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MULTIPLE L_p FOURIER-FEYNMAN TRANSFORM ON THE FRESNEL CLASS

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Abstract. In this paper, we introduce the concepts of multiple L_p analytic Fourier-Feynman transform ($1 \le p < \infty$) and a convolution product of functionals on abstract Wiener space and verify the existence of the multiple L_p analytic Fourier-Feynman transform for functionls in the Fresnel class. Moreover, we verify that the Fresnel class is closed under the L_p analytic Fourier-Feynman transformation and the convolution product, respectively. And we establish some relationships among the multiple L_p analytic Fourier-Feynman transform and the convolution product on the Fresnel class.

1. Introduction

In [2], Brue investigated initially the theory of an L_1 analytic Fourier-Feynman transform on a classical Wiener space, and in [3], Cameron and Storvick introduced the concept of an L_2 analytic Fourier-Feynman transform on a classical Wiener space. In [10], Johnson and Skoug developed an L_p analytic Fourier-Feynman transform theory for $1 \le p \le 2$ which extended the results in [2;3]. In [8;9], Huffman, Park and Skoug developed an L_p analytic Fourier-Feynman transform theory on certain classes of functionals defined on a classical Wiener space and they defined a convolution product of two functionals on the classical Wiener space and then found several interesting properties for the Fourier-Feynman transform and the convolution product on a classical Wiener space. In [1], the author investigated the L_1 analytic Fourier-Feynman transform theory on the Fresnel class of an abstract Wiener space.

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This paper is organized as follows. In Section 2, we introduce the basic concepts and the notations for our research. In Section 3, we investigate the essential properties for the multiple L_p analytic Fourier-Feynman transform and the convolution product on the Fresnel class of an abstract Wiener space. Finally, we establish some relationships among the L_p Fourier-Feynman transform and the convolution product on the Fresnel class.

2. Definitions and Preliminaries

Let H be a real separable infinite dimensional Hilbert space with norm $|\cdot| = \sqrt{\langle \cdot, \cdot \rangle}$, where $\langle \cdot, \cdot \rangle$ is an inner product on H. Let $\|\cdot\|_o$ be a fixed measurable norm on H (for definition see [13]). Let B be the completion of H with respect to the measurable norm $\|\cdot\|_o$ and $\mu_t (\, t > 0\,)$ the Gauss measure on H with variance t. Then μ_t induces a cylinder set measure $\tilde{\mu}_t$ on B which in turn extends to a countably additive measure ω_t on $(B, \mathcal{B}(B))$, where $\mathcal{B}(B)$ is the Borel σ-algebra of B. Then ω_t is called the *Wiener measure* with variance t and it has the following properties:

$$\left\{ \begin{array}{l} \omega_{st}(E) = \omega_t(s^{-1/2}E), \quad s > 0, \\ \omega_t(-E) = \omega_t(E). \end{array} \right. \tag{2.1}$$

From now on, we shall use ω instead of ω_1, by identifying ω with ω_1.

Let $\{e_n\}$ denote a complete orthonormal set of H such that e_n's are in B^*, the topological dual space of B. For each $h \in H$ and $x \in B$, we define a *stochastic inner product* $(\cdot, \cdot)^\sim$ between H and B as follows:

$$(h, x)^\sim = \left\{ \begin{array}{ll} \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \langle h, e_k \rangle (e_k, x) & , \text{ if the limit exists} \\ 0 & , \text{ otherwise,} \end{array} \right. \tag{2.2}$$

where $(\cdot, \cdot)$ is the natural dual pairing between B^* and B.

It is well known [11,12] that for every $h \in H, (h, x)^\sim$ exists for ω_t-a.e. $x \in B$, and $(h, \cdot)^\sim$ is a Borel measurable functional on B having a Gaussian distribution with mean zero and variance $t|h|^2$ with respect to ω_t. Furthermore, it is obvious that for each real number $\alpha, (\alpha h, x)^\sim = \alpha (h, x)^\sim = (h, \alpha x)^\sim$ holds for every $h \in H$ and $x \in B$. And we can show that $(h, x)^\sim = \langle h, x \rangle$, whenever h and x are elements of H.

Let (B, H, ω_t) be an abstract Wiener space. For each $\lambda > 0$, let $\mathcal{S}_\lambda(B)$ be the completion of $\mathcal{B}(B)$ with respect to ω_λ, and let $\mathcal{N}_\lambda(B) =$

$\{A \in \mathcal{S}_\lambda(B) : \omega_\lambda(A) = 0\}$. Let $\mathcal{S}(B) = \bigcap_{\lambda>0}\mathcal{S}_\lambda(B)$, and $\mathcal{N}(B) = \bigcap_{\lambda>0}\mathcal{N}_\lambda(B)$. Every set in $\mathcal{S}(B)$ (or $\mathcal{N}(B)$) is called a *scale-invariant measurable* (or *scale-invariant null*) set. A real (or complex)-valued functional F on B is called *scale-invariant measurable* if F is measurable with respect to $\mathcal{S}(B)$. A property that holds except on a scale- invariant null set is said to hold *scale-invariant almost everywhere* (briefly, s-a.e.). If two functionals F and G are equal s-a.e., then we write $F \approx G$. It is easy to show that this relation $\approx$ is an equivalence relation on the class of functionals on B. For a functional F on B, we will denote by $[F]$ the equivalence class of functionals which are equal to F s-a.e..

DEFINITION 2.1. Let (B, H, ω) be an abstract Wiener space and $\mathcal{M}(H)$ the space of all complex-valued countably additive Borel measures on H. Consider the functional F defined for s-a.e. $x \in B$ by the formula

$$F(x) = \int_H \exp\{i(h, x)^\sim\}\, df(h), \tag{2.3}$$

where f is in $\mathcal{M}(H)$. Let $\mathcal{F}(B)$ denote the collection of equivalence classes $[F]$ of functionals which are equal to F s-a.e. on B. Then we say that $\mathcal{F}(B)$ is the *Fresnel class* on the abstract Wiener space (B, H, ω).

REMARK 2.2. (1) As is customary, we will identify a functional with its equivalence class and think of $\mathcal{F}(B)$ as a class of functionals on B rather than as a class of equivalence classes.

(2) $\mathcal{M}(H)$ is a Banach algebra over the complex fields under the total variation norm $\|\cdot\|$, where the convolution is taken as the multiplication (see [7]). There exists an isomorphism of Banach algebras between $\mathcal{M}(H)$ and $\mathcal{F}(B)$ (see [11; Proposition 2.1].).

Throughout this paper, let $\mathbb{R}$ and $\mathbb{C}$ denote the real numbers and the complex numbers, respectively, and put $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ and $\widetilde{\mathbb{C}}_+ = \{z \in \mathbb{C} : z \neq 0, \mathrm{Re}(z) \geq 0\}$, where $\mathrm{Re}(z)$ means the real part of the complex number z.

Let F be a complex-valued scale-invariant measurable functional on the abstract Wiener space (B, H, ω) such that the Wiener integral

$$J(F; \lambda) = \int_B F(\lambda^{-1/2}x)\, d\omega(x)$$

exists as a finite number for all $\lambda > 0$. If there exists an analytic function $J^*(F; z)$ of z in the half-plane $\mathbb{C}_+$ such that $J^*(F; \lambda) = J(F; \lambda)$ for all

$\lambda > 0$, then we define this analytic extension $J^*(F;z)$ of $J(F;\lambda)$ to be the *analytic Wiener integral of F over B with parameter z* and we write

$$\int_B^{anw_z} F(x)\,d\omega(x) \equiv \mathcal{I}^{anw}(F;z) = J^*(F;z)$$

for all $z \in \mathbb{C}_+$.

Let q be a non-zero real number and F a functional on B such that the analytic Wiener integral $\mathcal{I}^{anw}(F;z)$ exists for all $z \in \mathbb{C}_+$. If the following limit exists, then we call it the *analytic Feynman integral of F over B with parameter q* and we write

$$\int_B^{anf_q} F(x)\,d\omega(x) \equiv \mathcal{I}^{anf}(F;q) = \lim_{z\to -iq} \mathcal{I}^{anw}(F;z),$$

where z approaches $-iq$ through $\mathbb{C}_+$.

Definition 2.3. Let $1 < p < \infty$ and let $\{F_n\}$ and F be scale-invariant measurable functionals on the abstract Wiener space (B,H,ω) such that for each $\rho > 0$,

$$\lim_{n\to\infty} \int_B |F_n(\rho x) - F(\rho x)|^{p'}\,d\omega(x) = 0. \tag{2.4}$$

Then we write

$$\underset{n\to\infty}{\mathrm{l.i.m.}}(w_s^{p'})(F_n) \approx F, \tag{2.5}$$

and we call F the *scale-invariant limit in the mean of order p'*, where p and p' are related by $1/p + 1/p' = 1$.

A similar definition is understood when n is replaced by the continuously varying parameter z.

Now we are ready to define an L_p analytic Fourier-Feynman transform $(1 \le p < \infty)$ on abstract Wiener space.

Definition 2.4. For each $z \in \mathbb{C}_+$, we define a transform $\mathcal{F}_z(F)$ of a functional F on the abstract Wiener space (B,H,ω) as follows:

$$\big(\mathcal{F}_z(F)\big)(y) = \mathcal{I}^{anw}(F(\cdot + y);z)\,, \quad y \in B. \tag{2.6}$$

Let q be a non-zero real number. In case that $1 < p < \infty$, we define the *L_p analytic Fourier-Feynman transform* $\mathcal{F}_{(q;p)}(F)$ for a functional F on (B,H,ω) by

$$(\mathcal{F}_{(q;p)}(F))(y) = \underset{z\to -iq}{\mathrm{l.i.m.}}(w_s^{p'})(\mathcal{F}_z(F))(y) \tag{2.7}$$

for s-a.e. $y \in B$, whenever this limit exists, where z approaches $-iq$ through $\mathbb{C}_+$.

Let q be a non-zero real number. In case that $p = 1$, we define the L_1 *analytic Fourier-Feynman transform* $\mathcal{F}_{(q;1)}(F)$ of F by

$$(\mathcal{F}_{(q;1)}(F))(y) = \lim_{z \to -iq} (\mathcal{F}_z(F))(y), \tag{2.8}$$

for s-a.e. $y \in B$, where z approaches $-iq$ through $\mathbb{C}_+$.

We note that for $1 \le p < \infty$, $\mathcal{F}_{(q;p)}(F)$ is defined only s-a.e.. We also note that if $\mathcal{F}_{(q;p)}(F)$ exists and if $F \approx G$, then $\mathcal{F}_{(q;p)}(G)$ exists and $\mathcal{F}_{(q;p)}(F) \approx \mathcal{F}_{(q;p)}(G)$.

We finish this section by giving the definition of the convolution product of two functionals on the abstract Wiener space (B, H, ω).

Definition 2.5. Let F and G be two complex-valued functionals on the abstract Wiener space (B, H, ω). For each $z \in \widetilde{\mathbb{C}}_+$, we define their *convolution product* $(F * G)_z$ as follows :

In case that z belongs to $\mathbb{C}_+$,

$$(F * G)_z(y) = \mathcal{I}^{anw}\Big[F\Big(\frac{1}{\sqrt{2}}(y + \cdot)\Big)\, G\Big(\frac{1}{\sqrt{2}}(y - \cdot)\Big)\, ;\, z\Big] \tag{2.9}$$

for $y \in B$, if it exists.

In case that $z = -iq$ ($q \in \mathbb{R} - \{0\}$),

$$(F * G)_q(y) = \mathcal{I}^{anf}\Big[F\Big(\frac{1}{\sqrt{2}}(y + \cdot)\Big)\, G\Big(\frac{1}{\sqrt{2}}(y - \cdot)\Big)\, ;\, q\Big] \tag{2.10}$$

for $y \in B$, if it exists.

3. Multiple L_p Analytic Fourier-Feynman Transform and Convolution

We begin this section by showing the existence of the L_p analytic Fourier-Feynman transform for every functional in the Fresnel class $\mathcal{F}(B)$.

Theorem 3.1. *Let $F \in \mathcal{F}(B)$ be given by* (2.3) *and let* $1 \le p < \infty$. *Then the transform $\mathcal{F}_z(F)$ exists for all $z \in \mathbb{C}_+$, it belongs to $\mathcal{F}(B)$, and the following formula*

$$\big(\mathcal{F}_z(F)\big)(y) = \int_H \exp\Big\{-\frac{1}{2z}|\,h\,|^2 + i(h, y)^\sim\Big\}\, df(h) \tag{3.1}$$

holds for s-a.e. $y \in B$, where f is in $\mathcal{M}(H)$.

Moreover, the L_p analytic Fourier-Feynman transform $\mathcal{F}_{(q;p)}(F)$ exists for all $q \in \mathbb{R} - \{0\}$, it belongs to $\mathcal{F}(B)$, and the following formula

$$(3.2) \qquad \big(\mathcal{F}_{(q;p)}(F)\big)(y) = \int_H \exp\Big\{-\frac{i}{2q}|\,h\,|^2 + i(h,y)^\sim\Big\}\, df(h)$$

holds for s-a.e. $y \in B$, where f is in $\mathcal{M}(H)$.

Proof. First of all, we shall calculate the transform $\mathcal{F}_t(F)$ for $t > 0$. By using Fubini's Theorem and the following integral formula :

$$(3.3) \qquad \int_B \exp\{i\,t\,(h,x)^\sim\}\, d\omega(x) = \exp\Big\{-\frac{t^2}{2}|\,h\,|^2\Big\}, \quad h \in H,\ t \in \mathbb{R},$$

we have, for each $t > 0$,

$$(3.4) \qquad \begin{aligned} \big(\mathcal{F}_t(F)\big)(y) &= \textstyle\int_B \int_H \exp\big\{i\,\big(h, \frac{x}{\sqrt{t}} + y\big)^\sim\big\}\, df(h)\, d\omega(x) \\ &= \textstyle\int_H \exp\big\{-\frac{1}{2t}|\,h\,|^2 + i(h,y)^\sim\big\}\, df(h) \end{aligned}$$

for s-a.e. $y \in B$.

By using Morera's Theorem, we can verify that the last expression of (3.4) is an analytic function of t throughout $\mathbb{C}_+$, and is a bounded continuous function of t throughout $\widetilde{\mathbb{C}}_+$ for all $y \in B$, because f is in $\mathcal{M}(H)$. Therefore the transform $\mathcal{F}_z(F)$ exists for all $z \in \mathbb{C}_+$, and finally we can show that (3.1) and (3.2) hold.

Finally we shall show that $\mathcal{F}_z(F)$ belongs to $\mathcal{F}(B)$ for every $z \in \mathbb{C}_+$. Let z be in $\mathbb{C}_+$ and define a set function $\eta : \mathcal{B}(H) \longrightarrow \mathbb{C}$ as follows :

$$\eta(E) = \int_E \exp\big\{-\frac{1}{2z}|\,h\,|^2\big\}\, df(h), \quad E \in \mathcal{B}(H),$$

where $\mathcal{B}(H)$ is the Borel σ-algebra of H. Then it is obvious that η belongs to the Banach algebra $\mathcal{M}(H)$. Moreover, (3.1) is expressed as follows :

$$(\mathcal{F}_z(F))(y) = \int_H \exp\big\{i(h,y)^\sim\big\}\, d\eta(h).$$

Hence $\mathcal{F}_z(F)$ belongs to $\mathcal{F}(B)$.

Similarly, we can show that $\mathcal{F}_{(q;p)}(F)$ belongs to $\mathcal{F}(B)$. □

Remark 3.2. Note that taking $y = 0$ in (3.2), we can obtain the analytic Feynman integral $\int_B^{anf_q} F(x)\, d\omega(x)$ for every element $F \in \mathcal{F}(B)$ given by (2.3) as follows :

$$\int_B^{anf_q} F(x)\, d\omega(x) = \mathcal{F}_{(q;p)}(F)(0) = \int_H \exp\Big\{-\frac{i}{2q}|\,h\,|^2\Big\}\, df(h).$$

This coincides with the result given in [11 ; Proposition 2.2].

DEFINITION 3.3. Let F be a functional defined on the abstract Wiener space (B,H,ω) and define a transform $\mathcal{F}_t^{(n)}(F)(t>0)$ of F as follows :

$$\mathcal{F}_t^{(n)}(F) = \big(\underbrace{\mathcal{F}_t \circ \cdots \circ \mathcal{F}_t}_{n}\big)(F),$$

that is, $\mathcal{F}_t^{(n)}$ means the n-times composition of $\mathcal{F}_t$, where $\mathcal{F}_t$ is equal to $\mathcal{F}_z$ for $z>0$ in (2.6) of Definition 2.4, and n is a natural number.

Let $\mathcal{F}_z^{(n)}(F)$ be an analytic extension of $\mathcal{F}_t^{(n)}(F)$ as a function of $z \in \mathbb{C}_+$. In case that $1<p<\infty$, for each $q \in \mathbb{R}-\{0\}$, we define the *multiple L_p analytic Fourier-Feynman transform* $\mathcal{F}_{(q;p)}^{(n)}(F)$ of F by

$$\mathcal{F}_{(q;p)}^{(n)}(F) = \underset{z\to -iq}{\mathrm{l.\,i.\,m.}}(w_s^{p'})(\mathcal{F}_z^{(n)}(F)), \tag{3.5}$$

where z approaches $-iq$ through $\mathbb{C}_+$.

In case that $p=1$, for each $q \in \mathbb{R}-\{0\}$, we define the *multiple L_1 analytic Fourier-Feynman transform* $\mathcal{F}_{(q;1)}^{(n)}(F)$ of F by

$$\mathcal{F}_{(q;1)}^{(n)}(F) = \lim_{z\to -iq}(\mathcal{F}_z^{(n)}(F)), \tag{3.6}$$

where z approaches $-iq$ through $\mathbb{C}_+$.

Note that $\mathcal{F}_z^{(0)}(F) \equiv F \equiv \mathcal{F}_{(q;p)}^{(0)}(F)$, $\mathcal{F}_z^{(1)}(F) \equiv \mathcal{F}_z(F)$, and $\mathcal{F}_{(q;p)}^{(1)}(F) \equiv \mathcal{F}_{(q;p)}(F)$.

By using the mathematical induction and proceeding as in the proof of Theorem 3.1, we can obtain the following theorem.

THEOREM 3.4. *Let $F \in \mathcal{F}(B)$ be given by* (2.3) *and let $1 \le p < \infty$. Then the transform $\mathcal{F}_z^{(n)}(F)$ exists for all $z \in \mathbb{C}_+$, it belongs to $\mathcal{F}(B)$, and the following formula*

$$\big(\mathcal{F}_z^{(n)}(F)\big)(y) = \int_H \exp\Big\{-\frac{n}{2z}|\,h\,|^2 + i(h,y)^{\sim}\Big\}\,df(h) \tag{3.7}$$

holds for s-a.e. $y \in B$, where f is in $\mathcal{M}(H)$ and $n=0,1,2,\cdots$.

Moreover, for each $q \in \mathbb{R}-\{0\}$, the multiple L_p analytic Fourier-Feynman transform $\mathcal{F}_{(q;p)}^{(n)}(F)$ exists, it belongs to $\mathcal{F}(B)$, and the following formula

$$\big(\mathcal{F}_{(q;p)}^{(n)}(F)\big)(y) = \int_H \exp\Big\{-\frac{i\,n}{2q}|\,h\,|^2 + i(h,y)^{\sim}\Big\}\,df(h) \tag{3.8}$$

holds for s-a.e. $y \in B$*, where* f *is in* $\mathcal{M}(H)$ *and* $n = 0, 1, 2, \cdots$.

Note that (3.7) and (3.8) are reduced to (3.1) and (3.2), respectively, if we take $n = 1$ in (3.7) and (3.8).

Theorem 3.5. *Let* F *and* G *be in* $\mathcal{F}(B)$ *which are given by* (2.3). *Then the convolution product* $\big((\mathcal{F}_z^{(n)}F) * (\mathcal{F}_z^{(m)}G)\big)_z$ *exists for each* $z \in \mathbb{C}_+$*, it belongs to* $\mathcal{F}(B)$*, and the following formula*

(3.9)

$$\begin{aligned}&\big((\mathcal{F}_z^{(n)}F) * (\mathcal{F}_z^{(m)}G)\big)_z(y)\\ &= \int_{H^2} \exp\Big\{-\tfrac{n}{2z}|\,u\,|^2 - \tfrac{m}{2z}|\,v\,|^2 - \tfrac{1}{4z}|\,u-v\,|^2 + \tfrac{i}{\sqrt{2}}(u+v,y)^\sim\Big\}\, df(u)\, dg(v)\end{aligned}$$

holds for s-a.e. $y \in B$*, where* f *and* g *are in* $\mathcal{M}(H)$ *and* $m, n = 0, 1, 2, \cdots$.

Furthermore, the convolution product $\big((\mathcal{F}_{(q;p)}^{(n)}F) * (\mathcal{F}_{(q;p)}^{(m)}G)\big)_q$ *exists for every* $q \in \mathbb{R} - \{0\}$*, it belongs to* $\mathcal{F}(B)$*, and it is given by*

(3.10)

$$\begin{aligned}&\big((\mathcal{F}_{(q;p)}^{(n)}F) * (\mathcal{F}_{(q;p)}^{(m)}G)\big)_q(y)\\ &= \int_{H^2} \exp\Big\{-\tfrac{i\,n}{2q}|\,u\,|^2 - \tfrac{i\,m}{2q}|\,v\,|^2 - \tfrac{i}{4q}|\,u-v\,|^2 + \tfrac{i}{\sqrt{2}}(u+v,y)^\sim\Big\}\, df(u)\, dg(v)\,,\end{aligned}$$

for s-a.e. $y \in B$*, where* f *and* g *are in* $\mathcal{M}(H)$ *and* $m, n = 0, 1, 2, \cdots$.

Proof. By using Fubini's Theorem, Definition 2.5, (3.3), and (3.7), we first calculate $\big((\mathcal{F}_t^{(n)}F) * (\mathcal{F}_t^{(m)}G)\big)_t$ for every $t > 0$ as follows :

$$\begin{aligned}&\big((\mathcal{F}_t^{(n)}F) * (\mathcal{F}_t^{(m)}G)\big)_t(y)\\ &= \int_B \big(\mathcal{F}_t^{(n)}F\big)\big(\tfrac{1}{\sqrt{2}}\big(y + \tfrac{x}{\sqrt{t}}\big)\big)\big(\mathcal{F}_t^{(m)}G\big)\big(\tfrac{1}{\sqrt{2}}\big(y - \tfrac{x}{\sqrt{t}}\big)\big)\, d\omega(x)\\ &= \int_{H^2} \exp\Big\{-\tfrac{n}{2t}|\,u\,|^2 - \tfrac{m}{2t}|\,v\,|^2 + \tfrac{i}{\sqrt{2}}(u+v,y)^\sim\Big\}\\ &\quad \cdot\Big[\int_B \exp\Big\{\tfrac{i}{\sqrt{2t}}(u-v,x)^\sim\Big\}\, d\omega(x)\Big]\, df(u)\, dg(v)\\ &= \int_{H^2} \exp\Big\{-\tfrac{n}{2t}|\,u\,|^2 - \tfrac{m}{2t}|\,v\,|^2 + \tfrac{i}{\sqrt{2}}(u+v,y)^\sim - \tfrac{1}{4t}|\,u-v\,|^2\Big\}\, df(u)\, dg(v).\end{aligned}$$

By using Morera's Theorem, we can verify that the last expression is an analytic function of t throughout $\mathbb{C}_+$, and is a bounded continuous function of t over $\widetilde{\mathbb{C}}_+$ for all y in B, because f and g are in $\mathcal{M}(H)$. Therefore, we can show that (3.9) and (3.10) hold.

Next we shall show that $\big((\mathcal{F}_z^{(n)}F) * (\mathcal{F}_z^{(m)}G)\big)_z$ belongs to $\mathcal{F}(B)$ for every $z \in \mathbb{C}_+$. Let z be in $\mathbb{C}_+$ and define a set function $\nu : \mathcal{B}(H^2) \to \mathbb{C}$ by

$$\nu(E) = \int_E \exp\Big\{-\frac{n}{2z}|\,u\,|^2 - \frac{m}{2z}|\,v\,|^2 - \frac{1}{4z}|\,u-v\,|^2\Big\}\, df(u)\, dg(v), \quad E \in \mathcal{B}(H^2).$$

Then it is obvious that ν is a complex-valued countably additive Borel measure on $\mathcal{B}(H^2)$.

Now define a function $\varphi : H^2 \to H$ as follows :

$$\varphi(u,v) = \frac{1}{\sqrt{2}}(u+v), \quad (u,v) \in H^2.$$

Then φ is continuous , and so it is a Borel measurable function. Hence $\mu = \nu \cdot \varphi^{-1}$ belongs to the Banach algebra $\mathcal{M}(H)$. By using the Change of Variable Formula, we have

$$\big((\mathcal{F}_z^{(n)}F) * (\mathcal{F}_z^{(m)}G)\big)_z(y) = \int_H \exp\{i(w,y)^\sim\}\, d\mu(w).$$

Hence $\big((\mathcal{F}_z^{(n)}F) * (\mathcal{F}_z^{(m)}G)\big)_z$ belongs to $\mathcal{F}(B)$.

Similarly, we can show that $\big((\mathcal{F}_{(q;p)}^{(n)}F) * (\mathcal{F}_{(q;p)}^{(m)}G)\big)_q$ belongs to $\mathcal{F}(B)$. □

Note that taking $m = n = 0$ in (3.10), we obtain the convolution product $\big(F * G\big)_q(y)$ for two functions F and G in the Fresnel class $\mathcal{F}(B)$ as follows :

$$\begin{aligned}&\big(F * G\big)_q(y)\\ &= \int_{H^2} \exp\{-\tfrac{i}{4q}|\, u - v\,|^2 + \tfrac{i}{\sqrt{2}}(u+v,y)^\sim\}\, df(u)\, dg(v)\,.\end{aligned}$$

This coincides with the formula (3.7) given in [1 ; Theorem 3.2].

Note that taking $m = n = 1$ in (3.10), we obtain the convolution product $\big((\mathcal{F}_{(q;p)}F) * (\mathcal{F}_{(q;p)}G)\big)_q(y)$ for two Fourier-Feynman transforms $\mathcal{F}_{(q;p)}F$ and $\mathcal{F}_{(q;p)}G$ as follows :

$$\begin{aligned}&\big((\mathcal{F}_{(q;p)}F) * (\mathcal{F}_{(q;p)}G)\big)_q(y)\\ &= \int_{H^2} \exp\{-\tfrac{i}{4q}(2\,|\, u\,|^2 + 2\,|\, v\,|^2 + |\, u - v\,|^2) + \tfrac{i}{\sqrt{2}}(u+v,y)^\sim\}\, df(u)\, dg(v)\,.\end{aligned}$$

Our next theorem shows that the L_p analytic Fourier-Feynman transform of the convolution product for two functionals in the Fresnel class $\mathcal{F}(B)$ is a product of Fourier-Feynman transforms for each functional.

THEOREM 3.6. *Let F and G be as in Theorem 3.5 and let $1 \le p < \infty$. Then the transform $\mathcal{F}_z\big((\mathcal{F}_z^{(n)}F) * (\mathcal{F}_z^{(m)}G)\big)_z$ exists for all $z \in \mathbb{C}_+$, and the following formula*

(3.11)
$$\begin{aligned}&\big(\mathcal{F}_z((\mathcal{F}_z^{(n)}F) * (\mathcal{F}_z^{(m)}G))_z\big)(y) = \big(\mathcal{F}_z^{(n+1)}F\big)\big(\tfrac{y}{\sqrt{2}}\big)\, \big(\mathcal{F}_z^{(m+1)}G\big)\big(\tfrac{y}{\sqrt{2}}\big)\\ &= \int_{H^2} \exp\{-\tfrac{(n+1)}{2z}|\, u\,|^2 - \tfrac{(m+1)}{2z}|\, v\,|^2 + \tfrac{i}{\sqrt{2}}(u+v,y)^\sim\}\, df(u)\, dg(v)\end{aligned}$$

holds for s-a.e. $y \in B$, *where* $m, n = 0, 1, 2, \cdots$.

Furthermore, for each $q \in \mathbb{R} - \{0\}$, *the* L_p *analytic Fourier-Feynman transform* $\mathcal{F}_{(q;p)}\big((\mathcal{F}^{(n)}_{(q;p)}F) * (\mathcal{F}^{(m)}_{(q;p)}G)\big)_q$ *is given by*

(3.12)

$$\begin{aligned}&\big(\mathcal{F}_{(q;p)}((\mathcal{F}^{(n)}_{(q;p)}F) * (\mathcal{F}^{(m)}_{(q;p)}G))_q\big)(y) = \big(\mathcal{F}^{(n+1)}_{(q;p)}F\big)\big(\tfrac{y}{\sqrt{2}}\big)\big(\mathcal{F}^{(m+1)}_{(q;p)}G\big)\big(\tfrac{y}{\sqrt{2}}\big)\\ &= \int_{H^2} \exp\big\{-\tfrac{i\,(n+1)}{2q}|\,u\,|^2 - \tfrac{i\,(m+1)}{2q}|\,v\,|^2 + \tfrac{i}{\sqrt{2}}(u+v,y)^\sim\big\}\, df(u)\, dg(v)\end{aligned}$$

where $m, n = 0, 1, 2, \ldots$.

Proof. By using Fubini's Theorem, (3.3) and (3.9), we first calculate the transform $\mathcal{F}_t\big((\mathcal{F}^{(n)}_t F) * (\mathcal{F}^{(m)}_t G)\big)_t$ for all $t > 0$ as follows:

(3.13)

$$\begin{aligned}&\big(\mathcal{F}_t((\mathcal{F}^{(n)}_t F) * (\mathcal{F}^{(m)}_t G))_t\big)(y)\\ &= \int_B \big((\mathcal{F}^{(n)}_t F) * (\mathcal{F}^{(m)}_t G)\big)_t\big(\tfrac{x}{\sqrt{t}} + y\big)\, d\omega(x)\\ &= \int_{H^2} \exp\big\{-\tfrac{n}{2t}|\,u\,|^2 - \tfrac{m}{2t}|\,v\,|^2 - \tfrac{1}{4t}|\,u-v\,|^2 + \tfrac{i}{\sqrt{2}}(u+v,y)^\sim\big\}\\ &\quad \cdot\big[\int_B \exp\big\{\tfrac{i}{\sqrt{2t}}(u+v,x)^\sim\big\}\, d\omega(x)\big]\, df(u)\, dg(v)\\ &= \int_{H^2} \exp\big\{-\tfrac{(n+1)}{2t}|\,u\,|^2 - \tfrac{(m+1)}{2t}|\,v\,|^2 + \tfrac{i}{\sqrt{2}}(u+v,y)^\sim\big\} df(u) dg(v).\end{aligned}$$

By using Morera's Theorem, we can verify that the last expression in (3.13) is an analytic function of t throughout $\mathbb{C}_+$, and is a bounded continuous function of t over $\widetilde{\mathbb{C}}_+$ for all $y \in B$, because f and g are in $\mathcal{M}(H)$. Therefore, for each $z \in \mathbb{C}_+$, the following formula

(3.14)

$$\begin{aligned}&\big(\mathcal{F}_z((\mathcal{F}^{(n)}_z F) * (\mathcal{F}^{(m)}_z G))_z\big)(y)\\ &= \int_{H^2} \exp\big\{-\tfrac{(n+1)}{2z}|\,u\,|^2 - \tfrac{(m+1)}{2z}|\,v\,|^2 + \tfrac{i}{\sqrt{2}}(u+v,y)^\sim\big\}\, df(u)\, dg(v)\end{aligned}$$

holds for s-a.e. $y \in B$.

On the other hand, using (3.7), we can show that for every $z \in \mathbb{C}_+$, the following formula

(3.15)

$$\begin{aligned}&\big(\mathcal{F}^{(n+1)}_z F\big)\big(\tfrac{y}{\sqrt{2}}\big)\big(\mathcal{F}^{(m+1)}_z G\big)\big(\tfrac{y}{\sqrt{2}}\big)\\ &= \int_{H^2} \exp\big\{-\tfrac{(n+1)}{2z}|\,u\,|^2 - \tfrac{(m+1)}{2z}|\,v\,|^2 + \tfrac{i}{\sqrt{2}}(u+v,y)^\sim\big\} df(u) dg(v)\end{aligned}$$

holds for s-a.e. $y \in B$.

Therefore, (3.11) follows from (3.14) and (3.15), and finally (3.12) comes from (3.11) with the help of Definition 2.3. □

Note that taking $m = n = 0$ in (3.12), we obtain

$$\begin{aligned}&\big(\mathcal{F}_{(q;p)}(F * G)_q\big)(y) = \big(\mathcal{F}_{(q;p)}F\big)\big(\tfrac{y}{\sqrt{2}}\big)\big(\mathcal{F}_{(q;p)}G\big)\big(\tfrac{y}{\sqrt{2}}\big)\\ &= \int_{H^2} \exp\big\{-\tfrac{i}{2q}(|\,u\,|^2 + |\,v\,|^2) + \tfrac{i}{\sqrt{2}}(u+v, y)^\sim\big\}df(u)dg(v)\,,\end{aligned}$$

which is similar to the result given in [1 ; Theorem 3.3].

REMARK 3.7. Let F and G be as in Theorem 3.5 and let $1 \le p < \infty$. By applying the formula (3.8) in Theorem 3.4 for the formula (3.10) in Theorem 3.5, we can show that a multiple L_p analytic Fourier-Feynman transform $\mathcal{F}^{(k)}_{(q;p)}\big((\mathcal{F}^{(n)}_{(q;p)}F) * (\mathcal{F}^{(m)}_{(q;p)}G)\big)_q$ exists for each $q \in \mathbb{R} - \{0\}$, and it is given by

$$(3.16)\quad \big(\mathcal{F}^{(k)}_{(q;p)}((\mathcal{F}^{(n)}_{(q;p)}F) * (\mathcal{F}^{(m)}_{(q;p)}G))_q\big)(y) = \int_{H^2} \exp\left\{-\frac{i\,(n+1)}{2q}|\,u\,|^2 -\frac{i\,(m+1)}{2q}|\,v\,|^2 - \frac{i\,(k-1)}{4q}\,|\,u\,+\,v\,|^2 + \frac{i}{\sqrt{2}}(u+v,y)^\sim\right\}\cdot\, df(u)\,dg(v)$$

where $k,\, m,\, n = 0, 1, 2, \ldots$. But by using the formula (3.8) in Theorem 3.4, we can show that the product $\big(\mathcal{F}^{(k+n)}_{(q;p)}F\big)\Big(\frac{y}{\sqrt{2}}\Big)\big(\mathcal{F}^{(k+m)}_{(q;p)}G\big)\Big(\frac{y}{\sqrt{2}}\Big)$ is given by

(3.17)

$$\begin{aligned}&\big(\mathcal{F}^{(k+n)}_{(q;p)}F\big)\big(\tfrac{y}{\sqrt{2}}\big)\big(\mathcal{F}^{(k+m)}_{(q;p)}G\big)\big(\tfrac{y}{\sqrt{2}}\big)\\ &= \int_{H^2} \exp\big\{-\tfrac{i\,(k+n)}{2q}|\,u\,|^2 - \tfrac{i\,(k+m)}{2q}|\,v\,|^2 + \tfrac{i}{\sqrt{2}}(u+v, y)^\sim\big\}\,df(u)dg(v)\end{aligned}$$

for s-a.e. $y \in B$, where $k,\, m,\, n, = 0,\, 1,\, 2,\, \cdots$. If we take $k = 1$ in both (3.16) and (3.17), we obtain the formula (3.12) given in Theorem 3.6. Note that

$$\big(\mathcal{F}^{(k)}_{(q;p)}((\mathcal{F}^{(n)}_{(q;p)}F) * (\mathcal{F}^{(m)}_{(q;p)}G))_q\big)(y) \neq \big(\mathcal{F}^{(k+n)}_{(q;p)}F\big)\big(\frac{y}{\sqrt{2}}\big)\big(\mathcal{F}^{(k+m)}_{(q;p)}G\big)\big(\frac{y}{\sqrt{2}}\big)$$

for s-a.e. $y \in B$, whenever k is a nonnegative integer with $k \neq 1$.

Our next theorem shows that an interesting Parseval's identity holds on the Fresnel class $\mathcal{F}(B)$.

THEOREM 3.8. *Let F and G be as in Theorem 3.5 and let $1 \le p < \infty$. Then for each $q \in \mathbb{R} - \{0\}$, the following Parseval's identity holds :*

$$(3.18)\quad \begin{aligned}&\mathcal{F}_{(-q;p)}\big(\mathcal{F}_{(q;p)}((\mathcal{F}^{(n)}_{(q;p)}F) * (\mathcal{F}^{(m)}_{(q;p)}G))_q\big)(0)\\ &= \mathcal{F}_{(q;p)}\big(\big(\mathcal{F}^{(n)}_{(q;p)}F\big)\big(\tfrac{\cdot}{\sqrt{2}}\big)\big(\mathcal{F}^{(m)}_{(q;p)}G\big)\big(-\tfrac{\cdot}{\sqrt{2}}\big)\big)(0)\\ &= \int_{H^2} \exp\big\{-\tfrac{i\,n}{2q}|\,u\,|^2 - \tfrac{i\,m}{2q}|\,v\,|^2 - \tfrac{i}{4q}|\,u - v\,|^2\big\}\,df(u)\,dg(v)\end{aligned}$$

where m, $n = 0, 1, 2, \cdots$.

Proof. We first calculate the transform $\mathcal{F}_t\big(\mathcal{F}_{(q;p)}((\mathcal{F}^{(n)}_{(q;p)}F)*(\mathcal{F}^{(m)}_{(q;p)}G))_q\big)$ for all $t > 0$, where q is a non-zero real number. Using Fubini's Theorem, (3.3), (3.8), and (3.12), we have, for all $t > 0$,

$$\begin{aligned}
&\mathcal{F}_t\big(\mathcal{F}_{(q;p)}((\mathcal{F}^{(n)}_{(q;p)}F) * (\mathcal{F}^{(m)}_{(q;p)}G))_q\big)(0)\\
&= \mathcal{F}_t\big(\big(\mathcal{F}^{(n+1)}_{(q;p)}F\big)\big(\tfrac{\cdot}{\sqrt{2}}\big)\big(\mathcal{F}^{(m+1)}_{(q;p)}G\big)\big(\tfrac{\cdot}{\sqrt{2}}\big)\big)(0)\\
&= \int_B\big(\mathcal{F}^{(n+1)}_{(q;p)}F\big)\big(\tfrac{x}{\sqrt{2t}}\big)\big(\mathcal{F}^{(m+1)}_{(q;p)}G\big)\big(\tfrac{x}{\sqrt{2t}}\big)\,d\omega(x)\\
&= \int_{H^2}\exp\big\{-\tfrac{i(n+1)}{2q}|\,u\,|^2 - \tfrac{i(m+1)}{2q}|\,v\,|^2)\big\}\big[\int_B\exp\big\{\tfrac{i}{\sqrt{2t}}(u+v,x)^\sim\big\}\,d\omega(x)\big]\\
&\quad\cdot df(u)\,dg(v)\\
&= \int_{H^2}\exp\big\{-\tfrac{i(n+1)}{2q}|\,u\,|^2 - \tfrac{i(m+1)}{2q}|\,v\,|^2 - \tfrac{1}{4t}|\,u+v\,|^2\big\}\,df(u)\,dg(v).
\end{aligned}$$

Since the last expression has an analytic extension for t over $\mathbb{C}_+$, and is a bounded continuous function of t over $\widetilde{\mathbb{C}}_+$, we can show that the following formula

$$\begin{aligned}
&\mathcal{F}_{(-q;p)}\big(\mathcal{F}_{(q;p)}((\mathcal{F}^{(n)}_{(q;p)}F) * (\mathcal{F}^{(m)}_{(q;p)}G))_q\big)(0)\\
&= \underset{t\to iq}{\mathrm{l.i.m}}(w_s^{p'})\mathcal{F}_t\big(\mathcal{F}_{(q;p)}((\mathcal{F}^{(n)}_{(q;p)}F) * (\mathcal{F}^{(m)}_{(q;p)}G))_q\big)(0)\\
&= \int_{H^2}\exp\big\{-\tfrac{i\,n}{2q}|\,u\,|^2 - \tfrac{i\,m}{2q}|\,v\,|^2 - \tfrac{i}{4q}|\,u-v\,|^2\big\}\,df(u)\,dg(v)
\end{aligned}\tag{3.19}$$

holds.

Next we calculate the transform $\mathcal{F}_t\big((\mathcal{F}^{(n)}_{(q;p)}F)(\frac{\cdot}{\sqrt{2}})(\mathcal{F}^{(m)}_{(q;p)}G)(-\frac{\cdot}{\sqrt{2}})\big)(0)$ for all $t > 0$. Using (3.3) and Fubini's Theorem, we obtain the following formula

$$\begin{aligned}
&\mathcal{F}_t\big(\big(\mathcal{F}^{(n)}_{(q;p)}F\big)\big(\tfrac{\cdot}{\sqrt{2}}\big)\,\big(\mathcal{F}^{(m)}_{(q;p)}G\big)\big(-\tfrac{\cdot}{\sqrt{2}}\big)\big)(0)\\
&= \int_B\big(\mathcal{F}^{(n)}_{(q;p)}F\big)\big(\tfrac{x}{\sqrt{2t}}\big)\,\big(\mathcal{F}^{(m)}_{(q;p)}G\big)\big(-\tfrac{x}{\sqrt{2t}}\big)\,d\omega(x)\\
&= \int_{H^2}\exp\big\{-\tfrac{i\,n}{2q}|\,u\,|^2 - \tfrac{i\,m}{2q}|\,v\,|^2\big\}\big[\int_B\exp\big\{\tfrac{i}{\sqrt{2t}}(u-v,x)^\sim\big\}d\omega(x)\big]df(u)\,dg(v)\\
&= \int_{H^2}\exp\big\{-\tfrac{i\,n}{2q}|\,u\,|^2 - \tfrac{i\,m}{2q}|\,v\,|^2 - \tfrac{1}{4t}|\,u-v\,|^2\big\}\,df(u)\,dg(v).
\end{aligned}$$

Since the last expression has an analytic extension for t over $\mathbb{C}_+$, and is a bounded continuous function of t throughout $\widetilde{\mathbb{C}}_+$, we can show that the following formula

$$\begin{aligned}
&\mathcal{F}_{(q;p)}\big(\big(\mathcal{F}^{(n)}_{(q;p)}F\big)\big(\tfrac{\cdot}{\sqrt{2}}\big)\,\big(\mathcal{F}^{(m)}_{(q;p)}G\big)\big(-\tfrac{\cdot}{\sqrt{2}}\big)\big)(0)\\
&= \underset{t\to -iq}{\mathrm{l.i.m}}(w_s^{p'})\mathcal{F}_t\big(\big(\mathcal{F}^{(n)}_{(q;p)}F\big)\big(\tfrac{\cdot}{\sqrt{2}}\big)\,\big(\mathcal{F}^{(m)}_{(q;p)}G\big)\big(-\tfrac{\cdot}{\sqrt{2}}\big)\big)(0)\\
&= \int_{H^2}\exp\big\{-\tfrac{i\,n}{2q}|\,u\,|^2 - \tfrac{i\,m}{2q}|\,v\,|^2 - \tfrac{i}{4q}|\,u-v\,|^2\big\}\,df(u)\,dg(v)
\end{aligned}\tag{3.20}$$

holds.

Therefore, (3.18) comes from (3.19) and (3.20). □

Note that taking $m = n = 0$ in (3.18), we obtain

$$\begin{aligned}\mathcal{F}_{(-q;p)}\big\{\big(\mathcal{F}_{(q;p)}(F*G)_q\big)\big\}(0) &= \mathcal{F}_{(q;p)}\big(F\big(\tfrac{\cdot}{\sqrt{2}}\big)\,G\big(-\tfrac{\cdot}{\sqrt{2}}\big)\big)(0)\\ &= \int_{H^2}\exp\big\{-\tfrac{i}{4q}|\,u-v\,|^2\big\}\,df(u)\,dg(v)\,,\end{aligned}$$

which is similar to the result given in [1; Theorem 3.4].

THEOREM 3.9. *Let F and G be as in Theorem 3.5 and let $1 \le p < \infty$. Then for each non-zero real number q, the following formula*

(3.21)

$$\begin{aligned}&\Big(\big(\mathcal{F}^{(n)}_{(q;p)}F\big)*\big(\mathcal{F}^{(m)}_{(q;p)}G\big)\Big)_{-q}(y) = \mathcal{F}_{(q;p)}\Big(\big(\mathcal{F}^{(n-1)}_{(q;p)}F\big)\big(\tfrac{\cdot}{\sqrt{2}}\big)\big(\mathcal{F}^{(m-1)}_{(q;p)}G\big)\big(\tfrac{\cdot}{\sqrt{2}}\big)\Big)(y)\\ &= \int_{H^2}\exp\big\{\tfrac{i}{\sqrt{2}}(u+v,y)^\sim - \tfrac{i\,n}{2q}|\,u\,|^2 - \tfrac{i\,m}{2q}|\,v\,|^2 + \tfrac{i}{4q}|\,u-v\,|^2\big\}\,df(u)dg(v).\end{aligned}$$

holds for s-a.e. $y \in B$, where $m, n = 1, 2, 3, \cdots$.

Proof. Let q be any non-zero real number. Using (3.3), (3.8) and Fubini's Theorem, for each $t > 0$ we first calculate the expression $\big(\big(\mathcal{F}^{(n)}_{(q;p)}F\big)*\big(\mathcal{F}^{(m)}_{(q;p)}G\big)\big)_t(y)$ for each $y \in B$ as follows

(3.22)

$$\begin{aligned}&\big(\big(\mathcal{F}^{(n)}_{(q;p)}F\big)*\big(\mathcal{F}^{(m)}_{(q;p)}G\big)\big)_t(y)\\ &= \int_B\big(\mathcal{F}^{(n)}_{(q;p)}F\big)\big(\tfrac{1}{\sqrt{2}}\big(y+\tfrac{x}{\sqrt{t}}\big)\big)\,\big(\mathcal{F}^{(m)}_{(q;p)}G\big)\big(\tfrac{1}{\sqrt{2}}\big(y-\tfrac{x}{\sqrt{t}}\big)\big)\,d\omega(x)\\ &= \int_{H^2}\exp\big\{\tfrac{i}{\sqrt{2}}(u+v,y)^\sim - \tfrac{i\,n}{2q}|\,u\,|^2 - \tfrac{i\,m}{2q}|\,v\,|^2\big\}\\ &\quad\cdot\big[\int_B\exp\big\{\tfrac{i}{\sqrt{2t}}(u-v,x)^\sim\big\}\,d\omega(x)\big]\,df(u)\,dg(v)\\ &= \int_{H^2}\exp\big\{\tfrac{i}{\sqrt{2}}(u+v,y)^\sim - \tfrac{i\,n}{2q}|\,u\,|^2 - \tfrac{i\,m}{2q}|\,v\,|^2 - \tfrac{1}{4t}|\,u-v\,|^2\big\}\,df(u)\,dg(v).\end{aligned}$$

By using Morera's Theorem, we can verify that the last expression in (3.22) is an analytic function of t throughout $\mathbb{C}_+$, and is a bounded continuous function of t throughout $\widetilde{\mathbb{C}}_+$. Therefore, for each non-zero real number q, we have the following formula

(3.23)

$$\begin{aligned}&\big(\big(\mathcal{F}^{(n)}_{(q;p)}F\big)*\big(\mathcal{F}^{(m)}_{(q;p)}G\big)\big)_{-q}(y)\\ &= \lim_{t\to iq}\big(\big(\mathcal{F}^{(n)}_{(q;p)}F\big)*\big(\mathcal{F}^{(m)}_{(q;p)}G\big)\big)_t(y)\\ &= \int_{H^2}\exp\big\{\tfrac{i}{\sqrt{2}}(u+v,y)^\sim - \tfrac{i\,n}{2q}|\,u\,|^2 - \tfrac{i\,m}{2q}|\,v\,|^2 + \tfrac{i}{4q}|\,u-v\,|^2\big\}\,df(u)dg(v).\end{aligned}$$

for s-a.e $y \in B$.

Next, for each $t > 0$, we obtain the following formula

(3.24)

$$\begin{aligned}
&\mathcal{F}_t\big(\big(\mathcal{F}^{(n-1)}_{(q;p)}F\big)\big(\tfrac{\cdot}{\sqrt{2}}\big)\big(\mathcal{F}^{(m-1)}_{(q;p)}G\big)\big(\tfrac{\cdot}{\sqrt{2}}\big)\big)(y)\\
&= \int_B\big(\mathcal{F}^{(n-1)}_{(q;p)}F\big)\big(\tfrac{1}{\sqrt{2}}\big(\tfrac{x}{\sqrt{t}}+y\big)\big)\big(\mathcal{F}^{(m-1)}_{(q;p)}G\big)\big(\tfrac{1}{\sqrt{2}}\big(\tfrac{x}{\sqrt{t}}+y\big)\big)\,d\omega(x)\\
&= \int_{H^2}\exp\big\{\tfrac{i}{\sqrt{2}}(u+v,y)^{\sim}-\tfrac{i(n-1)}{2q}|\,u\,|^2-\tfrac{i(m-1)}{2q}|\,v\,|^2\big\}\\
&\quad\cdot\big[\int_B\exp\big\{\tfrac{i}{\sqrt{2t}}(u+v,x)^{\sim}\big\}\,d\omega(x)\big]\,df(u)dg(v)\\
&= \int_{H^2}\exp\big\{\tfrac{i}{\sqrt{2}}(u+v,y)^{\sim}-\tfrac{i(n-1)}{2q}|\,u\,|^2-\tfrac{i(m-1)}{2q}|\,v\,|^2-\tfrac{1}{4t}|\,u+v\,|^2\big\}\\
&\quad\cdot df(u)\,dg(v)
\end{aligned}$$

for s-a.e. $y \in B$.

By using Morera's Theorem, we can verify that the last expression in (3.24) is an analytic function of t throughout $\mathbb{C}_+$, and is a bounded continuous function of t throughout $\widetilde{\mathbb{C}}_+$. Therefore, for each non-zero real number, we have the following formula

(3.25)

$$\begin{aligned}
&\mathcal{F}_{(q;p)}\big(\big(\mathcal{F}^{(n-1)}_{(q;p)}F\big)\big(\tfrac{\cdot}{\sqrt{2}}\big)\big(\mathcal{F}^{(m-1)}_{(q;p)}G\big)\big(\tfrac{\cdot}{\sqrt{2}}\big)\big)(y)\\
&= \underset{t\to -iq}{\mathrm{l.\,i.\,m}}(w^{p'}_s)\mathcal{F}_t\big(\big(\mathcal{F}^{(n-1)}_{(q;p)}F\big)\big(\tfrac{\cdot}{\sqrt{2}}\big)\big(\mathcal{F}^{(m-1)}_{(q;p)}G\big)\big(\tfrac{\cdot}{\sqrt{2}}\big)\big)(y)\\
&= \int_{H^2}\exp\big\{\tfrac{i}{\sqrt{2}}(u+v,y)^{\sim}-\tfrac{i\,n}{2q}|\,u\,|^2-\tfrac{i\,m}{2q}|\,v\,|^2+\tfrac{i}{4q}|\,u-v\,|^2\big\}\,df(u)dg(v)
\end{aligned}$$

for s-a.e. $y \in B$.

Therefore, (3.21) comes from (3.23) and (3.25). □

Note that taking $m = n = 1$ in (3.21), we obtain

$$\begin{aligned}
&\Big(\mathcal{F}_{(q;p)}(F)*\mathcal{F}_{(q;p)}(G)\Big)_{-q}(y) = \mathcal{F}_{(q;p)}\Big(F\big(\tfrac{\cdot}{\sqrt{2}}\big)\,G\big(\tfrac{\cdot}{\sqrt{2}}\big)\Big)(y)\\
&= \int_{H^2}\exp\big\{\tfrac{i}{\sqrt{2}}(u+v,y)^{\sim}-\tfrac{i}{4q}|\,u+v\,|^2\big\}\,df(u)dg(v)\,,
\end{aligned}$$

which is similar to the result given in [1; Theorem 3.6].

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