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ON APPROXIMATIONS BY IRRATIONAL SPLINES

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Abstract. A problem of approximation by irrational splines is considered. These splines have a constant curvature between interpolation nodes and need only one additional boundary condition for derivatives, which should be set only at one of two boundary nodes, that is impossible for usual polynomial splines required boundary conditions at both boundary nodal points. Some estimations for numerical differentiation and rounding error analysis are presented.

1. Introduction

Although in recent years new various approaches in approximation of data by convex and positivity preserving splines have been proposed (see for instance [1-3]) a problem of data interpolation by smooth functions with a constant curvature between interpolation nodes is important to date. This problem is especially topical in Computational Fluid Dynamics in transonic and supersonic cases and in some other applications. It is well-known that this interpolation problem can not be solved by usual polynomial splines because the curvature of cubic and other higher order polynomial splines between the interpolation is not constant. As to the quadratic polynomial splines it is known that these splines have a constant second derivative or curvature only between the splines nodes, but for these splines their nodes do not coincide with the interpolation nodes and usually are located at the middle of the interpolation nodes. This is necessary to provide a stability of the algorithm for evaluation of quadratic spline coefficients [4-5].

Another topical problem consists in setting of auxiliary boundary conditions for spline derivatives only at one of two boundary nodes. For polynomial splines this is impossible, because the algorithm for evaluation of spline coefficients is also unstable [4-5] in this case.

In this paper one class of irrational splines is considered. In this class on each segment, restricted by adjacent interpolation nodes, splines are described by circle arcs passing through the interpolation nodes. In all internal interpolation nodes the

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condition of smoothing for the first order spline derivatives is provided. It is shown that for considering splines, it is enough to set the boundary conditions for derivatives only at one of two boundary points.

2. Definitions

Let us consider any grid function y_i, $(i = 0, 1, 2, 3, ..., N)$ defined in nodes x_0, x_1, x_2, x_3, ..., x_N. Let $y_0^{'}$ be a known derivative of the data grid function y_i. Then we will show that these data is enough to construct an irrational spline function.

Let us consider the i-th segment $[x_i, x_{i+1}]$ and define a circle arc passing through points (x_i, y_i) and (x_{i+1}, y_{i+1}) with tangent $y_i^{'}$ at the first point (x_i, y_i). We present the equation of the circle arc in usual form

$$(x - x_{ci})^2 + (y - y_{ci})^2 = R_i^2 \ . \tag{1}$$

Here (x_{ci}, y_{ci}) is a center and R_i is a radius of the circle constructed for the i-th interpolation segment.

Implicit differentiation of equation (1) with respect to x yields

$$(x - x_{ci}) + (y - y_{ci})y^{'} = 0 \ . \tag{2}$$

Values x_{ci}, y_{ci} and R_i are unknown and our goal is to evaluate these values by the known values y_i, y_{i+1} and $y_i^{'}$ at points x_i and y_{i+1}.

Let us take two interpolation conditions satisfying to equation (1) at points x_i and x_{i+1} and the interpolation condition for expression (2) taken at point x_i. Then we obtain a system of three equations

$$\begin{aligned} &(x_i - x_{ci})^2 + (y_i - y_{ci})^2 = R_i^2 \ . \\ &(x_{i+1} - x_{ci})^2 + (y_{i+1} - y_{ci})^2 = R_i^2 \ . \\ &(x_i - x_{ci}) + (y_i - y_{ci})y_i^{'} = 0 \ . \end{aligned} \tag{3}$$

To construct a solution of (3), let us introduce new searching variables $\xi_i = x_{ci} - x_i$ and $\eta_i = y_{ci} - y_i$ and denote $h_i = x_{i+1} - x_i$, $H_i = y_{i+1} - y_i$. Then formulas (3) can be presented as follows

$$\begin{aligned} &\xi_i^2 + \eta_i^2 - R_i^2 = 0 \ , \\ &(h_i - \xi_i)^2 + (H_i - \eta_i)^2 - R_i^2 = 0 \ , \\ &\xi_i + \eta_i y_i^{'} = 0 \ . \end{aligned} \tag{4}$$

Solving this system we obtain

$$\begin{aligned} \eta_i &= \frac{h_i^2 + H_i^2}{2(H_i - h_i y_i^{'})} \ , \\ \xi_i &= -\frac{h_i^2 + H_i^2}{2(H_i - h_i y_i^{'})} y_0^{'} \ , \\ R_i^2 &= \frac{(h_i^2 + H_i^2)^2 [1 + (y_i^{'})^2]}{4(H_i - h_i y_i^{'})^2} \ , \end{aligned} \tag{5}$$

According to (5) we can find

$$x_{ci} = x_i + \xi_i \ , \qquad y_{ci} = y_i + \eta_i \ . \tag{6}$$

and can present the equation of the circle arc in one of the following forms

$$(x - x_i)(x - x_i - 2\xi_i) + (y - y_i)(y - y_i - 2\eta_i) = 0 \tag{7a}$$

or

$$\frac{y - y_i}{x - x_i} = -\frac{x - x_i - 2\xi_i}{y - y_i - 2\eta_i} \ , \tag{7b}$$

or

$$\frac{y - y_i}{x - x_i} = -\frac{(x - x_i)(H_i - h_i y_i^{'}) + (h_i^2 + H_i^2) y_i^{'}}{(y - y_i)(H_i - h_i y_i^{'}) - (h_i^2 + H_i^2)} \ . \tag{7c}$$

Therefore the solution of interpolation problem on the segment $[x_i, x_{i+1}]$ consists in solution of non-linear equation (7) for any data value $x_* \in [x_i, x_{i+1}]$. For this purpose it is possible to apply, for instance, well-known Newton method or one of its modifications.

3. Numerical Differentiation

Now we consider a problem of evaluation the first derivative of the data grid function at the second point x_{i+1} at the considering segment $[x_i, x_{i+1}]$. For this purpose we use the following geometry property of the tangential lines passing through the first and through the end points of the circle arc and the secant line passing also through these points

$$\varphi_{si} = \frac{1}{2}(\varphi_i + \varphi_{i+1}) \ ,$$

where $\varphi_i = arctg(y_i^{'})$, $\varphi_{i+1} = arctg(y_{i+1}^{'})$ are angles between tangential lines to the considering arc at points x_i and x_{i+1} and x-axes, $\varphi_{si} = arctg(\frac{H_i}{h_i})$ is an angle between the secant line passing through the points (x_i, y_i) and (x_{i+1}, y_{i+1}).

In this case, since

$$tg(\varphi_i + \varphi_{i+1}) = tg(2\varphi_{si}) \ ,$$

$$tg(2\varphi_{si}) = \frac{2tg(\varphi_{si})}{1 - tg^2(\varphi_{si})} = \frac{2H_i h_i}{h_i^2 - H_i^2} \ ,$$

$$tg(\varphi_i + \varphi_{i+1}) = \frac{tg\varphi_i + tg\varphi_{i+1}}{1 - tg\varphi_i tg\varphi_{i+1}} = \frac{y_i^{'} + y_{i+1}^{'}}{1 - y_i^{'} y_{i+1}^{'}},$$

we can express the first derivative at the point $i + 1$ by the following formula

$$y_{i+1}^{'} = \frac{2H_i h_i + (H_i^2 - h_i^2) y_i^{'}}{h_i^2 - H_i^2 + 2H_i h_i y_i^{'}} \ . \tag{8}$$

Thus, following to the formula (8) and step by step procedure starting from the first interpolation segment, we can construct the irrational spline for all considering interpolation segments $[x_i, x_{i+1}]$, $i = 0, 1, 2, 3, ..., (N - 1)$. Since we take $y_i^{'}$ evaluated by (8) at the $(i - 1)$-th step as initial data for calculation $y_{i+1}^{'}$ at the i-th step, we provide smoothing conditions for the first derivative of the considering spline at all internal interpolation nodes.

According to above mentioned we can see that considering irrational splines don't need solution of linear algebra equations systems for evaluation of their coefficients as usual polynomial splines need. All coefficients of these splines can be computed by the recurrent formulas (5,6,8)

Using a Taylor-series expansion we can estimate the accuracy of the formula (8) intending for the numerical differentiation of the data grid function. As a result this estimation can be presented as follows

$$y_i^{'} - \widetilde{y}_i^{'} = (\frac{\widetilde{y}_i^{'} (\widetilde{y}_i^{''})^2}{2[1 + (\widetilde{y}_i^{'})^2]} - \frac{\widetilde{y}_i^{'''}}{6}) h^2 + O(h^3) \ . \tag{9}$$

Here $\widetilde{y}_i^{'}$ is an exact value of the first derivative, $\widetilde{y}_i^{''}$ is an exact value of the second derivative and $\widetilde{y}_i^{'''}$ is an exact value of the third derivative of the considering function taken at the nodal point i. Thus the formula (8) has the second order approximation error.

4. Degeneration Case

Now we consider a degeneration case $y_i^{'} = \frac{H_i}{h_i}$. In this case denominators of fractions in right-hand sides of formulas (5) are equal to zero and hence the appropriate spline parameters η_i, ξ_i and R_i are undefined.

However, in this case according to the formula (8) we have

$$y_{i+1}^{'} = y_i^{'} = \frac{H_i}{h_i}. \tag{10}$$

This means that the considering circle arc element is singular with $R_i = \infty$ and $\eta_i = \xi_i = \infty$. Thus in considering case according to the formula (10) the circle arc spline element degenerates into the straight line element.

5. Estimation of Rounding Errors Spreading

Now we provide a rounding error analysis of the numerical differentiation formula (8). Let suppose that at any point i we know a perturbed value of the first derivative $y_i^{'} = \widetilde{y}_i^{'} + \varepsilon_i$. Here $\widetilde{y}_i^{'}$ is the exact value of the first derivative and ε_i is an rounding error. Then we consider the grid value of the first derivative taken at the next point $i+1$ and evaluated by the exact data

$$\widetilde{D}_{i+1} = \frac{2H_i h_i + (H_i^2 - h_i^2)\widetilde{y}_i^{'}}{h_i^2 - H_i^2 + 2H_i h_i \widetilde{y}_i^{'}} \ .$$

For comparison we also consider the appropriate value evaluated by the perturbed data

$$D_{i+1} = \frac{2H_i h_i + (H_i^2 - h_i^2)(\widetilde{y}_i^{'} + \varepsilon_i)}{h_i^2 - H_i^2 + 2H_i h_i (\widetilde{y}_i^{'} + \varepsilon_i)} \ .$$

Applying the Taylor-series expansion of the last expression with respect to ε_i, we obtain the following formula

$$D_{i+1} - \widetilde{D}_{i+1} = -\varepsilon_i \left(\frac{h_i^2 + H_i^2}{h_i^2 - H_i^2 + 2H_i h_i \widetilde{y}_i^{'}}\right)^2 + O(\varepsilon_i^2) \ . \tag{11}$$

According to the formula (11) perturbations in initial data or rounding errors are strictly damping, if the following inequality satisfies

$$\left|\frac{h_i^2 + H_i^2}{h_i^2 - H_i^2 + 2H_i h_i \widetilde{y}_i^{'}}\right| < 1 \ , \tag{12a}$$

or

$$-h_i^2 + H_i^2 - 2H_i h_i \widetilde{y}_i^{'} < h_i^2 + H_i^2 < h_i^2 - H_i^2 + 2H_i h_i \widetilde{y}_i^{'} \ . \tag{12b}$$

The inequality (12) can be presented as a system of two inequalities

$$\begin{aligned} H_i^2 &< H_i h_i \widetilde{y}_i^{'} \ , \\ - H_i h_i \widetilde{y}_i^{'} &< h_i^2 \ . \end{aligned} \tag{13}$$

Let us suppose $h_i > 0$ for the considering grid, otherwise we could change enumeration of grid points to provide this condition. Then (13) turns as follows

$$\begin{aligned} \frac{H_i^2}{h_i^2} &< \frac{H_i}{h_i}\widetilde{y}_i^{'} \ , \\ - \frac{H_i}{h_i}\widetilde{y}_i^{'} &< 1 \ . \end{aligned} \tag{14}$$

Let us consider case $H_i \geq 0$. Then according to this condition and the first inequality of (14) we have

$$0 \leq \frac{H_i}{h_i} < \widetilde{y_i'} \ ,$$

and then the second inequality of (14) is satisfied identically, because $-\widetilde{y_i'} < 0$ and $\frac{h_i}{H_i} > 0$.

In case $H_i < 0$, we have $\frac{H_i}{h_i} < 0$, and dividing both inequalities (14) by this negative term, we obtain instead of the first inequality

$$0 > \frac{H_i}{h_i} > \widetilde{y_i'}$$

and then the second inequality in the form $-\frac{h_i}{H_i} > \widetilde{y_i'}$ is satisfied identically.

In both cases considered above, we have

$$|\frac{H_i}{h_i}| < |\widetilde{y_i'}|, \tag{15a}$$

or

$$|y_{si}'| < |\widetilde{y_i'}|. \tag{15b}$$

Here we denote a tangent of the secant line slope $tg(\varphi_{si}) = \frac{H_i}{h_i}$ as y_{si}'.

We could present y_{si}' also as follows

$$\begin{aligned} y_{si}' &= tg[\frac{1}{2}(arctg\ y_i' + arctg\ y_{i+1}')] \\ &= \frac{\sqrt{[1+(y_i')^2][1+(y_{i+1}')^2]} + y_i'y_{i+1}' - 1}{y_i' + y_{i+1}'} \ . \end{aligned} \tag{16}$$

Let us write $y_{i+1}' = y_i' + (y_{i+1}' - y_i')$ and take a Taylor series expansion of right-hand side of the formula (16) with respect to $\triangle y_i' = y_{i+1}' - y_i'$. In result we obtain

$$y_{si}' = y_i' + \frac{1}{2}\triangle y_i' + O[(\triangle y_i')^2]$$

or

$$y_{si}' = \frac{1}{2}(y_i' + y_{i+1}') + O[(\triangle y_i')^2] \ .$$

Since

$$|\frac{1}{2}(y_i^{'} + y_{i+1}^{'}) + O[(\triangle y_i^{'})^2]| \leq \frac{1}{2}(|y_i^{'}| + |y_{i+1}^{'}|) + |O[(\triangle y_i^{'})^2]| \ ,$$

then the condition (12) is automatically satisfied, if the function y satisfies to the following condition

$$\frac{1}{2}|y_{i+1}^{'}| + O[(\triangle y_i^{'})^2] < \frac{1}{2}|y_i^{'}| \ .$$

Then taking a limit of the both sides of last inequality, divided by h_i, as h_i goes to zero, we obtain the following condition

$$\frac{d|y^{'}|}{dx} < 0 \ . \tag{17}$$

Thus, if any function $y(x)$ has a negative derivative of the absolute value of its first derivative, then the condition (12) holds and the rounding error arising in evaluation of the first derivative by formula (8) is strictly damping.

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