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ON THE OSTROWSKI INEQUALITY FOR THE RIEMANN-STIELTJES INTEGRAL $\int_a^b f(t)\,du(t)$, WHERE f IS OF HÖLDER TYPE AND u IS OF BOUNDED VARIATION AND APPLICATIONS

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Abstract. In this paper we point out an Ostrowski type inequality for the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$, where f is of $p-H-$Hölder type on $[a,b]$, and u is of bounded variation on $[a,b]$. Applications for the approximation problem of the Riemann-Stieltjes integral in terms of Riemann-Stieltjes sums are also given.

1. Introduction

In 1938, A. Ostrowski proved the following integral inequality [1, p. 468]:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$, differentiable on (a,b), with its first derivative $f':(a,b)\to\mathbb{R}$ bounded on (a,b), that is, $\|f'\|_\infty := \sup_{t\in(a,b)}|f'(t)| < \infty$. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\|f'\|_\infty (b-a), \tag{1.1}$$

for all $x\in[a,b]$.
The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

For a different proof than the original one provided by Ostrowski in 1938 as well as applications for special means (identric mean, logarithmic mean, $p-$logarithmic mean, etc.) and in *Numerical Analysis* for quadrature formulae of Riemann type, see the recent paper [2].

In [3], the following version of Ostrowski's inequality for the 1-norm of the first derivatives has been given.

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$, differentiable on (a,b), with its first derivative $f':(a,b)\to\mathbb{R}$ integrable on (a,b), that is, $\|f'\|_1 := \int_a^b |f'(t)|\,dt < \infty$. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \left[\frac{1}{2}+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right]\|f'\|_1, \tag{1.2}$$

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for all $x \in [a, b]$.
The constant $\frac{1}{2}$ *is sharp.*

Note that the sharpness of the constant $\frac{1}{2}$ in the class of differentiable mappings whose derivatives are integrable on (a, b) has been proven in the paper [5].

In [3], the authors applied (1.2) for special means and for quadrature formulae of Riemann type.

The following natural extension of Theorem 2 has been pointed out by S.S. Dragomir in [6].

Theorem 3. *Let* $f : [a, b] \to \mathbb{R}$ *be a mapping of bounded variation on* $[a, b]$ *and* $\bigvee_a^b (f)$ *its total variation on* $[a, b]$. *Then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \left[\frac{1}{2} + \frac{\left| x - \frac{a+b}{2} \right|}{b-a} \right] \bigvee_a^b (f), \tag{1.3}$$

for all $x \in [a, b]$. *The constant* $\frac{1}{2}$ *is sharp.*

In [6], the author applied (1.3) for quadrature formulae of Riemann type as well as for Euler's Beta mapping.

In this paper we point out some generalizations of (1.3) for the Riemann-Stieltjes integral $\int_a^b f(t)\, du(t)$ where f is of Hölder type and u is of bounded variation. Applications to the problem of approximating the Riemann-Stieltjes integral in terms of Riemann-Stieltjes sums are also given.

2. Some Integral Inequalities

The following theorem holds.

Theorem 4. *Let* $f : [a, b] \to \mathbb{R}$ *be a* $p-H-$*Hölder type mapping, that is, it satisfies the condition*

$$|f(x) - f(y)| \leq H |x - y|^p, \text{ for all } x, y \in [a, b]; \tag{2.1}$$

where $H > 0$ *and* $p \in (0, 1]$ *are given, and* $u : [a, b] \to \mathbb{R}$ *is a mapping of bounded variation on* $[a, b]$. *Then we have the inequality*

$$\begin{aligned} & \left| f(x)(u(b) - u(a)) - \int_a^b f(t)\, du(t) \right| \\ \leq\ & H \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]^p \bigvee_a^b (u), \end{aligned} \tag{2.2}$$

for all $x \in [a, b]$, *where* $\bigvee_a^b (u)$ *denotes the total variation of* u *on* $[a, b]$. *Furthermore, the constant* $\frac{1}{2}$ *is the best possible, for all* $p \in (0, 1]$.

Proof. It is well known that if $g : [a,b] \to \mathbb{R}$ is continuous and $v : [a,b] \to \mathbb{R}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_a^b g(t)\,dv(t)$ exists and the following inequality holds:

$$\left| \int_a^b g(t)\,dv(t) \right| \le \sup_{t\in[a,b]} |g(t)| \bigvee_a^b (v). \tag{2.3}$$

Using this property, we have

$$\begin{aligned} \left| f(x)(u(b)-u(a)) - \int_a^b f(t)\,du(t) \right| &= \left| \int_a^b (f(x)-f(t))\,du(t) \right| \\ &\le \sup_{t\in[a,b]} |f(x)-f(t)| \bigvee_a^b (u). \end{aligned} \tag{2.4}$$

As f is of $p-H-$Hölder type, we have

$$\begin{aligned} \sup_{t\in[a,b]} |f(x)-g(t)| &\le \sup_{t\in[a,b]} \left[H|x-t|^p \right] \\ &= H \max\{(x-a)^p, (b-x)^p\} \\ &= H[\max\{x-a, b-x\}]^p \\ &= H\left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right|\right]^p. \end{aligned}$$

Using (2.4), we deduce (2.2).
To prove the sharpness of the constant $\frac{1}{2}$ for any $p \in (0,1]$, assume that (2.2) holds with a constant $C > 0$, that is,

$$\begin{aligned} &\left| f(x)(u(b)-u(a)) - \int_a^b f(t)\,du(t) \right| \\ &\le H\left[C(b-a) + \left| x - \frac{a+b}{2} \right|\right]^p \bigvee_a^b (u), \end{aligned} \tag{2.5}$$

for all f, $p-H-$Hölder type mappings on $[a,b]$ and u of bounded variation on the same interval.
Choose $f(x) = x^p$ $(p \in (0,1])$, $x \in [0,1]$ and $u : [0,1] \to [0,\infty)$ given by

$$u(x) = \begin{cases} 0 \text{ if } x \in [0,1) \\ 1 \text{ if } x = 1 \end{cases}.$$

As

$$|f(x) - f(y)| = |x^p - y^p| \le |x-y|^p$$

for all $x, y \in [0,1]$, $p \in (0,1]$, it follows that f is of $p-H-$Hölder type with the constant $H = 1$.

By using the integration by parts formula for Riemann-Stieltjes integrals, we have:

$$\begin{aligned}\int_0^1 f(t)\,du(t) &= f(t)\,u(t)]_0^1 - \int_0^1 u(t)\,df(t)\\ &= 1-0=1\end{aligned}$$

and

$$\bigvee_0^1 (u) = 1.$$

Consequently, by (2.5), we get

$$|x^p - 1| \le \left[C + \left|x - \frac{1}{2}\right|\right]^p, \text{ for all } x \in [0,1].$$

For $x = 0$, we get $1 \le \left(C + \frac{1}{2}\right)^p$, which implies that $C \ge \frac{1}{2}$, and the theorem is completely proved. ∎

The following corollaries are natural.

Corollary 1. *Let u be as in Theorem 4 and $f : [a,b] \to \mathbb{R}$ an $L-$Lipschitzian mapping on $[a,b]$, that is,*

$$\text{(L)} \qquad |f(t) - f(s)| \le L\,|t-s| \text{ for all } t,s \in [a,b]$$

where $L > 0$ is fixed.
Then, for all $x \in [a,b]$, we have the inequality

$$\begin{aligned}&|\Theta(f,u,a,b)| \\ \le\ & L\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]\bigvee_a^b(u)\end{aligned} \tag{2.6}$$

where

$$\Theta(f,u,x,a,b) = f(x)\,(u(b) - u(a)) - \int_a^b f(t)\,du(t)$$

is the Ostrowski's functional associated to f and u as above. The constant $\frac{1}{2}$ is the best possible.

Remark 1. *If u is monotonic on $[a,b]$ and f is of $p-H-$Hölder type, then, by* (2.2) *we get*

$$\begin{aligned}&|\Theta(f,u,a,b)| \\ \le\ & H\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]|u(b) - u(a)|,\ x \in [a,b],\end{aligned} \tag{2.7}$$

and if we assume that f is $L-$Lipschitzian, then (2.6) *becomes*

$$\begin{aligned}&|\Theta(f,u,a,b)| \\ \le\ & L\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]|u(b) - u(a)|,\ x \in [a,b].\end{aligned} \tag{2.8}$$

Remark 2. *If u is $K-$Lipschitzian, then obviously u is of bounded variation on $[a,b]$ and $\bigvee_a^b (u) \leq L(b-a)$. Consequently, if f is of $p-H-$Hölder type, then*

$$\begin{aligned} &|\Theta(f,u,a,b)| \\ \leq\ & HK\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^p (b-a),\ x\in[a,b] \end{aligned} \tag{2.9}$$

and if f is $L-$Lipschitzian, then

$$\begin{aligned} &|\Theta(f,u,a,b)| \\ \leq\ & LK\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right](b-a),\ x\in[a,b]. \end{aligned} \tag{2.10}$$

The following corollary concerning a generalization of the mid-point inequality holds:

Corollary 2. *Let f and u be as defined in Theorem 4. Then we have the generalized mid-point formula*

$$|\Upsilon(f,u,a,b)| \leq \frac{H}{2^p}(b-a)^p \bigvee_a^b (u), \tag{2.11}$$

where

$$\Upsilon(f,u,a,b) = f\left(\frac{a+b}{2}\right)(u(b)-u(a)) - \int_a^b f(t)\,du(t)$$

is the mid point functional associated to f and u as above. In particular, if f is $L-$Lipschitzian, then

$$|\Upsilon(f,u,a,b)| \leq \frac{L}{2}(b-a) \bigvee_a^b (u). \tag{2.12}$$

Remark 3. *Now, if in (2.11) and (2.12) we assume that u is monotonic, then we get the midpoint inequalities*

$$|\Upsilon(f,u,a,b)| \leq \frac{H}{2^p}(b-a)^p |u(b)-u(a)| \tag{2.13}$$

and

$$|\Upsilon(f,u,a,b)| \leq \frac{L}{2}(b-a)|u(b)-u(a)| \tag{2.14}$$

respectively.

In addition, if in (2.11) and (2.12) we assume that u is $K-$Lipschitzian, then we obtain the inequalities

$$|\Upsilon(f,u,a,b)| \leq \frac{HK}{2^p}(b-a)^{p+1} \tag{2.15}$$

and

$$|\Upsilon(f,u,a,b)| \leq \frac{LK}{2}(b-a)^2. \tag{2.16}$$

The following inequalities of "*rectangle type*" also hold:

Corollary 3. *Let f and u be as in Theorem 4. Then we have the generalized "left rectangle" inequality*

$$\left| f(a)(u(b)-u(a)) - \int_a^b f(t)\,du(t) \right| \leq H(b-a)^p \bigvee_a^b (u) \tag{2.17}$$

and the "right rectangle" inequality

$$\left| f(b)(u(b)-u(a)) - \int_a^b f(t)\,du(t) \right| \leq H(b-a)^p \bigvee_a^b (u). \tag{2.18}$$

Remark 4. *If we add* (2.17) *and* (2.18), *then, by the triangle inequality, we end up with the following generalized trapezoidal inequality*

$$\left| \frac{f(a)+f(b)}{2}(u(b)-u(a)) - \int_a^b f(t)\,du(t) \right| \leq H(b-a)^p \bigvee_a^b (u). \tag{2.19}$$

In what follows, we point out some results for the Riemann integral of a product.

Corollary 4. *Let $f:[a,b]\to\mathbb{R}$ be a $p-H-$Hölder type mapping and $g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$. Then we have the inequality*

$$\begin{aligned} &\left| f(x)\int_a^b g(s)\,ds - \int_a^b f(t)g(t)\,dt \right| \\ &\leq H\left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right|\right]^p \int_a^b |g(s)|\,ds \end{aligned} \tag{2.20}$$

for all $x\in[a,b]$.

Proof. Define the mapping $u:[a,b]\to\mathbb{R}$, $u(t)=\int_a^t g(s)\,ds$. Then u is differentiable on (a,b) and $u'(t)=g(t)$. Using the properties of the Riemann-Stieltjes integral, we have

$$\int_a^b f(t)\,du(t) = \int_a^b f(t)g(t)\,dt$$

and

$$\bigvee_a^b (u) = \int_a^b |u'(t)|\,dt = \int_a^b |g(t)|\,dt.$$

Therefore, by the inequality (2.2), we deduce (2.20). ∎

Remark 5. *The best inequality we can get from* (2.20) *is that one for which $x=\frac{a+b}{2}$, obtaining the midpoint inequality*

$$\left| f\left(\frac{a+b}{2}\right)\int_a^b g(s)\,ds - \int_a^b f(t)g(t)\,dt \right| \leq \frac{1}{2^p}H(b-a)^p \int_a^b |g(s)|\,ds. \tag{2.21}$$

We now give some examples of weighted Ostrowski inequalities for some of the most popular weights.

Example 1. ***(Legendre)*** *If* $g(t) = 1,$ *and* $t \in [a,b]$*, then we get the following Ostrowski inequality for Hölder type mappings* $f : [a,b] \to \mathbb{R}$

$$\left|(b-a) f(x) - \int_a^b f(t)\,dt\right| \leq H\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^p (b-a) \tag{2.22}$$

for all $x \in [a,b]$*, and, in particular, the mid-point inequality*

$$\left|(b-a) f\left(\frac{a+b}{2}\right) - \int_a^b f(t)\,dt\right| \leq \frac{1}{2^p} H (b-a)^{p+1}. \tag{2.23}$$

Example 2. ***(Logarithm)*** *If* $g(t) = \ln\left(\frac{1}{t}\right)$*,* $t \in (0,1]$*,* f *is of* $p-$*Hölder type on* $[0,1]$ *and the integral* $\int_0^1 f(t)\ln\left(\frac{1}{t}\right)dt$ *is finite, then we have*

$$\left|f(x) - \int_0^1 f(t)\ln\left(\frac{1}{t}\right)dt\right| \leq H\left[\frac{1}{2} + \left|x - \frac{1}{2}\right|\right]^p \tag{2.24}$$

for all $x \in [0,1]$ *and, in particular,*

$$\left|f\left(\frac{1}{2}\right) - \int_0^1 f(t)\ln\left(\frac{1}{t}\right)dt\right| \leq \frac{1}{2^p} H. \tag{2.25}$$

Example 3. ***(Jacobi)*** *If* $g(t) = \frac{1}{\sqrt{t}}, t \in (0,1]$*,* f *is as above and the integral* $\int_0^1 \frac{f(t)}{\sqrt{t}}dt$ *is finite, then we have*

$$\left|f(x) - \frac{1}{2}\int_0^1 \frac{f(t)}{\sqrt{t}}dt\right| \leq H\left[\frac{1}{2} + \left|x - \frac{1}{2}\right|\right]^p, \tag{2.26}$$

for all $x \in [0,1]$ *and, in particular,*

$$\left|f\left(\frac{1}{2}\right) - \frac{1}{2}\int_0^1 \frac{f(t)}{\sqrt{t}}dt\right| \leq \frac{1}{2^p} H. \tag{2.27}$$

Finally, we have the following:

Example 4. ***(Chebychev)*** *If* $g(t) = \frac{1}{\sqrt{1-t^2}}$*,* $t \in (-1,1)$*,* f *is of* $p-$*Hölder type on* $(-1,-1)$ *and the integral* $\int_{-1}^1 \frac{f(t)}{\sqrt{1-t^2}}dt$ *is finite, then*

$$\left|f(x) - \frac{1}{\pi}\int_{-1}^1 \frac{f(t)}{\sqrt{1-t^2}}dt\right| \leq H[1+|x|]^p \tag{2.28}$$

for all $x \in [-1,1]$*, and in particular,*

$$\left|f(0) - \frac{1}{\pi}\int_{-1}^1 \frac{f(t)}{\sqrt{1-t^2}}dt\right| \leq H. \tag{2.29}$$

3. An Approximation for the Riemann-Stieltjes Integral

Consider $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ to be a division of the interval $[a, b]$, $h_i := x_{i+1} - x_i$ $(i = 0, ..., n-1)$ and $\nu(h) := \max\{h_i | i = 0, ..., n-1\}$. Define the general Riemann-Stieltjes sum

$$S(f, u, I_n, \xi) := \sum_{i=0}^{n-1} f(\xi_i)(u(x_{i+1}) - u(x_i)). \tag{3.1}$$

In what follows, we point out some upper bounds for the error approximation of the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ by its Riemann-Stieltjes sum $S(f, u, I_n, \xi)$.

Theorem 5. *Let $u : [a, b] \to \mathbb{R}$ be a mapping of bounded variation on $[a, b]$ and $f : [a, b] \to \mathbb{R}$ a $p-H-$Hölder type mapping. Then*

$$\int_a^b f(t)\,du(t) = S(f, u, I_n, \xi) + R(f, u, I_n, \xi), \tag{3.2}$$

where $S(f, u, I_n, \xi)$ is as given in (3.1) *and the remainder $R(f, u, I_n, \xi)$ satisfies the bound*

$$\begin{aligned} |R(f, u, I_n, \xi)| &\leq H\left[\frac{1}{2}\nu(h) + \max_{i=\overline{0,n-1}}\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right]^p \bigvee_a^b (u) \\ &\leq H[\nu(h)]^p \bigvee_a^b (u). \end{aligned} \tag{3.3}$$

Proof. We apply Theorem 4 on the subintervals $[x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ to obtain

$$\begin{aligned} &\left|f(\xi_i)(u(x_{i+1}) - u(x_i)) - \int_{x_i}^{x_{i+1}} f(t)\,du(t)\right| \\ &\leq H\left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right]^p \bigvee_{x_i}^{x_{i+1}} (u), \end{aligned} \tag{3.4}$$

for all $i \in \{0, ..., n-1\}$.
Summing over i from 0 to $n-1$ and using the generalized triangle inequality, we deduce

$$\begin{aligned} |R(f, u, I_n, \xi)| &\leq \sum_{i=0}^{n-1}\left|f(\xi_i)(u(x_{i+1}) - u(x_i)) - \int_{x_i}^{x_{i+1}} f(t)\,du(t)\right| \\ &\leq H\sum_{i=0}^{n-1}\left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right]^p \bigvee_{x_i}^{x_{i+1}} (u) \\ &\leq H \sup_{i=\overline{0,n-1}}\left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right]^p \sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}} (u). \end{aligned}$$

However,

$$\sup_{i=\overline{0,n-1}}\left[\frac{1}{2}h_i+\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]^p\leq\left[\frac{1}{2}\nu\left(h\right)+\sup\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]^p$$

and

$$\sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}}\left(u\right)=\bigvee_a^b\left(u\right),$$

which completely proves the first inequality in (3.3).
For the second inequality, we observe that

$$\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\leq\frac{1}{2}\cdot h_i,$$

for all $i\in\{0,...,n-1\}$.
The theorem is thus proved. ∎

The following corollaries are natural.

Corollary 5. *Let u be as in Theorem 5 and f an $L-$Lipschitzian mapping. Then we have the formula* (3.2) *and the remainder $R\left(f,u,I_n,\xi\right)$ satisfies the bound*

$$\begin{aligned}(3.5)\qquad \left|R\left(f,u,I_n,\xi\right)\right| &\leq L\left[\frac{1}{2}\nu\left(h\right)+\max_{i=\overline{0,n-1}}\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]\bigvee_a^b\left(u\right)\\ &\leq H\nu\left(h\right)\bigvee_a^b\left(u\right).\end{aligned}$$

Remark 6. *If u is monotonic on $[a,b]$, then the error estimate* (3.3) *becomes*

$$\begin{aligned}(3.6)\qquad &\left|R\left(f,u,I_n,\xi\right)\right|\\ \leq\ & H\left[\frac{1}{2}\nu\left(h\right)+\max_{i=\overline{0,n-1}}\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]^p\left|u\left(b\right)-u\left(a\right)\right|\\ \leq\ & H\left[\nu\left(h\right)\right]^p\left|u\left(b\right)-u\left(a\right)\right|\end{aligned}$$

and (3.5) *becomes*

$$\begin{aligned}(3.7)\qquad &\left|R\left(f,u,I_n,\xi\right)\right|\\ \leq\ & L\left[\frac{1}{2}\nu\left(h\right)+\max_{i=\overline{0,n-1}}\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]\left|u\left(b\right)-u\left(a\right)\right|\\ \leq\ & L\nu\left(h\right)\left|u\left(b\right)-u\left(a\right)\right|.\end{aligned}$$

Using Remark 2, we can state the following corollary.

Corollary 6. *If $u:[a,b]\to\mathbb{R}$ is Lipschitzian with the constant K and $f:[a,b]\to\mathbb{R}$ is of $p-H-$Hölder type, then the formula* (3.2) *holds and the remainder $R\left(f,u,I_n,\xi\right)$*

satisfies the bound

$$|R(f,u,I_n,\xi)| \leq HK\sum_{i=0}^{n-1}\left[\frac{1}{2}h_i+\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]^p h_i \leq HK\sum_{i=0}^{n-1}h_i^{p+1}\leq HK(b-a)[\nu(h)]^p. \tag{3.8}$$

In particular, if we assume that f *is* $L-$*Lipschitzian, then*

$$|R(f,u,I_n,\xi)| \leq \frac{1}{2}LK\sum_{i=0}^{n-1}h_i^2+LK\sum_{i=0}^{n-1}\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|h_i \leq LK\sum_{i=0}^{n-1}h_i^2\leq LK(b-a)\nu(h). \tag{3.9}$$

The best quadrature formula we can get from Theorem 5 is that one for which $\xi_i=\frac{x_i+x_{i+1}}{2}$ for all $i\in\{0,...,n-1\}$. Consequently, we can state the following corollary.

Corollary 7. *Let* f *and* u *be as in Theorem 5. Then*

$$\int_a^b f(t)\,du(t)=S_M(f,u,I_n)+R_M(f,u,I_n) \tag{3.10}$$

where $S_M(f,u,I_n)$ *is the generalized midpoint formula, that is;*

$$S_M(f,u,I_n):=\sum_{i=0}^{n-1}f\left(\frac{x_i+x_{i+1}}{2}\right)(u(x_{i+1})-u(x_i))$$

and the remainder satisfies the estimate

$$|R_M(f,u,I_n)|\leq\frac{H}{2^p}[\nu(h)]^p\bigvee_a^b(u). \tag{3.11}$$

In particular, if f *is* $L-$*Lipschitzian, then we have the bound:*

$$|R_M(f,u,I_n)|\leq\frac{H}{2}\nu(h)\bigvee_a^b(u). \tag{3.12}$$

Remark 7. *If in* (3.11) *and* (3.12) *we assume that* u *is monotonic, then we get the inequalities*

$$|R_M(f,u,I_n)|\leq\frac{H}{2^p}[\nu(h)]^p|f(b)-f(a)| \tag{3.13}$$

and

$$|R_M(f,u,I_n)|\leq\frac{H}{2}\nu(h)|f(b)-f(a)|. \tag{3.14}$$

The case where f is $K-$Lipschitzian is embodied in the following corollary.

Corollary 8. *Let u and f be as in Corollary 6. Then we have the quadrature formula* (3.10) *and the remainder satisfies the estimate*

$$|R_M(f,u,I_n)| \le \frac{HK}{2^p}\sum_{i=0}^{n-1} h_i^{p+1} \le \frac{HK}{2^p}[\nu(h)]^p . \tag{3.15}$$

In particular, if f is $L-$Lipschitzian, then we have the estimate

$$|R_M(f,u,I_n)| \le \frac{1}{2}LK\sum_{i=0}^{n-1} h_i^2 \le \frac{1}{2}LK(b-a)\,\nu(h) . \tag{3.16}$$

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