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SUMS AND JOINS OF FUZZY FINITE STATE MACHINES

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Abstract: We introduce sums and joins of fuzzy finite state machines and investigate their algebraic structures.

1. Introduction

Since Wee [9] in 1967 introduced the concept of fuzzy automata following Zadeh [10], fuzzy automata theory has been developed by many researchers. Recently Malik et al. [5-8] introduced the concepts of fuzzy finite state machines and fuzzy transformation semigroups based on Wee's concept [9] of fuzzy automata and related concepts and applied algebraic technique. Cho et al. [2,4] introduced the notion of a T-fuzzy finite state machine that is an extension of a fuzzy finite state machine. Even if $T = \wedge$, our notion is different from the notion of Malik et al. [6,7]. In this paper, we introduce sums and joins of fuzzy finite state machines that are generalizations of crisp concepts in algebraic automata theory and investigate their algebraic structures.

For the terminology in (crisp) algebraic automata theory, we refer to [3].

2. Preliminaries

Definition 2.1 [1,4]. *A triple $\mathcal{M} = (Q, X, \tau)$ where Q and X are finite nonempty sets and τ is a fuzzy subset of $Q \times X \times Q$, i.e., τ is a function from $Q \times X \times Q$ to* $[0, 1]$, *is called a fuzzy finite state machine if $\sum_{q\in Q} \tau(p, a, q) \leq 1$ for all $p \in Q$ and $a \in X$.*

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Definition 2.2 [4]. *Let $\mathcal{M}_1 = (Q_1, X_1, \tau_1)$ and $\mathcal{M}_2 = (Q_2, X_2, \tau_2)$ be fuzzy finite state machines. Let $\alpha : Q_1 \longrightarrow Q_2$ and $\beta : X_1 \longrightarrow X_2$ be mappings. Then the pair (α, β) is called a fuzzy finite state machine homomorphism (which is written by (α, β)) if*

$$\tau_1(p, a, q) \leq \tau_2(\alpha(p), \beta(a), \alpha(q)), p, q \in Q_1, a \in X_1$$

The homomorphism $(\alpha, \beta) : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ is called isomorphism if α and β are bijective respectively.

Definition 2.3 [1,4]. *Let $\mathcal{M}_1 = (Q_1, X_1, \tau_1)$ and $\mathcal{M}_2 = (Q_2, X_2, \tau_2)$ be fuzzy finite state machines. If $\xi : X_1 \longrightarrow X_2$ is a function and $\eta : Q_2 \longrightarrow Q_1$ is a surjective partial function such that $\tau_1(\eta(p), a, \eta(q)) \leq \tau_2(p, \xi(a), q)$ for all p, q in the domain of η and $a \in X_1$, then we say that (η, ξ) is a covering of $\mathcal{M}_1$ by $\mathcal{M}_2$ and that $\mathcal{M}_2$ covers $\mathcal{M}_1$ and denote by $\mathcal{M}_1 \leq \mathcal{M}_2$. Moreover, if the inequality always turns out equality, then we say that (η, ξ) is a complete covering of $\mathcal{M}_1$ by $\mathcal{M}_2$ and that $\mathcal{M}_2$ completely covers $\mathcal{M}_1$ and denote by $\mathcal{M}_1 \leq_c \mathcal{M}_2$.*

Example 2.4. *Let $\mathcal{M} = (Q, X, \tau)$ be a fuzzy finite state machine. Define an equivalence relation $\sim$ on X by $a \sim b$ if and only if $\tau(p, a, q) = \tau(p, b, q)$ for all $p, q \in Q$. Construct a fuzzy finite state machine $\mathcal{M}_1 = (Q, X/\sim, \tau^{\sim})$ by defining $\tau^{\sim}(p, [a], q) = \tau(p, a, q)$. Now define $\xi : X \longrightarrow X/\sim$ by $\xi(a) = [a]$ and $\eta = 1_Q$. Then (η, ξ) is a complete covering of $\mathcal{M}$ by $\mathcal{M}_1$ clearly.*

Proposition 2.5. *Let $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ be fuzzy finite state machines. If $\mathcal{M}_1 \leq \mathcal{M}_2$ [resp. $\mathcal{M}_1 \leq_c \mathcal{M}_2$] and $\mathcal{M}_2 \leq \mathcal{M}_3$ [$\mathcal{M}_2 \leq_c \mathcal{M}_3$], then $\mathcal{M}_1 \leq \mathcal{M}_3$ [$\mathcal{M}_1 \leq_c \mathcal{M}_3$].*

Proof. It is straightforward.

3. SEVERAL PRODUCTS OF FUZZY FINITE STATE MACHINES

Several products of finite state machines are in [3]. Some of these products have been fuzzified in [1], [4] and [6]. In this section we introduce sums and joins of fuzzy finite state machines.

Definition 3.1 [1,6]. *Let $\mathcal{M}_1 = (Q_1, X_1, \tau_1)$ and $\mathcal{M}_2 = (Q_2, X_2, \tau_2)$ be fuzzy finite state machines. The cascade product $\mathcal{M}_1 \omega \mathcal{M}_2$ of $\mathcal{M}_1$ and $\mathcal{M}_2$ with respect to $\omega : Q_2 \times X_2 \longrightarrow X_1$ is the fuzzy finite state machine $(Q_1 \times Q_2, X_2, \tau_1 \omega \tau_2)$ with*

$$(\tau_1 \omega \tau_2)((p_1, p_2), b, (q_1, q_2)) = \wedge(\tau_1(p_1, \omega(p_2, b), q_1), \tau_2(p_2, b, q_2)).$$

Definition 3.2 [1,6]. *Let $\mathcal{M}_1 = (Q_1, X_1, \tau_1)$ and $\mathcal{M}_2 = (Q_2, X_2, \tau_2)$ be fuzzy finite state machines. The wreath product $\mathcal{M}_1 \circ \mathcal{M}_2$ of $\mathcal{M}_1$ and $\mathcal{M}_2$ is the fuzzy finite state machine $(Q_1 \times Q_2, X_1^{Q_2} \times X_2, \tau_1 \circ \tau_2)$ with*

$$(\tau_1 \circ \tau_2)((p_1, p_2), (f, b), (q_1, q_2)) = \wedge(\tau_1(p_1, f(p_2), q_1), \tau_2(p_2, b, q_2)).$$

Now we introduce sums and joins of fuzzy finite state machines.

Definition 3.3. *Let $\mathcal{M}_1 = (Q_1, X_1, \tau_1)$ and $\mathcal{M}_2 = (Q_2, X_2, \tau_2)$ be fuzzy finite state machines, where $Q_1 \cap Q_2 = \emptyset$ and $X_1 \cap X_2 = \emptyset$. The join $\mathcal{M}_1 \vee \mathcal{M}_2$ of $\mathcal{M}_1$ and $\mathcal{M}_2$ is the fuzzy finite state machine $(Q_1 \cup Q_2, X_1 \cup X_2, \tau_1 \vee \tau_2)$ with*

$$(\tau_1 \vee \tau_2)(p, a, q) = \begin{cases} \tau_1(p, a, q) \text{ if } (p, a, q) \in Q_1 \times X_1 \times Q_1 \\ \tau_2(p, a, q) \text{ if } (p, a, q) \in Q_2 \times X_2 \times Q_2 \\ 0, \text{ otherwise} \end{cases}$$

Definition 3.4. *Let $\mathcal{M}_1 = (Q_1, X_1, \tau_1)$ and $\mathcal{M}_2 = (Q_2, X_2, \tau_2)$ be fuzzy finite state machines, where $Q_1 \cap Q_2 = \emptyset$ and $X_1 \cap X_2 = \emptyset$. The join* $\mathcal{M}_1 \vee^* \mathcal{M}_2$ of $\mathcal{M}_1$ and $\mathcal{M}_2$ is the fuzzy finite state machine $(Q_1 \cup Q_2, X_1 \cup X_2, \tau_1 \vee^* \tau_2)$ with*

$$(\tau_1 \vee^* \tau_2)(p, a, q) = \begin{cases} \tau_1(p, a, q) \text{ if } (p, a, q) \in Q_1 \times X_1 \times Q_1 \\ \tau_2(p, a, q) \text{ if } (p, a, q) \in Q_2 \times X_2 \times Q_2 \\ 1 \text{ if } (p, a, q) \in (Q_1 \times X_1 \times Q_2) \cup (Q_2 \times X_2 \times Q_1) \\ 0, \text{ otherwise} \end{cases}$$

Definition 3.5. *Let $\mathcal{M}_1 = (Q_1, X_1, \tau_1)$ and $\mathcal{M}_2 = (Q_2, X_2, \tau_2)$ be fuzzy finite state machines, where $Q_1 \cap Q_2 = \emptyset$. The sum $\mathcal{M}_1 + \mathcal{M}_2$ of $\mathcal{M}_1$ and $\mathcal{M}_2$ is the fuzzy finite state machine $(Q_1 \cup Q_2, X_1 \times X_2, \tau_1 + \tau_2)$ with*

$$(\tau_1 + \tau_2)(p, (a, b), q) = \begin{cases} \tau_1(p, a, q) \text{ if } p, q \in Q_1 \\ \tau_2(p, b, q) \text{ if } p, q \in Q_2 \\ 0, \text{ otherwise} \end{cases}$$

4. Associative properties

The following proposition is in [4].

Proposition 4.1. *Let $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{M}_3$ be fuzzy finite state machines. Then the following are hold:*

(i) $(\mathcal{M}_1 \wedge \mathcal{M}_2) \wedge \mathcal{M}_3 = \mathcal{M}_1 \wedge (\mathcal{M}_2 \wedge \mathcal{M}_3)$.

(ii) $(\mathcal{M}_1 \times \mathcal{M}_2) \times \mathcal{M}_3 = \mathcal{M}_1 \times (\mathcal{M}_2 \times \mathcal{M}_3)$.

Now we prove that wreath product, join and sum of fuzzy finite state machines are associative.

Theorem 4.2. *Let $\mathcal{M}_1 = (Q_1, X_1, \tau_1)$, $\mathcal{M}_2 = (Q_2, X_2, \tau_2)$ and $\mathcal{M}_3 = (Q_3, X_3, \tau_3)$ be fuzzy finite state machines. Then the following are hold:*

(i) $(\mathcal{M}_1 \circ \mathcal{M}_2) \circ \mathcal{M}_3 \cong \mathcal{M}_1 \circ (\mathcal{M}_2 \circ \mathcal{M}_3)$

(ii) $(\mathcal{M}_1 \vee \mathcal{M}_2) \vee \mathcal{M}_3 \cong \mathcal{M}_1 \vee (\mathcal{M}_2 \vee \mathcal{M}_3)$, *where* $Q_1 \cap Q_2 \cap Q_3 = \emptyset$ *and* $X_1 \cap X_2 \cap X_3 = \emptyset$

(iii) $(\mathcal{M}_1 + \mathcal{M}_2) + \mathcal{M}_3 \cong \mathcal{M}_1 + (\mathcal{M}_2 + \mathcal{M}_3)$, *where* $Q_1 \cap Q_2 \cap Q_3 = \emptyset$

Proof. (i) Let $\alpha : (Q_1 \times Q_2) \times Q_3 \longrightarrow Q_1 \times (Q_2 \times Q_3)$ be the natural mapping. Then α is a bijective mapping. Let $g_1 : X_1^{Q_2} \times X_2 \longrightarrow X_1^{Q_2}$ and $g_2 : X_1^{Q_2} \times X_2 \longrightarrow X_2$ be the natural projection mappings. Define a mapping $f : Q_3 \longrightarrow X_1^{Q_2} \times X_2$ by $f(p_3) = (s, b_2)$ and let $f_1 = g_1 \circ f$ and $f_2 = g_2 \circ f$. Define $\beta : (X_1^{Q_2} \times X_2)^{Q_3} \times X_3 \longrightarrow X_1^{Q_2 \times Q_3} \times (X_2^{Q_3} \times X_3)$ by $\beta((f, b_3)) = (h, (f_2, b_3))$, where $h : Q_2 \times Q_3 \longrightarrow X_1$ by $h((p_2, p_3)) = f_1(p_3)(p_2)$. We can easily show that β is injective. Let $(w, (v, b_3)) \in X_1^{Q_2 \times Q_3} \times (X_2^{Q_3} \times X_3)$ and define $u : Q_3 \longrightarrow X_1^{Q_2 \times Q_3}$ by $u(p_3) = (v^{p_3}, w(p_3))$ where $v^{p_3}(p_2) = v(p_2, p_3)$. Then $\beta((u, b_3)) = (w, (v, b_3))$ and thus β is surjective. Now

$$
\begin{aligned}
&(\tau_1 \circ (\tau_2 \circ \tau_3))(\alpha(((p_1, p_2), p_3), \beta((f, b_3)), \alpha(((q_1, q_2), q_3)))\\
=&(\tau_1 \circ (\tau_2 \circ \tau_3))(\alpha(((p_1, p_2), p_3), (h, (f_2, b_3)), \alpha(((q_1, q_2), q_3)))\\
=& \wedge (\tau_1(p_1, h(p_2, p_3), q_1), (\tau_2 \circ \tau_3)((p_2, p_3), (f_2, b_3), (q_2, q_3)))\\
=& \wedge (\tau_1(p_1, h(p_2, p_3), q_1), \wedge(\tau_2(p_2, f_2(p_3), q_2), \tau_3(p_3, b_3, q_3)))\\
=& \wedge (\wedge(\tau_1(p_1, h(p_2, p_3), q_1), \tau_2(p_2, f_2(p_3), q_2)), \tau_3(p_3, b_3, q_3)))\\
=& \wedge (\wedge(\tau_1(p_1, f_1(p_3)(p_2), q_1), \tau_2(p_2, f_2(p_3), q_2)), \tau_3(p_3, b_3, q_3)))\\
=& \wedge ((\tau_1 \circ \tau_2)((p_1, p_2), (f_1(p_3), f_2(p_3)), (q_1, q_2)), , \tau_3(p_3, b_3, q_3)))\\
=& \wedge ((\tau_1 \circ \tau_2)((p_1, p_2), (s, b_2), (q_1, q_2)), \tau_3(p_3, b_3, q_3))) \qquad \text{where } f(p_3) = (s, b_2)\\
=& \wedge ((\tau_1 \circ \tau_2)((p_1, p_2), f(p_3), (q_1, q_2)), \tau_3(p_3, b_3, q_3)))\\
=&((\tau_1 \circ \tau_2) \circ \tau_3)(((p_1, p_2), p_3), (f, b_3), ((q_1, q_2), q_3)))
\end{aligned}
$$

(ii) Let α be an identity mapping on $Q_1 \cup Q_2 \cup Q_3$ and β be an identity mapping on $X_1 \cup X_2 \cup X_3$. Then (α, β) be a required isomorphism.

(iii) Let α be an identity mapping on $Q_1 \cup Q_2 \cup Q_3$ and $\beta : (X_1 \times X_2) \times X_3 \longrightarrow X_1 \times (X_2 \times X_3)$ be the natural mapping. Then (α, β) be a required isomorphism.

Remark. $\vee^*$ *is not an associative operation.*

5. Coverings

The following proposition is in [1,4].

Proposition 5.1. *Let $\mathcal{M}_1 = (Q_1, X_1, \tau_1)$ and $\mathcal{M}_2 = (Q_2, X_2, \tau_2)$ be fuzzy finite state machines. Then*

(i) $\mathcal{M}_1 \wedge \mathcal{M}_2 \leq_c \mathcal{M}_1 \times \mathcal{M}_2$ where $X_1 = X_2$.

(ii) $\mathcal{M}_1 \omega \mathcal{M}_2 \leq_c \mathcal{M}_1 \circ \mathcal{M}_2$.

Proposition 5.2. *Let $\mathcal{M}_1 = (Q_1, X_1, \tau_1)$ and $\mathcal{M}_2 = (Q_2, X_2, \tau_2)$ be fuzzy finite state machines such that $Q_1 \cap Q_2 = \emptyset$ and $X_1 \cap X_2 = \emptyset$. Then*

(i) $\mathcal{M}_1 \leq \mathcal{M}_1 \vee \mathcal{M}_2$

(ii) $\mathcal{M}_1 \leq \mathcal{M}_1 \vee^* \mathcal{M}_2$

Proof. We only prove (i).

(i) Let $\eta : Q_1 \cup Q_2 \longrightarrow Q_1$ be a partial surjective function defined by $\eta(p_1) = p_1$, where $p_1 \in Q_1$. And $\xi : X_1 \longrightarrow X_1 \cup X_2$ be the natural projection. Then (η, ξ) is a required covering of $\mathcal{M}_1$ by $\mathcal{M}_1 \vee \mathcal{M}_2$.

Theorem 5.3. *Let $\mathcal{M}_1 = (Q_1, X, \tau_1)$ and $\mathcal{M}_2 = (Q_2, X, \tau_2)$ be fuzzy finite state machines. Then*

(i) $\mathcal{M}_1 \vee \mathcal{M}_2 \leq \mathcal{M}_1 \vee^* \mathcal{M}_2$.

(ii) $\mathcal{M}_1 + \mathcal{M}_2 \leq \mathcal{M}_1 +^* \mathcal{M}_2$.

Proof. (i) Let η and ξ identity mappings on $Q_1 \cup Q_2$ and $X_1 \cup X_2$ respectively.

Case (a): If $(p, a, q) \in Q_1 \times X_1 \times Q_1$, then $(\tau_1 \vee \tau_2)(\eta(p), a, \eta(q)) = \tau_1(p, a, q) = (\tau_1 \vee^* \tau_2)(p, \xi(a), q)$.

Case (b): If $(p, a, q) \in Q_2 \times X_2 \times Q_2$, then $(\tau_1 \vee \tau_2)(\eta(p), a, \eta(q)) = \tau_2(p, a, q) = (\tau_1 \vee^* \tau_2)(p, \xi(a), q)$.

Case (c): If $(p, a, q) \in (Q_1 \times X_1 \times Q_2) \cup (Q_2 \times X_2 \times Q_1)$, then $(\tau_1 \vee \tau_2)(\eta(p), a, \eta(q)) = (\tau_1 \vee \tau_2)(p, a, q) = 0 \leq 1 = (\tau_1 \vee^* \tau_2)(p, \xi(a), q)$.

(ii) The proof is similar to the proof of (i).

Theorem 5.4. *Let $\mathcal{M}_1 = (Q_1, X_1, \tau_1)$, $\mathcal{M}_2 = (Q_2, X_2, \tau_2)$ and $\mathcal{M}_3 = (Q_3, X_3, \tau_3)$ be fuzzy finite state machines such that $\mathcal{M}_1 \leq \mathcal{M}_2$. Then*

(i) $\mathcal{M}_1 \vee \mathcal{M}_3 \leq \mathcal{M}_2 \vee \mathcal{M}_3$.

(ii) $\mathcal{M}_1 \vee^* \mathcal{M}_3 \leq \mathcal{M}_2 \vee^* \mathcal{M}_3$.

(iii) $\mathcal{M}_1 + \mathcal{M}_3 \leq \mathcal{M}_2 + \mathcal{M}_3$.

Proof. We only show that (i) is hold. Since $\mathcal{M}_1 \leq \mathcal{M}_2$, there exist a partial surjective mapping $\eta : Q_2 \longrightarrow Q_1$ and a mapping $\xi : X_1 \longrightarrow X_2$ such that $\tau_1(\eta(p), a, \eta(q)) \leq \tau_2(p, \xi(a), q)$. Define $\eta' : Q_2 \cup Q_3 \longrightarrow Q_1 \cup Q_3$ by $\eta'(p) = \begin{cases} p \text{ if } p \in Q_3 \\ \eta(p) \text{ if } p \in Q_2 \end{cases}$ and $\xi' : X_1 \cup X_3 \longrightarrow X_2 \cup X_3$ by $\xi'(a) = \begin{cases} a \text{ if } a \in X_3 \\ \xi(a) \text{ if } p \in X_1 \end{cases}$. Then η' is a partial surjective mapping and ξ' is a mapping. Show that $(\tau_1 \vee \tau_2)(\eta'(p), a, \eta'(q)) \leq (\tau_2 \vee \tau_3)(p, \xi'(a), q)$, where $p, q \in Q_2 \cup Q_3$ and $a \in X_1 \cup X_3$.

(i) If $p, q \in Q_2$ and $a \in X_1$, then

$$\begin{aligned}(\tau_1 \vee \tau_3)(\eta'(p), a, \eta'(q) &= \tau_1(\eta(p), a, \eta(q)) \\ &\leq \tau_2(p, \xi(a), q) \\ &= (\tau_2 \vee \tau_3)(p, \xi'(a), q)\end{aligned}$$

(ii) If $p, q \in Q_3$ and $a \in X_3$, then

$$\begin{aligned}(\tau_1 \vee \tau_3)(\eta'(p), a, \eta'(q) &= \tau_3(p, a, q) \\ &= \tau_3(p, \xi'(a), q) \\ &= (\tau_2 \vee \tau_3)(p, \xi'(a), q)\end{aligned}$$

(iii) In all other cases

$$\begin{aligned}(\tau_1 \vee \tau_3)(\eta'(p), a, \eta'(q)) &= 0 \\ &\leq (\tau_2 \vee \tau_3)(p, \xi'(a), q)\end{aligned}$$

This completes the proof.

Theorem 5.5. *Let $\mathcal{M}_1 = (Q_1, X_1, \tau_1)$, $\mathcal{M}_2 = (Q_2, X_2, \tau_2)$ and $\mathcal{M}_3 = (Q_3, X_3, \tau_3)$ be fuzzy finite state machines such that $\mathcal{M}_1 \leq \mathcal{M}_2$. Then*

(i) $\mathcal{M}_1 \circ \mathcal{M}_3 \leq \mathcal{M}_2 \circ \mathcal{M}_3$.

(ii) $\mathcal{M}_3 \circ \mathcal{M}_1 \leq \mathcal{M}_3 \circ \mathcal{M}_2$.

Proof. Since $\mathcal{M}_1 \leq \mathcal{M}_2$, there exist $\eta : Q_2 \longrightarrow Q_1$ and $\xi : X_1 \longrightarrow X_2$ such that $\tau_1(\eta(p_2), a_1, \eta(q_2)) \leq \tau_2(p_2, \xi(a_1), q_2)$.

(i) Define $\eta' : Q_2 \times Q_3 \longrightarrow Q_1 \times Q_3$ by $\eta'((p_2, p_3)) = (\eta(p_2), p_3)$ and define $\xi' : X_1^{Q_3} \times X_3 \longrightarrow X_2^{Q_3} \times X_3$ by $\xi'((f, a_3)) = (\xi \circ f, a_3)$. Then

$$\begin{aligned}
&(\tau_1 \circ \tau_3)(\eta'(p_2, p_3), (f, a_3), \eta'(q_2, q_3)) \\
=&(\tau_1 \circ \tau_3)((\eta(p_2), p_3), (f, a_3), (\eta(q_2), q_3)) \\
=&\wedge (\tau_1(\eta(p_2), f(p_3), \eta(q_2)), \tau_3(p_3, a_3, q_3)) \\
\leq&\wedge (\tau_2(p_2, (\xi \circ f)(p_3), q_2), \tau_3(p_3, a_3, q_3)) \\
=&(\tau_2 \circ \tau_3)((p_2, p_3), \xi'((f, a_3)), (q_2, q_3))
\end{aligned}$$

(ii) Define $\eta' : Q_3 \times Q_2 \longrightarrow Q_3 \times Q_1$ by $\eta'((p_3, p_2)) = (p_3, \eta(p_2))$ and define $\xi' : X_3^{Q_1} \times X_1 \longrightarrow X_3^{Q_1} \times X_2$ by $\xi'((f, a_1)) = (f \circ \eta, \xi(a_1))$. Then

$$\begin{aligned}
&(\tau_3 \circ \tau_1)(\eta'(p_3, p_2), (f, a_1), \eta'(q_3, q_2)) \\
=&(\tau_3 \circ \tau_1)((p_3, \eta(p_2)), (f, a_1), (q_3, \eta(q_2))) \\
=&\wedge (\tau_3(p_3, f(\eta(p_2)), q_3), \tau_1(\eta(p_2), a_1, \eta(q_2))) \\
\leq&\wedge (\tau_3(p_3, (f \circ \eta)(p_2), q_3), \tau_2(p_2, \xi(a_1), q_2)) \\
=&(\tau_3 \circ \tau_2)((p_3, p_2), \xi'((f, a_1)), (q_3, q_2))
\end{aligned}$$

This completes the proof.

Theorem 5.6. *Let $\mathcal{M}_1 = (Q_1, X_1, \tau_1)$, $\mathcal{M}_2 = (Q_2, X_2, \tau_2)$ and $\mathcal{M}_3 = (Q_3, X_3, \tau_3)$ be fuzzy finite state machines such that $Q_2 \cap Q_3 = \emptyset$. Then*

(i) $\mathcal{M}_1 \circ (\mathcal{M}_2 \vee \mathcal{M}_3) \leq_c (\mathcal{M}_1 \circ \mathcal{M}_2) \vee (\mathcal{M}_1 \circ \mathcal{M}_3)$ where $X_2 \cap X_3 = \emptyset$

(ii) $\mathcal{M}_1 \circ (\mathcal{M}_2 \vee^* \mathcal{M}_3) \leq (\mathcal{M}_1 \circ \mathcal{M}_2) \vee^* (\mathcal{M}_1 \circ \mathcal{M}_3)$ where $X_2 \cap X_3 = \emptyset$

(iii) $\mathcal{M}_1 \circ (\mathcal{M}_2 + \mathcal{M}_3) \leq_c (\mathcal{M}_1 \circ \mathcal{M}_2) + (\mathcal{M}_1 \circ \mathcal{M}_3)$

Proof. We only prove (i) and (iii).

(i) Recall $\mathcal{M}_1 \circ (\mathcal{M}_2 \vee \mathcal{M}_3) = (Q_1 \times (Q_2 \cup Q_3), X_1^{Q_2 \cup Q_3} \times (X_2 \cup X_3), \tau_1 \circ (\tau_2 \vee \tau_3))$ and $(\mathcal{M}_1 \circ \mathcal{M}_2) \vee (\mathcal{M}_1 \circ \mathcal{M}_3) = ((Q_1 \times Q_2) \cup (Q_1 \times Q_3), (X_1^{Q_2} \times X_2) \cup (X_1^{Q_3} \times X_3), (\tau_1 \circ \tau_2) \vee (\tau_1 \circ \tau_3))$. Define $\eta : (Q_1 \times Q_2) \cup (Q_1 \times Q_3) \longrightarrow Q_1 \times (Q_2 \cup Q_3)$ by $\eta((p, q)) = (p, q)$. And define $\xi : X_1^{Q_2 \cup Q_3} \times (X_2 \cup X_3) \longrightarrow (X_1^{Q_2} \times X_2) \cup (X_1^{Q_3} \times X_3)$ by

$$\xi((f, b)) = \begin{cases} (f|_{Q_2}, b) \text{ if } b \in X_2 \\ (f|_{Q_3}, b) \text{ if } b \in X_3 \end{cases}$$

Then

$$
\begin{aligned}
&\tau_1 \circ (\tau_2 \vee \tau_3)(\eta(p,p'),(f,b),\eta(q,q'))\\
=&\tau_1 \circ (\tau_2 \vee \tau_3)((p,p'),(f,b),(q,q'))\\
=&\wedge(\tau_1(p,f(p'),q),(\tau_2 \vee \tau_3)(p',b,q'))\\
=&\begin{cases} \wedge(\tau_1(p,f(p'),q),\tau_2(p',b,q')),\ (p',b,q') \in Q_2 \times X_2 \times Q_2\\ \wedge(\tau_1(p,f(p'),q),\tau_3(p',b,q')),\ (p',b,q') \in Q_3 \times X_3 \times Q_3\\ 0, \text{ otherwise} \end{cases}\\
=&\begin{cases} (\tau_1 \circ \tau_2)((p,p'),(f|_{Q_2},b),(q,q')),\ (p',b,q') \in Q_2 \times X_2 \times Q_2\\ (\tau_1 \circ \tau_3)((p,p'),(f|_{Q_3},b),(q,q')),\ (p',b,q') \in Q_3 \times X_3 \times Q_3\\ 0, \text{ otherwise} \end{cases}\\
=&((\tau_1 \circ \tau_2) \vee (\tau_1 \circ \tau_3))((p,p'),\xi(f,b),(q,q'))
\end{aligned}
$$

(iii) Recall $\mathcal{M}_1 \circ (\mathcal{M}_2 + \mathcal{M}_3) = (Q_1 \times (Q_2 \cup Q_3), X_1^{Q_2 \cup Q_3} \times (X_2 \times X_3), \tau_1 \circ (\tau_2 + \tau_3))$ and $(\mathcal{M}_1 \circ \mathcal{M}_2) + (\mathcal{M}_1 \circ \mathcal{M}_3) = ((Q_1 \times Q_2) \cup (Q_1 \times Q_3), (X_1^{Q_2} \times X_2) \times (X_1^{Q_3} \times X_3), (\tau_1 \circ \tau_2) + (\tau_1 \circ \tau_3))$. Define $\eta : (Q_1 \times Q_2) \cup (Q_1 \times Q_3) \longrightarrow Q_1 \times (Q_2 \cup Q_3)$ by $\eta((p,q)) = (p,q)$. And define $\xi : X_1^{Q_2 \cup Q_3} \times (X_2 \times X_3) \longrightarrow (X_1^{Q_2} \times X_2) \times (X_1^{Q_3} \times X_3)$ by $\xi((f,(b_2,b_3))) = ((f|_{Q_2},b_2),(f|_{Q_3},b_3))$ Then

$$
\begin{aligned}
&\tau_1 \circ (\tau_2 + \tau_3)(\eta(p,p'),(f,(b_2,b_3)),\eta(q,q'))\\
=&\tau_1 \circ (\tau_2 + \tau_3)((p,p'),(f,(b_2,b_3)),(q,q'))\\
=&\wedge(\tau_1(p,f(p'),q),(\tau_2 + \tau_3)(p',(b_2,b_3),q'))\\
=&\begin{cases} \wedge(\tau_1(p,f(p'),q),\tau_2(p',b_2,q')),\ p',q' \in Q_2\\ \wedge(\tau_1(p,f(p'),q),\tau_3(p',b_3,q')),\ p',q' \in Q_3\\ 0, \text{ otherwise} \end{cases}\\
=&\begin{cases} (\tau_1 \circ \tau_2)((p,p'),(f|_{Q_2},b_2),(q,q')),\ p',q' \in Q_2\\ (\tau_1 \circ \tau_3)((p,p'),(f|_{Q_3},b_3),(q,q')),\ p',q' \in Q_3\\ 0, \text{ otherwise} \end{cases}\\
=&((\tau_1 \circ \tau_2) + (\tau_1 \circ \tau_3))((p,p'),((f|_{Q_2},b_2),(f|_{Q_3},b_3)),(q,q'))\\
=&((\tau_1 \circ \tau_2) + (\tau_1 \circ \tau_3))((p,p'),\xi(f,(b_2,b_3)),(q,q'))
\end{aligned}
$$

This completes the proof.

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