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A GENERALIZATION OF OSTROWSKI INTEGRAL INEQUALITY FOR MAPPINGS WHOSE DERIVATIVES BELONG TO L_∞ AND APPLICATIONS IN NUMERICAL INTEGRATION

SEVER SILVESTRU DRAGOMIR

Abstract. A generalization of Ostrowski integral inequality for mappings whose derivatives are bounded and applications for general quadrature formulae are given.

1. Introduction

The following theorem contains the integral inequality which is known in the literature as Ostrowski's inequality [2, p. 469].

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative is bounded on (a,b) and denote $\|f'\|_\infty=\sup_{t\in(a,b)}|f'(t)|<\infty$. Then for all $x\in[a,b]$, we have the inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\le\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty. \tag{1.1}$$

The constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

In [1], S.S. Dragomir and S. Wang applied Ostrowski's inequality in Numerical Integration as follows.

Let $I_n: a=x_0<x_1<...<x_{n-1}<x_n=b$ be a division of the interval $[a,b]$ and $\xi_i\in[x_i,x_{i+1}]$ $(i=0,...,n-1)$ a sequence of intermediate points for I_n. Construct the Riemann sums

$$R_n(f,I_n,\boldsymbol{\xi})=\sum_{i=0}^{n-1}f(\xi_i)h_i$$

where $h_i:=x_{i+1}-x_i$.

We have the following quadrature formula [1].

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Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be as in Theorem 1 and I_n,ξ_i $(i=0,...,n-1)$ be as above. Then we have the Riemann quadrature formula*

$$\int_a^b f(x)dx = R_n(f,I_n,\boldsymbol{\xi}) + W_n(f,I_n,\boldsymbol{\xi}) \tag{1.2}$$

where the remainder satisfies the estimation

$$\begin{aligned} |W_n(f,I_n,\boldsymbol{\xi})| &\le \frac{1}{4}\|f'\|_\infty \sum_{i=0}^{n-1} h_i^2 + \|f'\|_\infty \sum_{i=0}^{n-1}\left(\xi_i - \frac{x_i+x_{i+1}}{2}\right)^2 \\ &\le \frac{1}{2}\|f'\|_\infty \sum_{i=0}^{n-1} h_i^2 \end{aligned} \tag{1.3}$$

for all ξ_i $(i=0,...,n-1)$ as above. The constant $\frac{1}{4}$ is sharp in (1.3).

Remark 1. *It is obvious that the best inequality we can get from Theorem 2 is the one for which $\xi_i = \frac{x_i+x_{i+1}}{2}$ $(i=0,\dots n-1)$ obtaining:*

$$\int_a^b f(x)dx = M_n(f,I_n) + V_n(f,I_n),$$

where $M_n(f,I_n)$ is the midpiont quadrature rule, i.e.,

$$M_n(f,I_n) = \sum_{i=0}^{n-1} h_i f\left(\frac{x_i+x_{i+1}}{2}\right)$$

and the remainder $V_n(f,I_n)$ satisfies the estimate

$$|V_n(f,I_n)| \le \frac{1}{4}\|f'\|_\infty \sum_{i=0}^{n-1} h_i^2.$$

In this paper we point out in the main another inequality generalising Ostrowski's result (1.1) for $k-$points $x_1,\dots,x_k$ for which the upper bound (see (2.1)) is similar to the bound (1.3). Note that the best inequality we can get from (2.1) is the trapezoid inequality (2.6). Applications for general quadrature formulae (see Theorem 4) and for certain particualr cases (see Section 4 and Section 5) are also given.

2. Some Integral Inequalities

We start with the following result.

Theorem 3. *Let $I_k: a=x_0<x_1<...<x_{k-1}<x_k=b$ be a division of the interval $[a,b]$, α_i $(i=0,...,k+1)$ be "$k+2$" points so that $\alpha_0=a, \alpha_i\in[x_{i-1},x_i]$ $(i=1,...,k)$*

and $\alpha_{k+1} = b$. *If* $f : [a,b] \to \mathbb{R}$ *is absolutely continuous on* $[a,b]$, *then we have the inequality:*

$$\left| \int_a^b f(x)\,dx - \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) \right| \tag{2.1}$$

$$\leq \left[\frac{1}{4} \sum_{i=0}^{k-1} h_i^2 + \sum_{i=0}^{k-1} \left(\alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right)^2 \right] \|f'\|_\infty$$

$$\leq \frac{1}{2} \|f'\|_\infty \sum_{i=0}^{k-1} h_i^2 \leq \frac{1}{2} (b-a) \|f'\|_\infty \nu(h)$$

where $h_i := x_{i+1} - x_i$ $(i = 0, ..., k-1)$ *and* $\nu(h) := \max\{h_i \mid i = 0, ..., k-1\}$. *The constant* $\frac{1}{4}$ *in the first inequality and the constant* $\frac{1}{2}$ *in the second and third inequality are the best possible.*

Proof. Define the mapping $K : [a,b] \to \mathbb{R}$ given by

$$K(t) := \begin{cases} t - \alpha_1, & t \in [a, x_1) \\ t - \alpha_2, & t \in [x_1, x_2) \\ \dots\dots\dots\dots & \\ t - \alpha_{k-1}, & t \in [x_{k-2}, x_{k-1}) \\ t - \alpha_k, & t \in [x_{k-1}, b] \end{cases}.$$

Integrating by parts, we have successively:

$$\begin{aligned}
&\int_a^b K(t) f'(t)\,dt = \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} K(t) f'(t)\,dt = \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} (t-\alpha_{i+1}) f'(t)\,dt \\
= &\sum_{i=0}^{k-1} \left[(t-\alpha_{i+1}) f(t) \Big|_{x_i}^{x_{i+1}} - \int_{x_i}^{x_{i+1}} f(t)\,dt \right] \\
= &\sum_{i=0}^{k-1} \left[(\alpha_{i+1}-x_i) f(x_i) + (x_{i+1}-\alpha_{i+1}) f(x_{i+1}) \right] - \int_a^b f(t)\,dt \\
= &(\alpha_1 - a) f(a) + \sum_{i=1}^{k-1} (\alpha_{i+1}-x_i) f(x_i) + \sum_{i=0}^{k-2} (x_{i+1}-\alpha_{i+1}) f(x_{i+1}) \\
&+ (b-\alpha_k) f(b) - \int_a^b f(t)\,dt \\
= &(\alpha_1 - a) f(a) + \sum_{i=1}^{k-1} (\alpha_{i+1}-x_i) f(x_i) + \sum_{i=1}^{k-1} (x_i-\alpha_i) f(x_i) \\
&+ (b-\alpha_k) f(b) - \int_a^b f(t)\,dt \\
= &(\alpha_1 - a) f(a) + \sum_{i=1}^{k-1} (\alpha_{i+1}-\alpha_i) f(x_i) + (b-\alpha_k) f(b) - \int_a^b f(t)\,dt \\
= &\sum_{i=0}^{k} (\alpha_{i+1}-\alpha_i) f(x_i) - \int_a^b f(t)\,dt
\end{aligned}$$

and then we have the integral equality:

$$\int_a^b f(t)\,dt = \sum_{i=0}^{k} (\alpha_{i+1}-\alpha_i) f(x_i) - \int_a^b K(t) f'(t)\,dt. \tag{2.2}$$

On the other hand, we have

$$\begin{aligned}
&\left| \int_a^b K(t) f'(t)\,dt \right| \\
= &\left| \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} K(t) f'(t)\,dt \right| \leq \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} |K(t)| \left| f'(t) \right| dt \\
= &\sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} |t-\alpha_{i+1}| \, |f'(t)| \, dt \leq \left\| f' \right\|_\infty \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} |t-\alpha_{i+1}|\,dt.
\end{aligned} \tag{2.3}$$

A simple calculation shows that

$$\begin{aligned} &\int_{x_i}^{x_{i+1}} |t-\alpha_{i+1}|\,dt \qquad (2.4)\\ &= \int_{x_i}^{\alpha_{i+1}} (\alpha_{i+1}-t)\,dt + \int_{\alpha_{i+1}}^{x_{i+1}} (t-\alpha_{i+1})\,dt \\ &= \frac{1}{2}\left[(x_{i+1}-\alpha_{i+1})^2+(\alpha_{i+1}-x_i)^2\right] = \frac{1}{4}h_i^2+\left(\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right)^2 \end{aligned}$$

for all $i=0,...,k-1$.

Now, by (2.2) – (2.4) we get the first inequality in (2.1).

Assume that the first inequality in (2.1) holds for a constant $c>0$, i.e.,

$$\begin{aligned} &\left|\int_a^b f(x)\,dx - \sum_{i=0}^{k-1}(\alpha_{i+1}-\alpha_i) f(x_i)\right| \qquad (2.5)\\ &\leq \left[c\sum_{i=0}^{k-1}h_i^2+\sum_{i=0}^{k-1}\left(\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right)^2\right]\|f'\|_\infty . \end{aligned}$$

If we choose $f:[a,b]\to\mathbb{R}$, $f(x)=x$, $\alpha_0=a$, $\alpha_1=b$, $x_0=a$, $x_1=b$ in (2.5) we obtain

$$\frac{(b-a)^2}{2}\leq c(b-a)^2+\frac{(b-a)^2}{4}$$

from where we get $c\geq\frac{1}{4}$ and the sharpness of the constant $\frac{1}{4}$ is proved.

The last two inequalities as well as the sharpness of the constant $\frac{1}{2}$ are obvious and we omit the details. ∎

Now, if we assume that the points of the division I_k are given, then the best inequality we can get from Theorem 3 is embodied in the following corollary:

Corollary 1. *Let f, I_k be as above. Then we have the inequality*

$$\begin{aligned} &\left|\int_a^b f(x)\,dx \right. \qquad (2.6)\\ &\left. -\frac{1}{2}\left[(x_1-a)f(a)+\sum_{i=1}^{k-1}(x_{i+1}-x_{i-1})f(x_i)+(b-x_{k-1})f(b)\right]\right| \\ &\leq \frac{1}{4}\|f'\|_\infty\sum_{i=0}^{k-1}h_i^2. \end{aligned}$$

The constant $\frac{1}{4}$ is the best possible one.

Proof. We choose in Theorem 3 :

$$\alpha_0=a,\alpha_1=\frac{a+x_1}{2},...,\alpha_{k-1}=\frac{x_{k-2}+x_{k-1}}{2},\alpha_k=\frac{x_{k-1}+b}{2},\alpha_{k+1}=b.$$

In this case

$$
\begin{aligned}
& \sum_{i=0}^{k}\left(\alpha_{i+1}-\alpha_{i}\right) f\left(x_{i}\right) \\
= & \left(\frac{a+x_{1}}{2}-a\right) f(a)+\left(\frac{x_{1}+x_{2}}{2}-\frac{a+x_{1}}{2}\right) f\left(x_{1}\right)+\ldots \\
& +\left(\frac{x_{k-1}+b}{2}-\frac{x_{k-2}+x_{k-1}}{2}\right) f\left(x_{k-1}\right)+\left(b-\frac{x_{k-1}+b}{2}\right) f(b) \\
= & \frac{1}{2}\left[\left(x_{1}-a\right) f(a)+\sum_{i=1}^{k-1}\left(x_{i+1}-x_{i-1}\right) f\left(x_{i}\right)+\left(b-x_{k-1}\right) f(b)\right]
\end{aligned}
$$

and then (2.6) is obvious by (2.1). ∎

The case of equidistant partitioning is important in practice.

Corollary 2. *Let* $I_k : x_i = a + i\frac{b-a}{k}$ $(i = 0, ..., k)$ *be an equidistant partitioning of* $[a, b]$. *If* f *is as above, then we have the inequality*

$$
\begin{aligned}
(2.7) \quad & \left|\int_{a}^{b} f(x)\, dx-\left[\frac{1}{k} \cdot \frac{f(a)+f(b)}{2}(b-a)+\frac{(b-a)}{k} \sum_{i=1}^{k-1} f\left(\frac{(k-i)\, a+i b}{k}\right)\right]\right| \\
\leq & \frac{1}{4k}(b-a)^{2}\left\|f^{\prime}\right\|_{\infty}.
\end{aligned}
$$

The constant $\frac{1}{4}$ *is the best possible one.*

Remark 2. *If* $k = 1$, *we have the inequality*

$$
(2.8) \qquad \left|\int_{a}^{b} f(x)\, dx-\frac{f(a)+f(b)}{2}(b-a)\right| \leq \frac{1}{4}(b-a)^{2}\left\|f^{\prime}\right\|_{\infty}.
$$

Choose $f : [a, b] \to \mathbb{R}, f(x) = \left|x - \frac{a+b}{2}\right|$ *which is* L*-lipschitzian with* $L = 1$ *and*

$$
f^{\prime}(x)=\left\{\begin{array}{ll}
-1 & \text{if } \; t \in\left[a, \frac{a+b}{2}\right) \\
1 & \text{if } \; t \in\left(\frac{a+b}{2}, b\right]
\end{array}\right. .
$$

Then $\|f'\|_\infty = 1$ *and*

$$
\int_{a}^{b} f(x)\, dx-\frac{f(a)+f(b)}{2}(b-a)=-\frac{(b-a)^{2}}{4}
$$

and the equality is obtained in (2.8) showing that the constant $\frac{1}{4}$ *is sharp.*

3. The Convergence of a General Quadrature Formula

Let $\Delta_n : a = x_0^{(n)} < x_1^{(n)} < ... < x_{n-1}^{(n)} < x_n^{(n)} = b$ be a sequence of divisions of $[a, b]$ and consider the sequence of numerical integration formulae

$$I_n(f, \Delta_n, w^{(n)}) := \sum_{j=0}^{n} w_j^{(n)} f(x_j^{(n)})$$

where $w_j^{(n)} (j = 0, ..., n)$ are the quadrature weights.

The following theorem contains a sufficient condition for the weights $w_j^{(n)}$ so that $I_n(f, \Delta_n, w^{(n)})$ approximates the integral $\int_a^b f(x)dx$.

Theorem 4. *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous mapping on $[a, b]$. If the quadrature weights $w_j^{(n)} (j = 0, ..., n)$ satisfy the condition $\sum_{j=0}^{n} w_j^{(n)} = b - a$ and*

$$x_i^{(n)} - a \leq \sum_{j=0}^{i} w_j^{(n)} \leq x_{i+1}^{(n)} - a \text{ for all } i = 0, ..., n - 1; \tag{3.1}$$

then we have the estimation

$$\begin{aligned} &\left| I_n(f, \Delta_n, w^{(n)}) - \int_a^b f(x)dx \right| \\ \leq\ & \left[\frac{1}{4} \sum_{i=0}^{n-1} \left[h_i^{(n)} \right]^2 + \sum_{i=0}^{n-1} \left[a + \sum_{j=0}^{i} w_j^{(n)} - \frac{x_i^{(n)} + x_{i+1}^{(n)}}{2} \right]^2 \right] \|f'\|_\infty \\ \leq\ & \frac{1}{2} \|f'\|_\infty \sum_{i=0}^{n} [h_i^{(n)}]^2 \leq \frac{1}{2} \|f'\|_\infty (b - a) \nu(h^{(n)}) \end{aligned} \tag{3.2}$$

where $\nu(h^{(n)}) := \max\{h_i^{(n)} : i = 0, ..., n - 1\}$ and $h_i^{(n)} := x_{i+1}^{(n)} - x_i^{(n)}$. Particularly, if $\|f'\|_\infty < \infty$, then

$$\lim_{\nu(h^{(n)}) \to 0} I_n(f, \Delta_n, w^{(n)}) = \int_a^b f(x)dx$$

uniformly by rapport of the weights $w^{(n)}$.

Proof. Define the sequence of real numbers:

$$\alpha_{i+1}^{(n)} := a + \sum_{j=0}^{i} w_j^{(n)}, i = 0, ..., n.$$

Note that

$$\alpha_{n+1}^{(n)} := a + \sum_{j=0}^{n} w_j^{(n)} = a + b - a = b.$$

By the assumption (3.1) we have $\alpha_{i+1}^{(n)} \in [x_i^{(n)}, x_{i+1}^{(n)}]$ for all $i = 0, ..., n-1$. Define $\alpha_0^{(n)} = a$ and compute

$$\alpha_1^{(n)} - \alpha_0^{(n)} = w_0^{(n)},$$

$$\alpha_{i+1}^{(n)} - \alpha_i^{(n)} = a + \sum_{j=0}^{i} w_j^{(n)} - a - \sum_{j=0}^{i-1} w_j^{(n)} = w_i^{(n)} \ (i = 1, ..., n-1)$$

and

$$\alpha_{n+1}^{(n)} - \alpha_n^{(n)} = a + \sum_{j=0}^{n} w_j^{(n)} - a - \sum_{j=0}^{n-1} w_j^{(n)} = w_n^{(n)}.$$

Consequently,

$$\sum_{i=0}^{n} \left(\alpha_{i+1}^{(n)} - \alpha_i^{(n)}\right) f\left(x_i^{(n)}\right) = \sum_{i=0}^{n} w_i^{(n)} f\left(x_i^{(n)}\right) = I_n\left(f, \Delta_n, w^{(n)}\right).$$

Applying the inequality (2.1) we get the estimation (3.2).

The uniform convergence by rapport of quadrature weights $w_j^{(n)}$ is obvious by the last inequality. ∎

The case when the partitioning is equidistant is important in practice. Consider then the partitioning

$$E_n : x_i^{(n)} := a + i \cdot \frac{b-a}{n} \ \ (i = 0, ..., n)$$

and define the sequence of numerical quadrature formulae

$$I_n\left(f, w^{(n)}\right) := \sum_{i=0}^{n} w_i^{(n)} f\left(a + i\frac{b-a}{n}\right).$$

The following result holds:

Corollary 3. *Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous on $[a,b]$. If the quadrature weights $w_j^{(n)}$ satisfy the condition:*

$$\frac{i}{n} \leq \frac{1}{b-a} \sum_{j=0}^{i} w_j^{(n)} \leq \frac{i+1}{n} (i = 0, ..., n-1);$$

then we have the estimation

$$\begin{aligned} & \left| I_n\left(f, w^{(n)}\right) - \int_a^b f(x)dx \right| \\ \leq \ & \|f'\|_\infty \left[\frac{1}{4n}(b-a)^2 + \sum_{i=0}^{n-1} \left[\sum_{j=0}^{i} w_j^{(n)} - \frac{2i+1}{2} \cdot \frac{b-a}{n} \right]^2 \right] \\ \leq \ & \frac{1}{2n} \|f'\|_\infty (b-a)^2. \end{aligned}$$

Particularly, if $\|f'\|_\infty < \infty$, *then*

$$\lim_{n\to\infty} I_n(f, w^{(n)}) = \int_a^b f(x)dx$$

uniformly by rapport of $w^{(n)}$.

4. Some Particular Integral Inequalities

In this section we point out some particular inequalities which generalize some classical results such as : rectangle inequality, trapezoid inequality, Ostrowski's inequality, midpoint inequality, Simpson's inequality and others.

Proposition 1. *Let* $f : [a, b] \to \mathbb{R}$ *be absolutely continuous on* $[a, b]$ *and* $\alpha \in [a, b]$. *Then we have the inequality:*

$$\left|\int_a^b f(x)dx - [(\alpha - a)f(a) + (b - \alpha)f(b)]\right| \leq \left[\frac{1}{4}(b-a)^2 + \left(\alpha - \frac{a+b}{2}\right)^2\right]\|f'\|_\infty \leq \frac{1}{2}(b-a)^2\|f'\|_\infty. \tag{4.1}$$

The constant $\frac{1}{4}$ *is the best possible one.*

Proof. Follows from Theorem 3 by choosing $x_0 = a, x_1 = b, \alpha_0 = a, \alpha_1 = \alpha \in [a, b]$ and $\alpha_2 = b$. ∎

Remark 3. *a) If in (4.1) we put* $\alpha = b$, *we get the "left rectangle inequality":*

$$\left|\int_a^b f(x)dx - (b-a)f(a)\right| \leq \frac{1}{2}(b-a)^2\|f'\|_\infty. \tag{4.2}$$

b) If $\alpha = a$, *then by (4.1) we get the "right rectangle inequality"*

$$\left|\int_a^b f(x)dx - (b-a)f(b)\right| \leq \frac{1}{2}(b-a)^2\|f'\|_\infty. \tag{4.3}$$

c). It is clear that the best estimation we can have in (4.1) is for $\alpha = \frac{a+b}{2}$ *getting the "trapezoid inequality"*

$$\left|\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}(b-a)\right| \leq \frac{1}{4}(b-a)^2\|f'\|_\infty. \tag{4.4}$$

Another particular integral inequality with many applications is the following one:

Proposition 2. *Let* $f : [a, b] \to \mathbb{R}$ *be an absolutely continuous mapping on* $[a, b]$ *and* $a \leq x_1 \leq b, a \leq \alpha_1 \leq x_1 \leq \alpha_2 \leq b$. *Then we have the inequality:*

$$\left|\int_a^b f(x)dx - [(\alpha_1 - a)f(a) + (\alpha_2 - \alpha_1)f(x_1) + (b - \alpha_2)\,f(b)]\right| \tag{4.5}$$
$$\leq \left[\frac{1}{8}(b-a)^2 + \frac{1}{2}\left(x_1 - \frac{a+b}{2}\right)^2 + \left(\alpha_1 - \frac{a+x_1}{2}\right)^2 + \left(\alpha_2 - \frac{x_1+b}{2}\right)^2\right]\|f'\|_\infty.$$

Proof. Consider the division $a = x_0 \leq x_1 \leq x_2 = b$ and the numbers $\alpha_0 = a, \alpha_1 \in [a, x_1], \alpha_2 \in [x_1, b], \alpha_3 = b$. Applying Theorem 3 for these particular choices, we get

$$\left|\int_a^b f(x)dx - [(\alpha_1 - a)\,f\,(a) + (\alpha_2 - \alpha_1)\,f(x_1) + (b - \alpha_2)\,f\,(b)]\right| \tag{4.6}$$
$$\leq \left[\frac{1}{4}\left[(x_1-a)^2 + (b-x_1)^2\right] + \left(\alpha_1 - \frac{a+x_1}{2}\right)^2 + \left(\alpha_2 - \frac{x_1+b}{2}\right)^2\right]\|f'\|_\infty.$$

As a simple computation shows that

$$\frac{1}{2}\left[(x_1-a)^2 + (b-x_1)^2\right] = \frac{1}{4}\,(b-a)^2 + \left(x_1 - \frac{a+b}{2}\right)^2$$

then we get the desired inequality (4.5). ∎

Corollary 4. *Let f be as above and $x_1 \in [a, b]$. Then we have Ostrowski's inequality:*

$$\left|\int_a^b f(x)dx - (b-a)f(x_1)\right| \leq \left[\frac{1}{4}(b-a)^2 + (x_1 - \frac{a+b}{2})^2\right]\|f'\|_\infty. \tag{4.7}$$

Proof. Indeed, if we put $\alpha_1 = a, \alpha_2 = b$ in the second part of the right membership of (4.5), we get

$$\left(\alpha_1 - \frac{a+x_1}{2}\right)^2 + \left(\alpha_2 - \frac{x_1+b}{2}\right)^2$$
$$= \left(a - \frac{a+x_1}{2}\right)^2 + \left(b - \frac{x_1+b}{2}\right)^2 = \frac{1}{4}\left[(x_1-a)^2 + (b-x_1)^2\right]$$
$$= \frac{1}{8}\,(b-a)^2 + \frac{1}{2}\left(x_1 - \frac{a+b}{2}\right)^2.$$

Now, using (4.5), we get (4.7). ∎

Remark 4. *If we choose* $x_1 = \frac{a+b}{2}$ *in (4.7), we get the "midpoint inequality"*

$$\left| \int_a^b f(x)dx - (b-a) f\left(\frac{a+b}{2}\right) \right| \leq \frac{1}{4}(b-a)^2 \|f'\|_\infty. \tag{4.8}$$

The following corollary generalizing Simpson's inequality holds:

Corollary 5. *Let* f *be as above and* $x_1 \in \left[\frac{5a+b}{6}, \frac{a+5b}{6}\right]$. *Then we have the inequality*

$$\begin{aligned} &\left| \int_a^b f(x)dx - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f(x_1)\right] \right| \\ \leq \; &\left[\frac{5}{36}(b-a)^2 + \left(x_1 - \frac{a+b}{2}\right)^2\right] \|f'\|_\infty. \end{aligned} \tag{4.9}$$

Proof. Indeed, if we put in the second membership of (4.5) $\alpha_1 = \frac{5a+b}{6}, \alpha_2 = \frac{a+5b}{6}$, we get

$$\begin{aligned} I \;:\; &= \frac{1}{8}(b-a)^2 + \frac{1}{2}\left(x_1 - \frac{a+b}{2}\right)^2 \\ &+ \left(\frac{5a+b}{6} - \frac{a+x_1}{2}\right)^2 + \left(\frac{a+5b}{6} - \frac{x_1+b}{2}\right)^2. \end{aligned}$$

Let us observe that

$$\begin{aligned} \frac{5a+b}{6} - \frac{a+x_1}{2} &= \frac{b+2a}{6} - \frac{1}{2}x_1, \\ \frac{a+5b}{6} - \frac{x_1+b}{2} &= \frac{a+2b}{6} - \frac{1}{2}x_1 \end{aligned}$$

and then

$$\begin{aligned} &\left(\frac{5a+b}{6} - \frac{a+x_1}{2}\right)^2 + \left(\frac{a+5b}{6} - \frac{x_1+b}{2}\right)^2 \\ = &\left(\frac{b+2a}{6} - \frac{1}{2}x_1\right)^2 + \left(\frac{a+2b}{6} - \frac{1}{2}x_1\right)^2 \\ = &\frac{1}{2}\left(\frac{a+2b}{6} - \frac{b+2a}{6}\right)^2 + 2\left(\frac{1}{2}x_1 - \frac{1}{2}\left(\frac{b+2a}{6} + \frac{a+2b}{6}\right)\right)^2 \\ = &\frac{1}{2}\frac{(b-a)^2}{36} + \frac{1}{2}\left(x_1 - \frac{a+b}{2}\right)^2 \end{aligned}$$

and then

$$\begin{aligned} I &= \frac{1}{8}(b-a)^2 + \frac{1}{2}\left(x_1 - \frac{a+b}{2}\right)^2 + \frac{1}{2}\frac{(b-a)^2}{36} + \frac{1}{2}\left(x_1 - \frac{a+b}{2}\right)^2 \\ &= \frac{5}{36}(b-a)^2 + \left(x_1 - \frac{a+b}{2}\right)^2 \end{aligned}$$

and the inequality (4.9) is proved. ∎

Remark 5. *Let observe that the best estimation we can get from (4.9) is that one for which* $x_1 = \frac{a+b}{2}$, *obtaining the "Simpson's inequality"*

$$\left|\int_a^b f(x)dx - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right]\right| \leq \frac{5}{36}(b-a)^2\|f'\|_\infty. \tag{4.10}$$

The following corollary also holds

Corollary 6. *Let* f *be as above and* $a \leq \alpha_1 \leq \frac{a+b}{2} \leq \alpha_2 \leq b$. *Then we have the inequality*

$$\begin{aligned} &\left|\int_a^b f(x)dx - \left[(\alpha_1 - a)f(a) + (\alpha_2 - \alpha_1)f\left(\frac{a+b}{2}\right) + (b-\alpha_2)f(b)\right]\right| \\ \leq\ & \left[\frac{1}{8}(b-a)^2 + \left(\alpha_1 - \frac{3a+b}{4}\right)^2 + \left(\alpha_2 - \frac{a+3b}{4}\right)^2\right]\|f'\|_\infty. \end{aligned} \tag{4.11}$$

The proof is obvious by Proposition 2 by choosing $x_1 = \frac{a+b}{2}$.

Remark 6. *The best estimation we can obtain from (4.11) is that one for which* $\alpha_1 = \frac{3a+b}{4}$ *and* $\alpha_2 = \frac{a+3b}{4}$ *obtaining the inequality*

$$\left|\int_a^b f(x)dx - \frac{b-a}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]\right| \leq \frac{1}{8}(b-a)^2\|f'\|_\infty. \tag{4.12}$$

The following proposition generalizes the *"three-eights rule"* of Newton-Cotes:

Proposition 3. *Let* f *be as above and* $a \leq x_1 \leq x_2 \leq b$ *and* $\alpha_1 \in [a, x_1]$, $\alpha_2 \in [x_1, x_2]$, $\alpha_3 \in [x_2, b]$. *Then we have the inequality*

$$\begin{aligned} &\left|\int_a^b f(x)dx - ((\alpha_1 - a)f(a) + (\alpha_2 - \alpha_1)f(x_1)\right. \\ &\left. + (\alpha_3 - \alpha_2)f(x_2) + (b - \alpha_3)f(b))\right| \\ \leq\ & \left[\frac{1}{4}\left[(x_1-a)^2 + (x_2-x_1)^2 + (b-x_2)^2\right]\right. \\ &\left. + \left(\alpha_1 - \frac{a+x_1}{2}\right)^2 + \left(\alpha_2 - \frac{x_1+x_2}{2}\right)^2 + \left(\alpha_3 - \frac{x_2+b}{2}\right)^2\right]\|f'\|_\infty. \end{aligned} \tag{4.13}$$

The proof is obvious by Theorem 3.

The next corollary contains a generalization of the *"three-eights rule"* of Newton-Cotes in the following way:

Corollary 7. *Let f be as above and $a \le \alpha_1 \le \frac{2a+b}{3} \le \alpha_2 \le \frac{2b+a}{3} \le \alpha_3 \le b$. Then we have the inequality:*

$$\begin{aligned}(4.14)\quad & \left|\int_a^b f(x)dx - \left[(\alpha_1 - a)f(a) + (\alpha_2 - \alpha_1) f\left(\frac{2a+b}{3}\right)\right.\right. \\ & \left.\left. +(\alpha_3 - \alpha_2) f\left(\frac{a+2b}{3}\right) + (b-\alpha_3)f(b)\right]\right| \\ \le & \left[\frac{(b-a)^2}{12} + \left(\alpha_1 - \frac{5a+b}{6}\right)^2\right. \\ & \left. + \left(\alpha_2 - \frac{a+b}{2}\right)^2 + \left(\alpha_3 - \frac{a+5b}{6}\right)^2\right] \|f'\|_\infty .\end{aligned}$$

The proof follows by the above proposition by choosing $x_1 = \frac{2a+b}{3}$ and $x_2 = \frac{a+2b}{3}$.

Remark 7. *a) If we choose $\alpha_1 = \frac{b+7a}{8}, \alpha_2 = \frac{a+b}{2}$ and $\alpha_3 = \frac{a+7b}{8}$ in (4.14), then we get the "three-eights rule" of Newton-Cotes*

$$\begin{aligned}(4.15)\quad & \left|\int_a^b f(x)dx - \frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right]\right| \\ \le & \frac{25}{288}(b-a)^2 \|f'\|_\infty .\end{aligned}$$

b) The best estimation we can get from (4.14) is that one for which $\alpha_1 = \frac{5a+b}{6}, \alpha_2 = \frac{a+b}{2}$, $\alpha_3 = \frac{a+5b}{6}$ obtaining the inequality

$$\begin{aligned}(4.16)\quad & \left|\int_a^b f(x)dx - \frac{b-a}{6}\left[f(a) + 2f\left(\frac{2a+b}{3}\right) + 2f\left(\frac{a+2b}{3}\right) + f(b)\right]\right| \\ \le & \frac{1}{12}(b-a)^2 \|f'\|_\infty .\end{aligned}$$

5. Some Composite Quadrature Formulae

Let us consider the partitioning of the interval $[a,b]$ given by $\Delta_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ and put $h_i := x_{i+1} - x_i (i = 0, ..., n-1)$.

The following theorem holds:

Theorem 5. *Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous on $[a,b]$ and $k \ge 1$. Then we have the composite quadrature formula*

$$(5.1)\quad \int_a^b f(x)dx = A_k(\Delta_n, f) + R_k(\Delta_n, f),$$

where

$$A_k(\Delta_n, f) := \frac{1}{k}\left[T(\Delta_n, f) + \sum_{i=0}^{n-1}\sum_{j=1}^{k-1} f\left(\frac{(k-j)x_i + jx_{i+1}}{k}\right)h_i\right] \tag{5.2}$$

and

$$T(\Delta_n, f) := \frac{1}{2}\sum_{i=0}^{n-1}\left[f(x_i) + f(x_{i+1})\right]h_i \tag{5.3}$$

is the trapezoid quadrature formula.

The remainder $R_k(\Delta_n, f)$ satisfies the estimation

$$|R_k(\Delta_n, f)| \le \frac{1}{4k}\|f'\|_\infty \sum_{i=0}^{n-1} h_i^2. \tag{5.4}$$

Proof. Applying Corollary 2 on the intervals $[x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ we get

$$\begin{aligned}&\left|\int_{x_i}^{x_{i+1}} f(x)dx - \left[\frac{1}{k}\cdot\frac{f(x_i) + f(x_{i+1})}{2}h_i + \frac{h_i}{k}\sum_{j=1}^{k-1} f\left(\frac{(k-j)x_i + jx_{i+1}}{k}\right)\right]\right| \\ \le\ & \frac{1}{4k}h_i^2\|f'\|_\infty.\end{aligned}$$

Summing over i from 0 to $n-1$ and using the generalized triangle inequality, we get the desired estimation (5.4). ∎

The following corollary holds:

Corollary 8. *Let f, Δ_n be as above. Then we have the quadrature formula*

$$\int_a^b f(x)dx = \frac{1}{2}\left[T(\Delta_n, f) + M(\Delta_n, f)\right] + R_2(\Delta_n, f),$$

where $M(\Delta_n, f)$ is the midpoint rule, we recall,

$$M(\Delta_n, f) := \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right)h_i.$$

The remainder satisfies the estimation:

$$|R_2(\Delta_n, f)| \le \frac{1}{8}\|f'\|_\infty \sum_{i=0}^{n-1} h_i^2. \tag{5.5}$$

The following corollary also holds:

Corollary 9. *Let f, Δ_n be as above. Then we have the quadrature formula*

$$\begin{aligned} \int_a^b f(x)dx &= \frac{1}{3}\left[T(\Delta_n, f) + \sum_{i=0}^{n-1} f\left(\frac{2x_i + x_{i+1}}{3}\right) h_i \right. \\ &\left. + \sum_{i=0}^{n-1} f\left(\frac{x_i + 2x_{i+1}}{3}\right) h_i \right] + R_3\left(\Delta_n, f\right). \end{aligned} \tag{5.6}$$

The remainder $R_3(\Delta_n, f)$ satisfies the estimation:

$$|R_2(\Delta_n, f)| \leq \frac{1}{12}\|f'\|_\infty \sum_{i=0}^{n-1} h_i^2. \tag{5.7}$$

The following theorem also holds:

Theorem 6. *Let f and Δ_n be as above. Suppose that $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$. Then we have the formula*

$$\int_a^b f(x)dx = \sum_{i=0}^{n-1} \left[(\xi_i - x_i) f(x_i) + (x_{i+1} - \xi_i) f(x_{i+1})\right] + R(\boldsymbol{\xi}_n, \Delta_n, f). \tag{5.8}$$

The remainder $R(\boldsymbol{\xi}, \Delta_n, f)$ satisfies the estimation:

$$\begin{aligned} |R(\boldsymbol{\xi}, \Delta_n, f)| &\leq \left[\frac{1}{4}\sum_{i=0}^{n-1} h_i^2 + \sum_{i=0}^{n-1}\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2\right]\|f'\|_\infty \\ &\leq \frac{1}{2}\|f'\|_\infty \sum_{i=0}^{n-1} h_i^2. \end{aligned} \tag{5.9}$$

Proof. Apply Proposition 1 on the intervals $[x_i, x_{i+1}](i = 0, ..., n-1)$ to get

$$\begin{aligned} &\left|\int_{x_i}^{x_{i+1}} f(x)dx - [(\xi_i - x_i)f(x_i) + (x_{i+1} - \xi_i)f(x_{i+1})]\right| \\ \leq &\left[\frac{1}{4}h_i^2 + \left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2\right]\|f'\|_\infty \leq \frac{1}{2}\|f'\|_\infty h_i^2. \end{aligned}$$

Summing over i from 0 to $n-1$ and using the generalized triangle inequality we deduce the desired estimation (5.9). ∎

Corollary 10. *Let f and Δ_n be as above. Then we have*

1) the "left rectangle rule":

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1} f(x_i)\,h_i + R_l\left(\Delta_n, f\right); \tag{5.10}$$

2) the "right rectangle rule":

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1} f(x_{i+1})\,h_i + R_r(\Delta_n, f)\,; \tag{5.11}$$

3) the "trapezoid rule":

$$\int_a^b f(x)\,dx = T(\Delta_n, f) + R_T(\Delta_n, f) \tag{5.12}$$

where

$$|R_l(\Delta_n, f)|\,, |R_r(\Delta_n, f)| \le \frac{1}{2}\|f'\|_\infty \sum_{i=0}^{n-1} h_i^2$$

and

$$|R_T(\Delta_n, f)| \le \frac{1}{4}\|f'\|_\infty \sum_{i=0}^{n-1} h_i^2.$$

The following theorem also holds.

Theorem 7. *Let f and Δ_n be as above. If $x_i \le \alpha_i^{(1)} \le \xi_i \le \alpha_i^{(2)} \le x_{i+1}$ $(i = 0, ..., n-1)$, then we have the formula:*

$$\begin{aligned} &\int_a^b f(x)\,dx \\ = &\sum_{i=0}^{n-1} \left(\alpha_i^{(1)} - x_i\right) f(x_i) + \sum_{i=0}^{n-1} \left(\alpha_i^{(2)} - \alpha_i^{(1)}\right) f(\xi_i) \\ &+ \sum_{i=0}^{n-1} \left(x_{i+1} - \alpha_i^{(2)}\right) f(x_{i+1}) + R\left(\boldsymbol{\xi}, \boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}, \Delta_n, f\right), \end{aligned} \tag{5.13}$$

where the remainder satisfies the estimation

$$\begin{aligned} &\left|R\left(\boldsymbol{\xi}, \boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}, \Delta_n, f\right)\right| \\ \le &\left[\frac{1}{8}\sum_{i=0}^{n-1} h_i^2 + \frac{1}{2}\sum_{i=0}^{n-1}\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2 \right. \\ &\left. + \sum_{i=0}^{n-1}\left(\alpha_i^{(1)} - \frac{x_i + \xi_i}{2}\right)^2 + \sum_{i=0}^{n-1}\left(\alpha_i^{(2)} - \frac{\xi_i + x_{i+1}}{2}\right)^2\right] \|f'\|_\infty. \end{aligned} \tag{5.14}$$

The proof follows from Proposition 2 applied on the intervals $[x_i, x_{i+1}]$ $(i = 0, ..., n-1)$.

The following corollary of the above theorem also holds

Corollary 11. *Let f, Δ_n be as above and $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$. Then we have the formula of Riemann type:*

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1} f(\xi_i)\,h_i + R_R(\boldsymbol{\xi}, \Delta_n, f). \tag{5.15}$$

The remainder $R_R(\boldsymbol{\xi}, \Delta_n, f)$ satisfies the estimation

$$\begin{aligned} &|R_R(\boldsymbol{\xi}, \Delta_n, f)| \\ \leq\ &\left[\frac{1}{4}\sum_{i=0}^{n-1} h_i^2 + \sum_{i=0}^{n-1}\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2\right]\|f'\|_\infty \leq \frac{1}{2}\|f'\|_\infty \sum_{i=0}^{n-1} h_i^2. \end{aligned} \tag{5.16}$$

Remark 8. *If we choose in (5.15), $\xi_i = \frac{x_i+x_{i+1}}{2}$, then we get the midpoint quadrature formula*

$$\int_a^b f(x)\,dx = M(\Delta_n, f) + R_M(\Delta_n, f),$$

where

$$|R_M(\Delta_n, f)| \leq \frac{1}{4}\|f'\|_\infty \sum_{i=0}^{n-1} h_i^2.$$

The following corollary also holds

Corollary 12. *Let f, Δ_n be as above and $\xi_i \in \left[\frac{5x_i+x_{i+1}}{6}, \frac{x_i+5x_{i+1}}{6}\right]$ $(i = 0, ..., n-1)$. Then we have the formula*

$$\int_a^b f(x)\,dx = \frac{1}{6}\sum_{i=0}^{n-1}\left[f(x_i) + f(x_{i+1})\right]h_i + \frac{2}{3}\sum_{i=0}^{n-1} f(\xi_i)\,h_i + R_S(\boldsymbol{\xi}, \Delta_n, f). \tag{5.17}$$

The remainder $R_S(\boldsymbol{\xi}, \Delta_n, f)$ satisfies the estimation:

$$|R_S(\boldsymbol{\xi}, \Delta_n, f)| \leq \left[\frac{5}{36}\sum_{i=0}^{n-1} h_i^2 + \sum_{i=0}^{n-1}\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2\right]\| f'\|_\infty \tag{5.18}$$

Remark 9. *If we choose above $\xi_i = \frac{x_i+x_{i+1}}{2}$ $(i = 0, ..., n-1)$, then we get*

$$\begin{aligned} &\int_a^b f(x)\,dx \\ =\ &\frac{1}{6}\sum_{i=0}^{n-1}\left[f(x_i) + f(x_{i+1})\right]h_i + \frac{2}{3}\sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right)h_i + R_S(\Delta_n, f), \end{aligned} \tag{5.19}$$

where the remainder satisfies the estimation

$$|R_S(\Delta_n, f)| \leq \frac{5}{36}\|f'\|_\infty \sum_{i=0}^{n-1} h_i^2. \tag{5.20}$$

The following corollary also holds.

Corollary 13. *Let* f, Δ_n *be as above and* $x_i \leq \alpha_i^{(1)} \leq \frac{x_i + x_{i+1}}{2} \leq \alpha_i^{(2)} \leq x_{i+1}$ $(i = 0, ..., n-1)$. *Then we have the formula*

$$\begin{aligned} & \int_a^b f(x)\,dx \\ = & \sum_{i=0}^{n-1} \left(\alpha_i^{(1)} - x_i\right) f(x_i) + \sum_{i=0}^{n-1} \left(\alpha_i^{(2)} - \alpha_i^{(1)}\right) f\left(\frac{x_i + x_{i+1}}{2}\right) \\ & + \sum_{i=0}^{n-1} \left(x_{i+1} - \alpha_i^{(2)}\right) f(x_{i+1}) + R_B\left(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}, \Delta_n, f\right). \end{aligned} \tag{5.21}$$

The remainder satisfies the estimation:

$$\begin{aligned} & \left| R_B\left(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}, \Delta_n, f\right) \right| \\ \leq & \left[\frac{1}{8} \sum_{i=0}^{n-1} h_i^2 + \sum_{i=0}^{n-1} \left(\alpha_i^{(1)} - \frac{3x_i + x_{i+1}}{4}\right)^2 + \sum_{i=0}^{n-1} \left(\alpha_i^{(2)} - \frac{x_i + 3x_{i+1}}{4}\right)^2 \right] \|f'\|_\infty. \end{aligned}$$

Finally, the following theorem holds:

Theorem 8. *Let* f, Δ_n *be as above and* $x_i \leq \xi_i^{(1)} \leq \xi_i^{(2)} \leq x_{i+1}$ *and* $\alpha_i^{(1)} \in \left[x_i, \xi_i^{(1)}\right]$, $\alpha_i^{(2)} \in \left[\xi_i^{(1)}, \xi_i^{(2)}\right]$ *and* $\alpha_i^{(3)} \in \left[\xi_i^{(2)}, x_{i+1}\right]$ *for* $i = 0, ..., n-1$. *Then we have the formula:*

$$\begin{aligned} & \int_a^b f(x)\,dx \\ = & \sum_{i=0}^{n-1} \left(\alpha_i^{(1)} - x_i\right) f(x_i) + \sum_{i=0}^{n-1} \left(\alpha_i^{(2)} - \alpha_i^{(1)}\right) f\left(\xi_i^{(1)}\right) \\ & + \sum_{i=0}^{n-1} \left(\alpha_i^{(3)} - \alpha_i^{(2)}\right) f\left(\xi_i^{(2)}\right) + \sum_{i=0}^{n-1} \left(x_{i+1} - \alpha_i^{(3)}\right) f(x_{i+1}) \\ & + R\left(\boldsymbol{\xi}^{(1)}, \boldsymbol{\xi}^{(2)}, \boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}, \boldsymbol{\alpha}^{(3)}, \Delta_n, f\right) \end{aligned} \tag{5.22}$$

and the remainder satisfies the estimation

$$\begin{aligned} & \left|R\left(\boldsymbol{\xi}^{(1)},\boldsymbol{\xi}^{(2)},\boldsymbol{\alpha}^{(1)},\boldsymbol{\alpha}^{(2)},\boldsymbol{\alpha}^{(3)},\Delta_n,f\right)\right| \\ \leq & \left[\frac{1}{4}\left[\sum_{i=0}^{n-1}\left(\xi_i^{(1)}-x_i\right)^2+\sum_{i=0}^{n-1}\left(\xi_i^{(2)}-\xi_i^{(1)}\right)^2+\sum_{i=0}^{n-1}\left(x_{i+1}-\xi_i^{(1)}\right)^2\right]\right. \\ & +\sum_{i=0}^{n-1}\left(\alpha_i^{(1)}-\frac{x_i+\xi_i^{(1)}}{2}\right)^2+\sum_{i=0}^{n-1}\left(\alpha_i^{(2)}-\frac{\xi_i^{(1)}+\xi_i^{(2)}}{2}\right) \\ & \left.+\sum_{i=0}^{n-1}\left(\alpha_i^{(3)}-\frac{\xi_i^{(2)}+x_{i+1}}{2}\right)\right]\|f'\|_\infty. \end{aligned}$$

The proof follows from Theorem 3. We shall omit the details.

Remark 10. *We note only that if we choose*

$$\alpha_i^{(1)}=\frac{x_{i+1}+7x_i}{8},\alpha_i^{(2)}=\frac{x_i+x_{i+1}}{2},$$

$$\alpha_i^{(3)}=\frac{x_i+7x_{i+1}}{8},\xi_i^{(1)}=\frac{2x_i+x_{i+1}}{3}$$

and

$$\xi_i^{(2)}=\frac{x_i+2x_{i+1}}{3}\,(i=0,...,n-2)$$

then we get the "three-eights formula" of Newton-Cotes:

$$\begin{aligned} \int_a^b f(x)\,dx &= \frac{1}{8}\sum_{i=0}^{n-1}\left[f(x_i)+3f\left(\frac{2x_i+x_{i+1}}{3}\right)+3f\left(\frac{x_i+2x_{i+1}}{3}\right)+f(x_{i+1})\right]h_i \\ & +R_{N-C}(\Delta_n,f), \end{aligned}$$

where the remainder satisfies the estimation

$$|R_{N-C}(\Delta_n,f)|\leq\frac{25}{288}\|f'\|_\infty\sum_{i=0}^{n-1}h_i^2.$$

References

[1] Dragomir, S.S. and Wang, S. : Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules, *Appl. Math. Lett.* **11**(1998)(1), 105-109.

[2] Mitrinović, D.S.; Pečarić, J.E. and Fink A.M.: *Inequalities for Functions and Their Integrals and Derivatives*, Kluwer Academic, Dordrecht, 1994.

School of Communications and Informatics
Victoria University of Technology
PO Box 14428
Melbourne City MC
Victoria 8001, Australia
email sever@matilda.vu.edu.au
http://rgmia.vu.edu.au/SSDragomirWeb.html