the first row of the two dimensional coefficient matrix in Eq. (29), and OSBC would do the same for the last row. All the elements in the intermediate rows can be fixed by using Eq. (28a, b). The Earth's atmosphere model will describe the extinction-related quantity, K, as a function of radial distance r, hence of light-path distance s. And the gridding system fixes the backward b and forward f step sizes. Therefore, there is no unknown information and $\mathcal{A}_i$, $\mathcal{B}_i$, and $\mathcal{C}_i$ can be specified with numerical values. In other words, the atmosphere model and the gridding system are what we need to establish the coefficient matrix.

However, the constant matrix of a single column is not at all easy to fix; as can be seen from the boundary conditions and Eq. (28b), the $\mathcal{D}_i$ column vector involves $\mathcal{E}$, hence an evaluation of η, which in turn requires $I(r, \mu)$ to be known at every grid points for the whole range of μ, *i.e.*, the entire intensity field. There is no way of knowing the entire field before solving the equation. Therefore, we have to rely on some sort of iterations.

a) Construction of the Grid Mesh

The grid mesh is the first thing to construct. An example of the mesh system is illustrated in Fig 3. In total 121 light rays are selected to trace; 30 out of the 121 are with projected distance shorter than the Earth's radius; one is tangent to the Earth's surface; and the remaining 90 have projected distance longer than the radius. The 30 rays

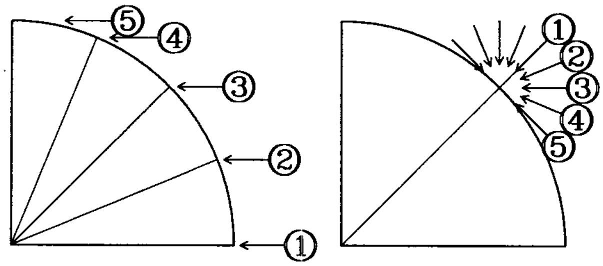

Fig. 5.— Construction of local intensity field from the two stream values. If all the inward intensities along a spherical shell are collected at a point on the shell, as is illustrated in this figure, they will fully describe the μ-dependence of the local inward intensity. The same type of collection can be done to construct the outward field.

are chosen in such a way that they meet the Earth's surface at integer latitudes 0^o through 87^o increasing in 3^o intervals. The entire extent of the Earth's atmosphere is divided into 90 layers of equal height, which automatically fixes the remaining 91 rays.

Let us assign index i to the light-path distance s, and j to the projected distance p. Please note that in our indexing system $s_{i=1} = 0$ for any j's, and that $p_{j=1} = 0$ and $p_{j=31} = \mathrm{R_E}$. The light path index i increases from 1 to N, and j takes 1 through 121. Obviously N depends on the j index: For $j \leq 31$, the i index takes 1 through $N = 91$; while for $j > 31$, it does from 1 up to $N = 121 - j + 1$. For any mesh system the backward $b(i,j)$ and forward $f(i,j)$ steps from every points of the grid intersection can be calculated, once and for all, as functions of i and j indices.

One may need to add a few more rays to the region where $p \sim \mathrm{R_E}$. For night sky observations, for example, we often need to observe all the way down to the horizon, from where one can retrieve many pieces of vital information about the interplanetary dust particles that are located close to the Sun. Furthermore, modern satellite platforms allow us to point telescope even below the Earth's horizon. Since atmospheric optical properties change steeply with altitude near the horizon, grid meshes there need to be finer than elsewhere. In these situations the entire atmosphere should be divided into layers of equal optical depth, rather than geometrical depth.

What the `QDM_sca` radiative code gives the user is the values of I^+ and I^- along each ray. One might wonder how to find out the μ-dependence of I from the intensity values only along each *fixed* direction of the rays. We show, in Fig 5, the conceptual basis of the ray tracing technique. Along a circle of fixed i the code calculates, for example, I^- values for all the j's. When all the I^-_j's with a fixed i are collected in the way illustrated in Fig 5, they completely describe how I^- varies with μ at that point of collection. The same can be understood for I^+. The spherical symmetry inherent to the problem has thus played a fundamental role in making the ray tracing a success.

This line of reasoning leads us to the question of μ and μ' combinations. If all the radial vectors starting from the Earth's center meet the rays exactly at the grid intersections, it would be suffice for us to calculate $PH(\mu, \mu')$ at only a fixed set of μ'_k 's. However, as can be seen from Fig 5, the set of μ'_k 's that can be collected from a circle of any i differs from the set collected from a circle of any other i's. We thus decided to calculate $PH(\mu, \mu')$ at a fixed set of finely distributed μ_k values and then to interpolate the resulting values as need arises for other values of μ.

In summary one can construct the data base for $b(i,j)$ and $f(i,j)$ by adopting a suitable grid system for a particular problem.

b) The Earth's Atmosphere Model

If we know how the volume densities of the atmospheric constituents vary with altitude, all the volume coefficients of scattering, absorption, and extinction, and the mean volume scattering phase function as well can be calculated as functions of radial distance by using Eq. (5), Eq. (6), and Eq. (7).

Although we limited our discussions in section IIc to the H-G representation, Legendre functions of low order are particularly useful for describing the scattering by molecular species. Where molecules are the only constituents of the atmosphere, one should of course utilize the Legendre function for the convenience of its addition theorem. In fact, Legendre expansion has been the standard practice in the field of radiative transfer. However, we prefer to integrate the H-G function directly without expanding it with Legendre functions, because the cases of strongly forward throwing scattering often require us to include the Legendre functions up to very high orders. In such cases the addition theorem would require an excessive amount of computations.

According to our experience, however, a linear combination of only *three* H-G functions suffices to faithfully portray the scattering phase functions due to mixtures of aerosol particles and molecules in the Earth's atmosphere. In general, the resulting scattering parameters, g and w_g, depend on the radial distance. This wouldn't introduce any conceptual difficulty to the calculation schemes for $PH(\mu, \mu')$ and η, but would certainly increase the amount of computations. So far we have considered the atmosphere in the sense of causing the scattering and absorption of light. In the atmosphere model we may introduce an airglow emitting layer at a very high altitude. This is pointed out in the source code by giving detailed comments. This part of the code will be revised extensively, when a more realistic model of the Earth's atmosphere is available.

In summary the atmosphere model will fix the opacity-related function $K(r)$ and the scattering phase function $\Phi(\Theta; r)$.

c) Iterations for η-Convergency

We have all the information to fill the three diagonals, $\mathcal{A}_i$, $\mathcal{B}_i$, and $\mathcal{C}_i$, of Eq. (29). However, to fill the column $\mathcal{D}_i$, we need to know $I(r,\mu)$ in addition to the scattering phase function. Since it is not known *a priori*, a trial function is assigned directly to $\eta(r;\mu)$ and we solve the tri-diagonal system of equations. The resulting solution for u is substituted in Eq. (16a), which yields v. By using Eq. (12) we can retrieve $I(r,\mu)$ from u and v, which enables us to calculate a new $\eta(r,\mu)$. When the new and old η's agree with each other, the iteration may be stopped. If not, the iteration is continued until the difference between the new and old becomes less than a pre-set criterion.

For an initial trial one usually ignores the μ-dependence of the volume emissivity and concentrates mostly on its r-dependence. First of all, the emissivity is directly proportional to the scattering coefficient $\kappa_{\rm sca}(r)$. Equation (3) or (9) indicates that the emissivity is a kind of "mean intensity." We said *a kind*, because it is not a true mean intensity. In this case the averaging is done with the scattering function being taken as a weight. Once we accept the idea of a mean intensity, we know that the integral will decrease from top of the atmosphere to the bottom following a relation something like $\exp[-\tau_{\rm eff}(r)]$. Here, optical depth $\tau_{\rm eff}(r)$ is measured from the outer boundary of the atmosphere to the local point $r = r$. If only singly scattered photons are considered, the effective extinction optical depth $\tau_{\rm eff}(r)$ would be the extinction optical depth $\tau_{\rm ext}(r)$ itself. Because of the contributions from multiply scattered photons to the local radiation field, $\tau_{\rm eff}(r)$ would be slightly less than $\tau_{\rm ext}(r)$. According to our experience, $\tau_{\rm eff}(r) \simeq (3/4)\,\tau_{\rm ext}(r)$ serves as a good approximation for the Earth's atmosphere (Hong et al. 1998).

The approximation based on the effective extinction optical depth grossly under-estimates the true mean intensity deep inside the atmosphere. As a trial function, however, it has all the qualifications needed for the emissivity due to scattering. We thus suggest the following function

$$\eta^{(o)}(r,\mu) = \kappa_{\rm sca}(r)\exp[-\epsilon\,\tau_{\rm ext}(r)] \tag{30}$$

as the zeroth order trial for the volume emissivity. To have a means of adjusting the dilution level due to multiple scattering we include a free parameter ϵ in the trial function. Users can adjust ϵ to a specific problem and may use values other than 3/4. Because this trial function results in an isotropic volume emissivity, $\eta^{(o)}_{\rm evn}(r,\mu) = \eta^{(o)}(r,\mu)$

and $\eta^{(o)}_{\rm odd}(r,\mu) = 0$. But from the next order iteration the volume emissivity becomes a μ-dependent quantity.

VI. AN APPLICATION OF THE QDM CODE

One of the most difficult tasks in isolating zodiacal light in the light of night sky is to remove atmosphere-related diffuse light from the observed brightness of the night sky. Since air molecules and aerosols divert light from off the telescope axis into the telescope beam through multiple scattering process, the brightness observed at a sky point has contributions from the point under consideration itself and its off-axis regions as well. In stellar photometry is no need to worry about multiply scattered photons, because for point sources there are no off-axis photons from the beginning. Therefore, reduction in starlight intensity is made by absorption and *single* scattering. This can be accurately taken into account by the usual $\exp[-\tau_{\rm ext}]$ factor with $\tau_{\rm ext}$ being total extinction optical depth. However, since in night sky photometry those off-axis photons that are scattered more than once get into the telescope beam, the exponential factor over-estimates the amount of net extinction. This in turn exaggerates the zodiacal light brightness.

If one wants to utilize a simplicity of the same exponential function in diffuse light photometry, he should use an effective extinction optical depth $\tau_{\rm eff}$ that is less than the total $\tau_{\rm ext}$ for such extended sources as zodiacal light, diffuse galactic light and airglow, i.e. $\tau_{\rm eff} = \epsilon\,\tau_{\rm ext}$ with ϵ being less than unity. It is difficult to pin-point a value for ϵ, since it depends on many scattering characteristics of the Earth's atmosphere. Only empirically it is known to be in the range 0.6 to 0.8 (Kwon 1990). This question was addressed in a recent paper by Hong et al. (1998). By using the code they rigorously solved the problem of radiative transfer in the Earth's scattering atmosphere and calculated the ratio, $\tau_{\rm eff}/\tau_{\rm ext}$, as a function of zenith distance for various values of total extinction optical depth over the zenith and albedo and scattering asymmetry factor of the aerosol particles. According to the model calculations the empirically known ratio holds true for zenith distances up to about 65°, beyond which it changes so significantly with zenith distance that a single value can not be assigned to ϵ. The QDM code thus has opened up a practical means for removing the atmospheric diffuse light accurately and led Kwon et al. (2000) to determine the zodiacal light brightness at solar elongation even down to 20 degrees.

VII. SUMMARY

The equation of radiative transfer in an anisotropically scattering spherical atmosphere has been transformed to a type of diffusion equation. To make transformation we have followed the intensity change along a set of carefully chosen rays and described the intensity field along each ray by its inward I^- and outward I^+ values. The sum, u, of the two intensity values is directly related to the difference, v, between the two through first order differential equations [Eq. (16a)]. From u and v we eliminate one in favor of the other and combine the two first order differential equations into a second order quasi-diffusion equation [Eq. (15a and b) or Eq. (16b)]. Both u and v are shown to satisfy the same type of second order differential equation.

By using the method of three point finite differencing we have converted the second order *differential* equation into a set of finite *difference* equations, which comprise a tri-diagonal system [Eq. (29)]. All the elements along the three diagonals of the coefficient matrix are fully specified by the geometry of the ray tracing meshes, the atmosphere model, and the boundary conditions [section IV]. However, evaluation of the constant column vector of the tri-diagonal system [Eq. (28b)] requires us to know the mean volume emissivity η due to scattering, which in turn involves an integral of the intensity field over the entire range of polar angle θ [Eq. (3) and Eq. (26)]. Because the intensity field is not known *a priori*, an r-dependent function [Eq. (30)] is tried first for the mean volume scattering emissivity, which fixes the column vector approximately. With the trial emissivity we solve the tri-diagonal system and obtain an approximate solution to the intensity field, which is then used to improve the scattering emissivity. In this way we can iterate the whole scheme of ray tracing until the intensity fields obtained in two consecutive iterations become identical to each other to a satisfactory degree.

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