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BOUNDARY CONTROLLABILITY OF SEMILINEAR SYSTEMS IN BANACH SPACES

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Abstract. Sufficient conditions for boundary controllability of semilinear systems in Banach spaces are established. The results are obtained by using the analytic semigroup theory and the Banach contraction principle. An example is provided to illustrate the theory.

1. Introduction

The problem of controllability of semilinear systems in Banach spaces has received much attention in recent years. Using the fractional power of operators Balachandran and Dauer[2] studied the controllability problem for semilinear evolution systems in Banach spaces. Rankin [10] discussed the solutions of a class of semilinear parabolic differential equations in which the nonlinear operator maps from a fractional power space into a Banach space. Hagen and Turic [6] studied the same class of equations by using the semigroup method. The motivation for an abstract theory of these type occur from the following partial differential equation:

$$\begin{aligned} z_t(x,t) - z_{xx}(x,t) &= \beta(z(x,t))_x, \\ z(0,t) &= z(1,t) = 0, \quad t > 0, \\ z(x,0) &= a(x), \quad 0 < x < 1. \end{aligned} \tag{1}$$

In general, it is not possible to consider $\frac{\partial}{\partial x} = A^{\frac{1}{2}}$, however we will show that for a special class of operator A there exists a bounded linear operator F such that $A^{\frac{1}{2}}F = \frac{\partial}{\partial x}$. Letting $G = F\beta$ we can fit the equation (1) into the abstract theory developed in [10].

It is interesting to study the controllability of such problems in which the control is acted through the boundary. But in these approaches we can encounter the difficulty for the existence of sufficiently regular solution to state space system, the control must be taken in a space of sufficiently smooth functions. Several authors [3,5] have

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discussed the general theory of a boundary control systems. Lasiecka [8] established the regularity of optimal boundary controls for parabolic equations. Han and Park [7] studied the boundary controllability of semilinear systems with nonlocal condition. Balachandran and Anandhi [1] investigated the same problem for integrodifferential systems in Banach spaces. The purpose of this paper is to derive a set of sufficient conditions for the boundary controllability of semilinear systems in Banach spaces by using the Banach fixed point theorem and fractional power of operators.

2. Preliminaries

Let E and U be a pair of real Banach spaces with $\|.\|_E$ and $|.|$, respectively. Let σ be a linear closed and densely defined operator with $D(\sigma) \subseteq E$ and let τ be a linear operator with $D(\tau) \subseteq E$ and $R(\tau) \subseteq X$, a Banach space.

Consider the boundary control semilinear system of the form

$$\begin{aligned} \dot{x}(t) &= \sigma x(t) + f(t, x(t)), \quad t \in J = [0, b], \\ \tau x(t) &= B_1 u(t), \\ x(0) &= x_0 \end{aligned} \tag{2}$$

where $B_1 : U \to X$ is a linear continuous operator, the control function $u \in L^1(J, U)$, a Banach space of admissible control functions, with U as a Banach space and the nonlinear operator $f : J \times E \to E$ is given.
Let $A : E \to E$ be the linear operator defined by

$$D(A) = \{x \in D(\sigma) : \tau x = 0\}, \quad Ax = \sigma x, \quad \text{for } \; x \in D(A).$$

Let A be the infinitesimal generator of an analytic semigroup $T(t)$. Then the fractional power $(-A)^\alpha$ can be defined for $0 \le \alpha \le 1$. $A^{-\alpha}$ for $0 < \alpha < 1$ is defined by the integral

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty s^{\alpha-1} T(s) ds,$$

where $\Gamma(\alpha)$ denotes the gamma function. $(-A)^\alpha = (-A^{-\alpha})^{-1}$ exists as a densely defined closed linear invertible operator with domain $D((-A)^\alpha)$ dense in E. The closedness of $(-A)^\alpha$ implies that $D((-A)^\alpha)$ endowed with the graph norm $|||x||| = \|x\|_E + \|A^\alpha x\|_E$ is a Banach space. Since $(-A)^\alpha$ is invertible its graph norm $|||.|||$ is equivalent to the norm $|x| = \|A^\alpha x\|_E$. Thus $D(-A^\alpha)$ equipped with the norm $|.|$ is a Banach space which we denote by E_α. Also it is clear that $0 < \alpha < \beta$ implies $E_\alpha \supset E_\beta$ and that the imbedding is continuous.

Let Y be a subspace of E with norm $\|.\|$ and $E_\alpha \subseteq Y \subseteq E$. Let $B_r = \{y \in Y : \|y - x_0\| \le r\}$, for some $r > 0$. We shall make the following hypotheses:

(i) $D(\sigma) \subset D(\tau)$ and the restriction of τ to $D(\sigma)$ is continuous relative to graph norm of $D(\sigma)$.

(iii) A is the infinitesimal generator of the analytic semigroup $T(t),\ t \geq 0$ and there exists a constant $M > 0$ such that $\|T(t)\| \leq M$ and $\|A^{-\alpha}T(t)\| \leq Ct^{-\alpha}$ for $t > 0$ and $C \geq 0$.

(iv) $f : J \times E_\alpha \to E$ is continuous and there exists $g : J \times E_\alpha \to E_\alpha$ such that $f(t, y) = A^\alpha g(t, y)$ for each $y \in E_\alpha$.

(v) $g : J \times Y \to E$ and there exists $L \geq 0$ such that

$$\|g(t, v) - g(t, w)\|_E \leq L\|v - w\|$$

for all $v, w \in B_r$ and $0 \leq t \leq b$.

(vi) There exists a linear continuous operator $B : U \to E$ such that $\sigma B \in L(U, E)$, $\tau(Bu) = B_1 u$, for all $u \in U$. Also $Bu(t)$ is continuously differentiable and $\|Bu\| \leq q\|B_1 u\|$ for all $u \in U$, where q is a constant.

(vii) For all $t \in (0, b]$ and $u \in U$, $T(t)Bu \in D(A)$. Moreover, there exists a positive function $\nu \in L^1(0, b)$ such that $\|AT(t)B\| \leq \nu(t)$, a.e. $t \in (0, b)$.

Let $x(t)$ be the solution of (1). Then we can define a function $z(t) = x(t) - Bu(t)$ and from our assumption we see that $z(t) \in D(A)$. Hence (1) can be written in terms of A and B as

$$\begin{aligned} \dot{x}(t) &= Az(t) + \sigma Bu(t) + A^\alpha g(t, x(t)), \quad t \in J, \\ x(t) &= z(t) + Bu(t), \\ x(0) &= x_0. \end{aligned} \tag{3}$$

If u is continuously differentiable on $[0, b]$ then z can be defined as a mild solution to the Cauchy problem

$$\begin{aligned} \dot{z}(t) &= Az(t) + \sigma Bu(t) - B\dot{u}(t) + A^\alpha g(t, x(t)), \\ z(0) &= x_0 - Bu(0) \end{aligned}$$

and the solution of (2) is given by

$$\begin{aligned} x(t) &= T(t)[x_0 - Bu(0)] + Bu(t) \\ &\quad + \int_0^t T(t - s)[\sigma Bu(s) - B\dot{u}(t) + A^\alpha g(s, x(s))]ds. \end{aligned} \tag{4}$$

Since the differentiability of the control u represents an unrealistic and severe requirement, it is necessary to extend the concept of the solution for the general inputs $u \in L^1(J, U)$. Integrating (4) by parts, we get

$$\begin{aligned} x(t) &= T(t)x_0 + \int_0^t [T(t - s)\sigma - AT(t - s)]Bu(s)ds \\ &\quad + \int_0^t A^\alpha T(t - s)g(s, x(s))ds. \end{aligned} \tag{5}$$

Thus (5) is well defined and it is called a mild solution of the system(2).

Definition: The system (2) is said to be controllable on the interval J if for every $x_0, x_1 \in Y$, there exists a control $u \in L^2(J, U)$ such that the solution $x(.)$ of (2) satisfies $x(b) = x_1$.

Further we assume the following conditions:

(viii) There exists a constant $K_1 > 0$ such that $\int_0^b \nu(t)dt \leq K_1$.

(ix) The linear operator W from $L^2(J, U)$ into E defined by

$$Wu = \int_0^b [T(b-s)\sigma - AT(b-s)]Bu(s)ds$$

induces an invertible operator $\tilde{W}$ defined on $L^2(J,U)/kerW$ and there exists a positive constant $K_2 > 0$ such that $\|\tilde{W}^{-1}\| \leq K_2$.

(x) Let $L^* = \frac{b^{(1-\alpha)}}{1-\alpha}[(bM\|\sigma B\| + K_1)K_2 + 1]$ be such that $0 \leq L^* < 1$.

(xi) $$(M+1)\|x_0\| + [bM\|\sigma B\| + K_1]K_2 \left[\|x_1\| + M\|x_0\| + (L_1 r + L_2)C\frac{b^{1-\alpha}}{1-\alpha}\right]. + (L_1 r + L_2)C\frac{b^{1-\alpha}}{1-\alpha} \leq \frac{r}{2}.$$

3. Main Result

Theorem: If the hypotheses (i) - (xii) are satisfied , then the boundary control semilinear system (2) is controllable on J.

Proof: Using the hypothesis (vi), for an arbitrary function $x(.)$ define the control

$$u(t) = \tilde{W}^{-1}[x_1 - T(b)x_0 - \int_0^b A^\alpha T(b-s)g(s,x(s))ds](t). \tag{6}$$

Let $Z = C(J, Y)$ be the space endowed with the supremun norm. Define the set

$$S = \{x \in Z : \quad x(0) = x_0, \quad \|x(t) - x_0\| \leq r\}.$$

We shall show that when using the above control the operator F defined by

$$\begin{aligned} Fx(t) \quad &= \quad T(t)x_0 + \int_0^t [T(t-s)\sigma - AT(t-s)]B\tilde{W}^{-1}[x_1 - T(b)x_0 \\ &\quad - \int_0^b A^\alpha T(b-\tau)g(\tau, x(\tau))](s)ds \\ &\quad + \int_0^t A^\alpha T(t-s)g(s,x(s))ds \end{aligned}$$

has a fixed point. First we show that F maps S into itself. For $x \in S$,

$$\begin{aligned}
\|Fx(t) - x_0\| &\leq \|T(t)x_0 - x_0\| + \| \int_0^t [T(t-s)\sigma - AT(t-s)]B\tilde{W}^{-1}[x_1 - T(b)x_0 \\
&\quad - \int_0^b A^\alpha T(b-\tau)g(\tau, x(\tau))d\tau](s)ds\| + \| \int_0^t A^\alpha T(t-s)g(s,x(s))ds\| \\
&\leq \|T(t) - I\|\|x_0\| + \int_0^t \|T(t-s)\|\|\sigma B\|\|\tilde{W}^{-1}\|[\|x_1\| + \|T(b)x_0\| \\
&\quad\quad + \int_0^b \|A^\alpha T(b-\tau)\|[\|g(\tau,x(\tau)) - g(\tau,x_0)\|_E + \|g(\tau,0)\|_E]d\tau]ds \\
&\quad + \int_0^t \|AT(t-s)B\|\|\tilde{W}^{-1}\|[\|x_1\| + \|T(b)x_0\| \\
&\quad\quad + \int_0^b \|A^\alpha T(b-\tau)\|[\|g(\tau,x(\tau) - g(\tau,x_0)\|_E + \|g(\tau,x_0)\|_E]d\tau]ds \\
&\quad + \int_0^t \|A^\alpha T(t-s)\|[\|g(s,x(s)) - g(s,x_0)\|_E + \|g(s,x_0)\|_E]ds \\
&\leq (M+1)\|x_0\| + bM\|\sigma B\|K_2[\|x_1\| + M\|x_0\| \\
&\quad\quad + [L_1 r + L_2]C\int_0^b (b-s)^{-\alpha}ds + KK_2[\|x_1\| + M\|x_0\| \\
&\quad\quad + [L_1 r + L_2]C\int_0^b (b-s)^{-\alpha}ds] + [L_1 r + L_2]C\int_0^t (t-s)^{-\alpha}ds \\
&\leq (M+1)\|x_0\| + [bM\|\sigma B\| + K]K_2[\|x_1\| + M\|x_0\| \\
&\quad\quad + (L_1 r + L_2)C\frac{b^{1-\alpha}}{1-\alpha}] + (L_1 r + L_2)C\frac{b^{1-\alpha}}{1-\alpha} \\
&\leq r
\end{aligned}$$

Thus F maps S into itself. Now, for $x_1, x_2 \in S$ we have

$$\begin{aligned}
\|Fx_1(t) - Fx_2(t)\| &\leq \| \int_0^t [T(t-s)\sigma - AT(t-s)]B\tilde{W}^{-1} \\
&\quad\quad [\int_0^b A^\alpha T(b-\tau)[g(\tau,x_1(\tau)) - g(\tau,x_2(\tau))]d\tau]ds\| \\
&\quad + \| \int_0^t A^\alpha T(t-s)[g(s,x_1(s)) - g(s,x_2(s))]ds\| \\
&\quad \leq \int_0^t [\|T(t-s)\|\|\sigma B\| + \|AT(t-s)B\|]\|\tilde{W}^{-1}\| \\
&\quad\quad [\int_0^b \|A^\alpha T(b-\tau)\|\|g(\tau,x_1(\tau)) - g(\tau,x_2(\tau))\|_E d\tau
\end{aligned}$$

$$+\int_0^t \|A^\alpha T(t-s)\| \|g(s,x_1(s)) - g(s,x_2(s))\|_E ds$$
$$\leq [bM\|\sigma B\| + K]K_2 L_1 \int_0^b \|A^\alpha T(b-s)\| \|x_1(\tau) - x_2(\tau)\| d\tau$$
$$+ L_1 \int_0^t \|A^\alpha T(t-s)\| \|x_1(s) - x_2(s)\| ds$$
$$\leq \frac{b^{1-\alpha}}{1-\alpha}[bM(\|\sigma B\| + K)K_2 + 1]L_1 \sup_{0\leq t\leq b} \|x_1(t) - x_2(t)\|$$
$$\leq L^* \|x_1(t) - x_2(t)\|$$

Therefore, F is a contraction mapping and hence there exists a unique fixed point $x \in Y$ such that $Fx(t) = x(t)$. Any fixed point of F is a mild solution of (1) on J which satisfies $x(b) = x_1$. Thus the system (2) is controllable on J.

3. Example

Let Ω be a bounded, open connected subset of R^n and let Γ be sufficiently smooth boundary of Ω. Consider the boundary control system of the form

$$\begin{aligned} z_t(t,y) - \nu\Delta z(t,y) + (z.\Delta)z + \nabla p &= 0 \\ div z(t,y) &= 0, \qquad (t,y) \in Q = (0,b)\times\Omega, \\ z(t,y) &= u(t,y), \quad on\ \Sigma = (0,b)\times\Gamma, \quad t \in J \\ z(0,y) &= \phi(y), \qquad y \in \Omega, \end{aligned} \tag{7}$$

where $u \in L^2(\Sigma)$ and $\phi(y) \in L^2(\Omega)$.
The above problem can be formulated as a boundary control problem of the form (2) by suitably choosing the spaces $E,\ X,\ U$ as follows:
Let $E = L^2(\Omega)$ be equipped with the usual norm

$$\|z\| = (\int_\Omega |z(y)|^2 dy)^{\frac{1}{2}}$$

and let $W^{m,2}(\Omega)$ be the Sobolev space of all functions on Ω whose distributional derivatives up through order m are in $L^2(\Omega)$ with norm given by

$$\|\phi\|_{m,2} = (\sum_{\alpha\leq m} \|\frac{\partial^\alpha \phi}{\partial y}\|^2)^{\frac{1}{2}}.$$

Take $X = H^{-\frac{1}{2}}(\Gamma)$, $U = L^2(\Gamma)$ and $B_1 = I$, the identity operator and

$$D(\sigma) = \{z \in L^2(\Omega);\ \Delta z \in L^2(\Omega)\}, \quad \sigma = \nu\Delta.$$

The operator τ is the "trace" operator such that $\tau z = z|_\Gamma$ is well defined and belongs to $H^{-\frac{1}{2}}(\Gamma)$ for each $z \in D(\sigma)$. (see[4]). Define $E^2 = (L^2(\Omega))^n$ and let the function $z(t,y) = (z^1(t,y), \dots, z^n(t,y))$.
Set

$$Z^2 = \text{Closure } \ of \ \ \{\phi \in (C_0(\Omega))^n, \ div\phi(y) = 0\} \ \text{ in } E^2$$

and

$$J_p = \{\nabla p; \ \ p \in W^{1,2}(\Omega)\}.$$

Then we have the well known decomposition $E^2 = Z^2 \oplus J_p$. Let P be continuous projection from E^2 to Z^2 and let Δ be the Laplace operator with

$$D(\Delta) = \{\phi \in (H^2(\Omega))^n : \ \ \phi(y) = 0, \ \ y \in \partial\Omega\}.$$

Define $A = -P\nu\Delta$ with domain $D(A) = Z^2 \cap D(\Delta)$. Then it is known that $-A$ is a closed densely defined linear operator in Z^2 with a bounded inverse and generates a bounded analytic semigroup $T(t)$ in Z^2.
Define the linear operator $B : L^2(\Gamma) \to L^2(\Omega)$ by $Bu = w_u$ where w_u is the unique solution to the Dirichlet boundary value problem,

$$\begin{aligned} \Delta w_u &= 0 \quad \text{in } \ \Omega, \\ w_u &= u \quad \text{in } \ \Gamma. \end{aligned}$$

In other words (see [9])

$$(8) \qquad \int_\Omega w_u \Delta\psi dx = \int_\Gamma u \frac{\partial\psi}{\partial n} dx, \quad \text{for all} \quad \psi \in H_0^1(\Omega) \cup H^2(\Omega),$$

where $\frac{\partial\psi}{\partial n}$ denotes the outward normal derivative of ψ which is well defined as an element of $H^{\frac{1}{2}}(\Gamma)$. ¿From (8), it follows that,

$$\|w_u\|_{L^2(\Omega)} \leq C_1 \|u\|_{H^{-\frac{1}{2}}(\Gamma)}, \quad \text{for all } \ u \in H^{-\frac{1}{2}}(\Gamma)$$

and

$$\|w_u\|_{H^1(\Omega)} \leq C_2 \|u\|_{H^{\frac{1}{2}}(\Gamma)}, \quad \text{for all } \ u \in H^{\frac{1}{2}}(\Gamma),$$

where C_i, $i = 1, 2$ are positive constants independent of u.
From the above estimates it follows by an interpolation argument [11] that

$$\|AT(t)B\|_{L(L^2(\Gamma), L^2(\Gamma))} \leq Ct^{-\frac{3}{4}}, \quad \text{for all } \ t > 0 \ \text{ with } \ \nu(t) = Ct^{-\frac{3}{4}}.$$

Applying the projection P to (7), we find that (2) is an abstract formulation of (7) where f is defined on $D(A^{\frac{1}{2}}) = {W_0}^{1,2}((\Omega))^n \cap E^2$ with range in E^2 by

$$f(\phi(y)) = P \sum_{k=1}^{n} \left(\frac{\partial\phi^k(y)\phi^1(y)}{\partial y_k}, \dots, \frac{\partial\phi^k(y)\phi^n(y)}{\partial y_k} \right).$$

It is clear from a known result [14] that there exists a bounded linear operator $B_k : E^2 \to X^2$ such that $P\frac{\partial}{\partial y_k} = A^{\frac{1}{2}} B_k$ and also the mapping

$$g(\phi(y)) = \sum_{k=1}^{n} B_k(\phi^k(x)\phi^1(x), \dots, \phi^k(x)\phi^n(x))$$

is locally Lipschitz. Further assume that the bounded invertible operator $\tilde{W}$ exists. Choose b and other constants such that the conditions (x) and (xi) are satisfied. Hence, we see that all the conditions stated in the theorem are satisfied and so the system (7) is controllable on $[0, b]$.

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