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MAX-NORM ERROR ESTIMATES FOR FINITE ELEMENT METHODS FOR NONLINEAR SOBOLEV EQUATIONS

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ABSTRACT. We consider the finite element method applied to nonlinear Sobolev equation with smooth data and demonstrate for arbitrary order ($k \geq 2$) finite element spaces the optimal rate of convergence in $L_\infty \; W^{1,\infty}(\Omega)$ and $L_\infty(L_\infty(\Omega))$ (quasi–optimal for $k = 1$). In other words, the nonlinear Sobolev equation can be approximated equally well as its linear counterpart. Furthermore, we also obtain superconvergence results in $L_\infty(W^{1,\infty}(\Omega))$ for the difference between the approximate solution and the generalized elliptic projection of the exact solution.

1. INTRODUCTION

Consider the nonlinear Sobolev equation on a bounded smooth domain $\Omega \subset R^2$

$$(1.1)\quad \begin{aligned} u_t &= \nabla \cdot \{a(x,u)\nabla u_t + b(x,u)\nabla u\} + f(x,u), \qquad (x,t) \in \Omega \times [0,T], \\ u(x,0) &= u_0(x), \qquad x \in \Omega, \\ u(x,t) &= 0, \qquad (x,t) \in \partial\Omega \times [0,T], \end{aligned}$$

where $u_t = \partial u/\partial t$, the functions a, b, f, u_0 are smooth enough for the ensuing analysis to be valid, and the function $a(x,u)$ is bounded below and above:

$$(1.2)\qquad 0 < a_* \leq a(x,u) \leq M, \qquad x \in \Omega, u \in R.$$

Since we shall show that the approximate solution is uniformly convergent to the exact solution of (1.1), assumption (1.2) needs hold only in a neighborhood of the exact solution. Problems of form (1.1) arise in many physical applications such as the flow of fluids through fissured rock [2] and dispersive waves [3]. For more detail the reader is referred to [1, 5] and the references therein.

We shall use $W^{m,p}(\Omega)$ to denote the usual Sobolev spaces and $\|\cdot\|_{m,p}$ the corresponding norms. When $p = 2$ we write $H^m(\Omega)$ for $W^{m,p}(\Omega)$ with $\|\cdot\|_{m,2} = \|\cdot\|_m$, and $\|\cdot\|_{0,2} = \|\cdot\|$. Let X be a Banach space with norm $\|\cdot\|_X$. For $\phi : [0,T] \to X$, define

$$\|\phi\|^p_{L_p(X)} := \int_0^T \|\phi(t)\|^p_X dt, \quad 1 \leq p < \infty, \; \|\phi\|_{L_\infty(X)} = \sup_{0 \leq t \leq T} \|\phi(t)\|_X.$$

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We use $(\cdot,\cdot)$ to denote $L_2(\Omega)$ or $L_2(\Omega)^2$ inner product. The symbol C will be used as a generic constant independent of the triangulation mesh gauge h and may have different values at different places.

Let S_h be a finite-dimensional subspace of $\subset H_0^1(\Omega)\cap W^{1,\infty}(\Omega)$ such that the following standard approximation and inverse properties hold:

$$\begin{aligned} \inf_{\chi\in S_h}\{||\phi-\chi||_{0,p}+h||\phi-\chi||_{1,p} &\le C||\phi||_{r,p}h^r, \qquad \phi\in H_0^1(\Omega)\cap W^{r,p}, \\ & \quad 1\le r\le k+1, \quad 2\le p\le\infty, \end{aligned} \tag{1.3}$$

and

$$||\chi||_{0,\infty} \le C|\ln h|^{1/2}||\chi||_1, \qquad \chi\in S_h. \tag{1.4}$$

Throughout the paper we shall refer to the integer k as the order of the approximation space.

Problem (1.1) has the following semidiscrete approximation:
Find $U(\cdot,t)\in S_h, t\in[0,T]$ such that

$$\begin{aligned} (U_t,\chi) &+ (a(U)\nabla U_t+b(U)\nabla U,\nabla\chi)=(f(U),\chi), \qquad \chi\in S_h, \\ U(0) &= R_hu_0, \qquad x\in\Omega, \end{aligned} \tag{1.5}$$

where $a(U)=a(x,U), b(U)=b(x,U), f(U)=f(x,U), U(0)=U(x,0)$ and R_h is the generalized elliptic projection operator satisfying $R_hu(\cdot,t)\in S_h, t\in[0,T]$ such that for all $\chi\in S_h$

$$\left(a(u)\nabla(u-R_hu)+\int_0^t[b(u(\tau))-a_u(u(\tau))u_t(\tau)]\nabla(u(\tau)-R_hu(\tau))d\tau,\nabla\chi\right)=0. \tag{1.6}$$

Differentiating (1.6) leads to

$$(a(u)\nabla(u-R_hu)_t+b(u)\nabla(u-R_hu),\nabla\chi)=0, \qquad \chi\in S_h, \tag{1.7}$$

and now setting $t=0$ gives

$$(a(u(0))\nabla(u(0)-R_hu(0)),\nabla\chi)=0. \tag{1.8}$$

Note that (1.7)-(1.8) is equivalent to (1.6). From (1.8), it is clear that R_h is the generalization of the usual elliptic projector [16] associated with the error analysis of parabolic problems. The projector R_h was first introduced in [4, 11] for integrodifferential equations to unify and obtain optimal error analysis of the associated Galerkin method. In this context, the partial derivative term a_u is not needed. Although the Sobolev equation (1.1) cannot be put into the general integrodifferential form studied in [4, 9, 11], Lin *et al.* [14, 15] introduced (1.6) above by including the partial derivative term.

Galerkin finite element methods for the linear and nonlinear Sobolev equations have been studied in [1, 5, 6, 10, 14, 15]. See also [4, 7, 11] for closely related integrodifferential equations. In [1, 10, 11] some optimal order H^1 and L_2 estimates were shown in special cases. In addition, One can find in [15] the quasi-optimal order L_∞ estimate for the linear element for (1.1). Some extensive results in one dimension case can be found in [12]. Up to now it is unclear if the higher order elements possess optimal L_∞ estimate for (1.1). We show in this paper that the answer is positive and demonstrate the optimal rate of convergence in $L_\infty(W^{1,\infty}(\Omega))$ and $L_\infty(L_\infty(\Omega))$ for arbitrary order ($k \geq 2$) finite element spaces. (Of course for $k=1$ one still has quasi-optimal.) In other words, the nonlinear Sobolev equation can be approximated equally well as its linear counterpart. Furthermore, we also obtain superconvergence results in the $L_\infty(W^{1,\infty}(\Omega))$ norm for the approximate spaces. The rest of this paper is organized as follows. In section 2 we derive some preliminary lemmas. In section 3 we demonstrate the main results of this paper. Max-norm error estimates in $W^{1,\infty}(\Omega)$ and L_∞ and superconvergence are given in Thms 3.1, 3.2 and 3.3 respectively. The main tool we used is the Green's functions method.

2. Preliminary Lemmas

In the remaining section we shall use u, U and $R_h u$ to denote, respectively, the solutions of (1.1), (1.5), and (1.6). Let

$$\eta = u - R_h u, \qquad \xi = U - R_h u.$$

The following lemma is contained in [4]

Lemma 2.1. *Assume that $u, u_t, u_{tt} \in L_1(H^{k+1}(\Omega))$. Then*

$$\begin{aligned} \|\eta(t)\| &+ \|\eta_t(t)\| + \|\eta_{tt}(t)\| \\ &\leq Ch^{k+1}\sum_{j=0}^{2}[\|\frac{\partial^j u}{\partial t^j}(t)\|_{k+1} + \int_0^t \|\frac{\partial^j u}{\partial t^j}(\tau)\|_{k+1} d\tau] \end{aligned} \tag{2.1}$$

The above lemma combined with the inverse properties and the interpolation theory give at once the following lemma.

Lemma 2.2. *Assume that $u, u_t, u_{tt} \in L_1(H^{k+1}(\Omega))$. Then*

$$\sum_{j=0}^{2} \|\frac{\partial^j}{\partial t^j} R_h u\|_{L_\infty(W^{1,\infty}(\Omega))} \leq C. \tag{2.2}$$

Applying Lemmas 2.1 and 2.2, we can prove the superconvergence estimates of ξ and ξ_t in H^1.

Lemma 2.3. *If $u(0), u_t(0) \in H^{k+1}(\Omega), u, u_t, u_{tt} \in L_2(H^{k+1}(\Omega))$ then the following superconvergence result holds:*

$$\|\xi(t)\|_1 + \|\xi_t(t)\|_1 \leq Ch^{k+1}. \tag{2.3}$$

Proof. From (1.1), (1.5) and (1.7),

$$
\begin{aligned}
(\xi_t,\chi) &+ (a(U)\nabla\xi_t,\nabla\chi)+(b(U)\nabla\xi,\nabla\chi)\\
&= (f(U)-f(u)-\eta_t,\chi)-((a(U)-a(u))\nabla R_h u_t,\nabla\chi)\\
&\quad -(((b(U)-b(u))\nabla R_h u_t,\nabla\chi), \qquad \chi\in S_h. \qquad (2.4)
\end{aligned}
$$

Differentiate the above with respect to t to obtain

$$
\begin{aligned}
(\xi_{tt},\chi) &+ (a(U)\nabla\xi_{tt},\nabla\chi)+\frac{1}{2}(a_u(U))U_t\nabla\xi_t,\nabla\chi)\\
&= (f_u(U)U_t-f_u(u)u_t-\eta_{tt},\chi)-\frac{1}{2}(a_u(U)U_t\nabla\xi_t,\nabla\chi)\\
&\quad -(a_u(U)U_t-a_u(u)u_t)\nabla R_h u_t,\nabla\chi)-((a(U)-a(u))\nabla R_h u_{tt},\nabla\chi)\\
&\quad -((b_u(U)U_t-b_u(u)u_t)\nabla R_h u,\nabla\chi)-((b(U)-b(u))\nabla R_h u_{tt},\nabla\chi)\\
&\quad -(b_u(U)U_t\nabla\xi,\nabla\chi)-(b(U)\nabla\xi_t,\nabla\chi)\\
&:= I_1+\ldots+I_8 \qquad (2.5)
\end{aligned}
$$

Set $\chi=\xi_t$ in (2.5) and proceed to estimate I_j's. First note that the left hand-side of (2.5)

$$
\begin{aligned}
(\xi_{tt},\xi_t) &+ (a(U)\nabla\xi_{tt},\nabla\xi_t)+\frac{1}{2}(a_u(U)U_t\nabla\xi_t,\nabla\xi_t)\\
&= \frac{1}{2}\frac{d}{dt}[||\xi_t||^2+(a(U)\nabla\xi_t,\nabla\xi_t)].
\end{aligned}
$$

It is easy to estimate I_j's using (2.1) and (2.2):

$$
\begin{aligned}
|I_1| &= ((f_u(U)(\xi_t+\eta_t)+(f_u(U)-f_u(u))u_t-\eta_{tt},\xi_t)\\
&\le C[||\xi_t||^2+||\eta_t||^2+||\xi||^2+||\eta||^2+||\eta_{tt}||^2]\\
&\le C[h^{2k+2}+||\xi||^2+||\xi_t||^2],
\end{aligned}
$$

$$
\begin{aligned}
|I_2| &= |\frac{1}{2}(a_u(U)(\xi_t+R_h u_t)\nabla\xi_t,\nabla\xi_t)|\\
&\le C(||\xi_t||_{L_\infty(L_\infty(\Omega))}+||R_h u_t||_{L_\infty(L_\infty(\Omega))})||\nabla\xi_t||^2\\
&\le C(||\xi_t||_{L_\infty(L_\infty(\Omega))}+1)||\nabla\xi_t||^2,
\end{aligned}
$$

$$
\begin{aligned}
|I_3| &= |(a_u(U)(\xi_t+\eta_t)+(a_u(U)-(a_u(u)))\nabla R_h u_t,\nabla\xi_t)\\
&\le C\left(||\nabla R_h u_t||_{L_\infty(L_\infty(\Omega))}+1\right)[||\xi_t||^2+||\eta_t||^2+||\xi||^2+||\eta||^2+||\nabla\xi_t||^2]\\
&\le C[h^{2k+2}+||\xi||^2+||\xi_t||_1^2].
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|I_4+I_5+I_6+I_8| &\le C[h^{2k+2}+||\xi||^2+||\xi_t||_1^2],\\
|I_7| &\le C(||\xi_t||_{L_\infty(L_\infty(\Omega))}+1)[||\nabla\xi||^2+||\nabla\xi_t||^2].
\end{aligned}
$$

Substitute the above into (2.5) to obtain

$$\begin{aligned} \frac{d}{dt} \quad [\quad & ||\xi_t||^2 + (a(U)\nabla\xi_t, \nabla\xi_t)] \\ \leq \quad & C(||\xi_t||_{L_\infty(L_\infty(\Omega))} + 1)[h^{2k+2} + ||\xi||_1^2 + ||\xi_t||_1^2]. \end{aligned} \tag{2.6}$$

From the inverse property (1.4) and the known estimate of ξ_t (see [14])

$$||\xi_t||_{L_\infty(L_\infty(\Omega))} \leq Ch^{-1}||\xi_t||_{L_\infty(L_2(\Omega))} \leq Ch^k \leq C$$

and hence from (2.6)

$$\begin{aligned} ||\xi_t(t)||_1^2 \quad &= \quad ||\xi_t(t)||^2 + ||\nabla\xi_t(t)||^2 \\ &\leq \quad C\left[||\xi_t(0)||_1^2 + h^{2k+2} + \int_0^t (||\xi(\tau)||_1^2 + ||\xi_t(\tau)||_1^2) d\tau\right]. \end{aligned} \tag{2.7}$$

Since $U(0) = R_h u(0)$ then $\xi(0) = 0$. Hence set $t = 0$ in (2.4) to obtain that for all $\chi \in S_h$

$$\begin{aligned} (\xi_t(0), \chi) \quad &+ \quad (a(R_h u(0))\nabla\xi_t(0), \nabla\chi) = -(\eta_t(0), \chi) \\ &+ \quad ([f(R_h u(0)) - f(u(0)], \chi) - ([a(R_h u(0)) - a(u(0)]\nabla R_h u_t(0), \nabla\chi) \\ &\quad -([b(R_h(u(0)) - b(u(0)]\nabla R_h u_t(0), \nabla\chi). \end{aligned}$$

Set $\chi = \xi_t(0)$ and use $ab \leq \epsilon a^2 + \frac{1}{4\epsilon}b^2$ to derive

$$||\xi_t(0)||_1^2 \quad \leq \quad C[||\eta_t(0)||^2 + ||\eta(0)||^2] + \epsilon||\xi_t(0)||_1^2$$

and so that by Lemma 2.1

$$||\xi_t(0)||_1 \leq Ch^{k+1}. \tag{2.8}$$

On the other hand,

$$||\xi(t)||_1^2 \leq C||\nabla\xi(t)||^2 \leq C\int_0^t ||\xi_t(\tau)||_1^2 d\tau \tag{2.9}$$

Add (2.9) to (2.7) and use (2.8) to obtain

$$||\xi(t)||_1^2 + ||\xi_t(t)||_1^2 \leq C[h^{2k+2} + \int_0^t (||\xi(\tau)||_1^2 + ||\xi_t(\tau)||_1^2)\, d\tau].$$

Now applying the Gronwall's inequality completes the proof of (2.3). □

We will use the Green's function method [17, 18] to derive max-norm error estimates. Let us introduce a discrete delta function: for a fixed z in $\bar{\Omega}$, define the discrete delta function $\delta_z^h \in S_h$ by

$$(\delta_z^h, \chi) = \chi(z), \qquad \chi \in S_h.$$

If $w = w(x) \in W^{1,\infty}(\Omega)$ then $a(w) \in W^{1,\infty}(\Omega)$. Given a $z \in \bar{\Omega}$, a function $G_z^h \in S_h$ is called a discrete Green's function if

$$(a(w)\nabla G_z^h, \nabla\chi) = \chi(z) = (\delta_z^h, \chi), \qquad \chi \in S_h. \tag{2.10}$$

A function $G_z^* \in H_0^1(\Omega)$ is called a pre-Green's function if

$$(a(w)\nabla G_z^*, \nabla v) = (\delta_z^h, v) = P_h v(z), \qquad v \in H_0^1(\Omega), \tag{2.11}$$

where $P_h : L_2(\Omega) \to S_h$ is the L_2 projection. A function $G_z \in W_0^{1,p}(\Omega)$ is called a Green's function if

$$(a(w)\nabla G_z, \nabla v) = v(z), \qquad v \in W^{1,p'}(\Omega), \tag{2.12}$$

where $\frac{1}{p} + \frac{1}{p'} = 1, \quad 1 \le p < 2$.

We shall also need the discrete Green's functions associated with the max-norm estimates of the partial derivatives. Given a fixed unit vector e, we define the directional derivative $D_{z,e}$ of $F : \Omega \to R$ as

$$D_{z,e}F := \lim_{t \to 0} \frac{F(z+te) - F(z)}{t}. \tag{2.13}$$

In this paper, the direction vector e will be taken as one of the unit coordinate vectors, i.e., $e = e_1 = (1,0)^t$ or $e = e_2 = (0,1)^t$, and so when there is no danger of confusion, we will simply write $D_{z,e} = \partial_z$. It is easy to see that

$$\partial_z (a(w)\nabla G_z^h, \nabla \chi) = (a(w)\nabla \partial_z G_z^h, \nabla \chi), \qquad \chi \in S_h$$

and

$$\partial_z (\delta_z^h, \chi) = (\partial_z \delta_z^h, \chi), \qquad \chi \in S_h.$$

Thus from (2.10) we have

$$(a(w)\nabla \partial_z G_z^h, \nabla \chi) = (\partial_z \delta_z^h, \chi), \qquad \chi \in S_h.$$

(Note that the above equation is valid only for z in the interior of elements.) Alternatively, one can simply define $\partial_z G_z^h = D_{z,e} G_z^h$ as the function $g_{h,i}^z$ (here $e = e_i, i = 1$, or 2) satisfying the equation

$$(a(w)\nabla g_{h,i}^z, \nabla \chi) = \frac{\partial}{\partial x_i}\chi(z) \qquad \chi \in S_h.$$

This is done in [13]. In other words, in terms of our notation $\partial_z G_z^h = g_{h,i}^z$. Similar comments can be made about $\partial_z G_z^*$ once one interprets (2.13) in the weak sense. Hence

$$\begin{aligned} &(a(w)\nabla \partial_z G_z^*, \nabla v) = P_h \partial_z v(z), \qquad v \in H_0^1(\Omega), \\ &(a(w)\nabla(\partial_z G_z^* - \partial_z G_z^h), \nabla \chi) = 0, \qquad \chi \in S_h. \end{aligned} \tag{2.14}$$

The following lemma is contained in [18] (see also [13]; bearing the above notation convention in mind).

Lemma 2.4. *The following properties hold:*

$$
\begin{aligned}
||G_z||_{1,1} \quad &+ \quad ||G_z^*||_{1,1} + ||\partial_z G_z^* - \partial_z G_z^h||_{1,1} + h||\partial_z G_z^*||_{2,1} \le C, \\
||\partial_z G_z^*||_{1,1} \quad &\le \quad C|\ln h|.
\end{aligned}
\tag{2.15}
$$

The next lemma concerns the stability of R_h.

Lemma 2.5. *Suppose that $v, v_t \in H_0^1(\Omega) \cap W^{1,\infty}(\Omega) \cap L_1(W^{1,\infty}(\Omega))$. Then the generalized elliptic projection of (1.7) has the following stability property:*

$$
||R_h v_t(t)||_{1,\infty} \le C[||v(0)||_{1,\infty} + ||v_t(t)||_{1,\infty} + \int_0^t ||v_t(\tau)||_{1,\infty} d\tau.]
\tag{2.16}
$$

Proof.
Set $w = v$ in the definition of the Green's function and let $\zeta = v - R_h v$. From $(2.14)_1$ and $(2.14)_2$

$$
\begin{aligned}
P_h \partial_z \zeta_t(z,t) \quad &= \quad (a(v)\nabla \zeta_t, \nabla \partial_z G_z^*) \\
&= \quad (a(v)\nabla \zeta_t, \nabla(\partial_z G_z^* - \partial_z G_z^h)) + (b(v)\nabla \zeta, \nabla(\partial_z G_z^* - \partial_z G_z^h)) \\
&\quad\quad -(b(v)\nabla \zeta, \nabla \partial_z G_z^*) \\
&= \quad (a(v)\nabla v_t, \nabla(\partial_z G_z^* - \partial_z G_z^h)) + (b(v)\nabla(v - P_h v), \nabla(\partial_z G_z^* - \partial_z G_z^h)) \\
&\quad\quad +(b(v)\nabla P_h \zeta, \nabla(\partial_z G_z^* - \partial_z G_z^h)) - (b(v)\nabla P_h \zeta, \nabla \partial_z G_z^*) \\
&\quad\quad -(b(v)\nabla(v - P_h v), \nabla \partial_z G_z^*) = Q_1 + \ldots + Q_5.
\end{aligned}
\tag{2.17}
$$

Use $(2.15)_1$ to obtain

$$
|Q_1| \le C||v_t||_{1,\infty}||\partial_z G_z^* - \partial_z G_z^h||_{1,1} \le C||v_t(t)||_{1,\infty}.
$$

From the property of P_h and bounds of the form (1.2) for $b(v)$ and its derivative, we have

$$
\begin{aligned}
|Q_2| \quad &\le \quad C||v(t) - P_h v(t)||_{1,\infty}||\partial_z G_z^* - \partial_z G_z^h||_{1,1} \\
&\le \quad C||v(t)||_{1,\infty} \le C[||v(0)||_{1,\infty} + ||\int_0^t v_t(\tau) d\tau||_{1,\infty}], \\
|Q_3| \quad &\le \quad C||P_h \zeta(t)||_{1,\infty}||\partial_z G_z^* - \partial_z G_z^h||_{1,1} \le C[||P_h \zeta(0)||_{1,\infty} + \int_0^t ||P_h \zeta_t(\tau)||_{1,\infty} d\tau].
\end{aligned}
$$

For Q_4 use Green's identity, $(2.14)_1$, and $(2.15)_2$ to obtain

$$\begin{aligned}
|Q_4| &= |(a(v)\nabla(\frac{b(v)}{a(v)}P_h\zeta), \nabla\partial_z G_z^*) - (a(v)P_h\zeta\nabla(\frac{b(v)}{a(v)}), \nabla\partial_z G_z^*)| \\
&\leq \left|P_h\partial_z\left(\frac{b(z,v(z,t))}{a(z,v(z,t))}P_h\zeta(z,t)\right)\right| + \left|\left(\nabla\cdot(a(v)P_h\zeta\nabla(\frac{b(v)}{a(v)})), \partial_z G_z^*\right)\right| \\
&\leq C||P_h\zeta(t)||_{1,\infty}(1+||\partial_z G_z^*||_{1,1}) \\
&\leq C[||P_h\zeta(0)||_{1,\infty} + \int_0^t ||P_h\zeta_t(\tau)||_{1,\infty}d\tau].
\end{aligned}$$

Use $v - P_h v$ for the ζ in Q_4 to have

$$\begin{aligned}
|Q_5| &\leq C[||v(0) - P_h v(0)||_{1,\infty} + \int_0^t ||v_t(\tau) - P_h v_t(\tau)||_{1,\infty}d\tau] \\
&\leq C[||v(0)||_{1,\infty} + \int_0^t ||v_t(\tau)||_{1,\infty}d\tau].
\end{aligned}$$

By Thm. 1 of [14],

$$||R_h v(t)||_{1,\infty} \leq C[||v(t)||_{1,\infty} + \int_0^t ||v(\tau)||_{1,\infty}]\,d\tau$$

and so that

$$||P_h\zeta(0)||_{1,\infty} \leq ||P_h v(0)||_{1,\infty} + ||R_h v(0)||_{1,\infty} \leq C||v(0)||_{1,\infty}. \tag{2.18}$$

Substitute the estimates for Q_1–Q_5 into (2.17) and combine (2.18) to derive

$$||P_h\zeta_t(t)||_{1,\infty} \leq C[||v(0)||_{1,\infty} + ||v_t(t)||_{1,\infty} + \int_0^t ||v_t(\tau)||_{1,\infty}d\tau + \int_0^t ||P_h\zeta_t(\tau)||_{1,\infty}d\tau].$$

Use Gronwall's inequality to obtain

$$||P_h\zeta_t(t)||_{1,\infty} \leq C[||v(0)||_{1,\infty} + ||v_t(t)||_{1,\infty} + \int_0^t ||v_t(\tau)||_{1,\infty}d\tau]. \tag{2.19}$$

Hence by the triangle inequality and the stability of P_h we have

$$\begin{aligned}
||R_h v_t(t)||_{1,\infty} &\leq ||P_h\zeta_t(t)||_{1,\infty} + ||P_h v_t(t)||_{1,\infty} \\
&\leq ||P_h\zeta_t(t)||_{1,\infty} + ||v_t(t)||_{1,\infty}.
\end{aligned}$$

Combining this with (2.19) completes the proof. □

We now show the max-norm estimates of η.

Lemma 2.6. *Suppose that $u(0) \in W^{k+1,\infty}$, $u(t), u_t(t) \in W^{k+1,\infty}(\Omega) \cap L_1(W^{k+1,\infty}(\Omega))$. Then the following estimates hold. For $k \geq 1$,*

$$\begin{aligned}
||\eta(t)||_{1,\infty} &+ ||\eta_t(t)||_{1,\infty} \\
&\leq Ch^k[||u(0)||_{k+1,\infty} + ||u_t(t)||_{k+1,\infty} + \int_0^t ||u_t(\tau)||_{k+1,\infty}d\tau],
\end{aligned} \tag{2.20}$$

$$(2.21) \qquad ||\eta(t)||_{0,\infty} + ||\eta_t(t)||_{0,\infty}$$
$$\leq Ch^{k+1}\left[|\ln h|^{\bar{r}}||u(0)||_{k+1,\infty} + ||u_(t)||_{k+1,\infty} + ||u_t(t)||_{k+1,\infty} + \int_0^t ||u_t(\tau)||_{k+1,\infty} d\tau\right],$$
$$\bar{r} = 1, \ \mathit{if}\ k = 1; \bar{r} = 0 \ \mathit{if}\ k \geq 2.$$

Proof.

Let $\Pi_h u$ be the usual interpolant of u in S_h. By Lemma 2.5 and the interpolation property we have

$$\begin{aligned}
||\eta_t(t)||_{1,\infty} &\leq ||u_t - \Pi_h u_t||_{1,\infty} + ||R_h(u_t - \Pi_h u_t)|| \\
&\leq ||u_t - \Pi_h u_t||_{1,\infty} \\
&\quad + C[||u(0) - \Pi_h u(0)||_{1,\infty} + ||u_t(t) - \Pi_h u_t(t)||_{1,\infty} + \int_0^t ||u_t(\tau) - \Pi_h u_t(\tau)||_{1,\infty} d\tau] \\
(2.22) \qquad &\leq Ch^k[||u(0)||_{k+1,\infty} + ||u_t(t)||_{k+1,\infty} + \int_0^t ||u_t(\tau)||_{k+1,\infty} d\tau].
\end{aligned}$$

Observe that

$$(2.23) \qquad ||\eta(t)||_{r,\infty} \leq ||\eta(0)||_{r,\infty} + \int_0^t ||\eta_t(\tau)||_{r,\infty} d\tau \qquad r = 0, 1.$$

Noting that when $t = 0$, the generalized elliptic projection operator R_h is the same as the Ritz projection operator, we have

$$(2.24) \qquad ||\eta(0)||_{1,\infty} \leq Ch^k ||u(0)||_{k+1,\infty}$$

and

$$(2.25) \qquad ||\eta(0)||_{0,\infty} \leq Ch^{k+1}|\ln h|^{\bar{r}}||u(0)||_{k+1,\infty}.$$

Combining (2.22), (2.24), with (2.23), we derive the assertion (2.20).

As for the second assertion of the theorem, the case of $k = 1$ has been demonstrated in [14]. Let us show the case $k \geq 2$. Set $w = u$ in the definitions of G_z and G_z^h to obtain from (1.7)

$$\begin{aligned}
\eta_t(z,t) &= (a(u)\nabla\eta_t, \nabla G_z) \\
&= (a(u)\nabla\eta_t, \nabla(G_z - G_z^h)) + (b(u)\nabla\eta, \nabla(G_z - G_z^h)) \\
&\quad -(b(u)\nabla(u - P_h u), \nabla G_z) - (b(u)\nabla P_h \eta, \nabla G_z) \\
(2.26) \qquad &= J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

Use (2.15) and the property of P_h to get

$$\begin{aligned}
|J_1| &\leq C||\eta_t||_{1,\infty}||G_z - G_z^h||_{1,1} \leq Ch||\eta_t||_{1,\infty}, \\
|J_2| &\leq C||\eta||_{1,\infty}||G_z - G_z^h||_{1,1} \leq Ch||\eta||_{1,\infty}, \\
|J_3| &= |(a(u)\nabla(\frac{b(u)}{a(u)}(u - P_h u)), \nabla G_z) - (a(u)(u - P_h u)\nabla(\frac{b(u)}{a(u)}), \nabla G_z)| \\
&= |\frac{b(u)}{a(u)}(u - P_h u)| + ||u - P_h u||_{0,\infty}||G_z||_{1,1} \\
&\leq C||u - P_h u||_{0,\infty}(1 + ||G_z||_{1,1}) \leq C||u - P_h u||_{0,\infty} \\
&\leq Ch^{k+1}||u(t)||_{k+1,\infty}.
\end{aligned}$$

Substitute $P_h\eta$ for $u - P_h u$ in the above inequality and use (2.23) to obtain

$$|J_4| \leq C||P_h\eta(t)||_{0,\infty} \leq ||\eta||_{0,\infty} \leq C[||\eta(0)||_{0,\infty} + \int_0^t ||\eta_t(\tau)||_{0,\infty}d\tau].$$

Substituting the estimates for J_1–J_4 into (2.26) to derive

(2.27)

$$||\eta_t(t)||_{0,\infty} \leq C\{h^{k+1}||u(t)||_{k+1,\infty}+||\eta(0)||_{0,\infty}+h[||\eta(t)||_{1,\infty}+||\eta_t(t)||_{1,\infty}]+\int_0^t ||\eta_t(\tau)||_{0,\infty}d\tau\}.$$

Finally the proof is complete by applying the Gronwall's inequality to (2.27) and combining (2.20), (2.25), and the property of P_h. □

3. Main Theorem

In this section, we will employ lemmas given in the previous section to derive the main theorems of this paper. Our first result deals with the error estimates of $U - u$ in $W^{1,\infty}$.

Theorem 3.1. *Suppose that $u, u_t \in L_\infty(W^{k+1,\infty}(\Omega)), u_{tt} \in L_2(H^{k+1}(\Omega))$ then the following estimate holds*

$$||U - u||_{L_\infty(W^{1,\infty}(\Omega))} + ||(U - u)_t||_{L_\infty(W^{1,\infty}(\Omega))} \leq Ch^k. \tag{3.1}$$

Proof.

¿From the inverse property, (2.2) and (2.3) we derive

$$\begin{aligned}
||U||_{L_\infty(W^{1,\infty}(\Omega))} &\leq ||R_h u||_{L_\infty(W^{1,\infty}(\Omega))} + ||\xi||_{L_\infty(W^{1,\infty}(\Omega))} \\
&\leq ||R_h u||_{L_\infty(W^{1,\infty}(\Omega))} + ch^{-1}||\xi||_{L_\infty(H^1(\Omega))} \\
&\leq ||R_h u||_{L_\infty(W^{1,\infty}(\Omega))} + Ch^k \leq C
\end{aligned}$$

Since $U \in L_\infty(W^{1,\infty}(\Omega))$, $a(U) \in L_\infty(W^{1,\infty}(\Omega))$. We can set $w = U$ in the definition of the Green's function and derive from $(2.14)_1$ and (2.4) that

$$\begin{aligned}\partial_z \xi_t(z,t) &= (a(U)\nabla \xi_t, \nabla \partial_z G_z^*) \\ &= ((f(U) - f(u) - \xi_t - \eta_t, \partial_z G_z^*) - ((a(U) - a(u))\nabla R_h u_t, \nabla \partial_z G_z^*) \\ &\quad -((b(U) - b(u))\nabla R_h u, \nabla \partial_z G_z^*) - (b(U)\nabla \xi, \nabla \partial_z G_z^*)\end{aligned} \tag{3.2}$$

Estimate the above equation by (2.1)-(2.3), imbedding theorem, and $(2.15)_1$ to obtain

$$\begin{aligned}||\xi_t(t)||_{1,\infty} &\leq C(||\nabla R_h u||_{L_\infty(L_\infty(\Omega))} + 1)[||\xi||_1 + ||\eta|| + ||\xi_t|| + ||\eta_t||]||\partial_z G_z^*||_1 \\ &\leq Ch^{k+1}||\partial_z G_z^*||_{2,1} \leq Ch^k\end{aligned} \tag{3.3}$$

On the other hand,

$$||\xi(t)||_{1,\infty} \leq \int_0^t ||\xi_t(\tau)||_{1,\infty} d\tau. \tag{3.4}$$

Combining (3.3), (3.4), and (2.20) completes the proof. □

Our next theorem is to derive the max-norm estimates of $U - u$ in L_∞.

Theorem 3.2. *Suppose the hypotheses of Thm. 3.1 hold. Then*

$$\begin{aligned}||U - u||_{L_\infty(L_\infty(\Omega))} \quad + \quad ||(U - u)_t||_{L_\infty(L_\infty(\Omega))} \leq Ch^{k+1}|\ln h|^{\bar{r}}, \\ \bar{r} = 1 \text{ if } k = 1, \bar{r} = 0 \text{ if } k \geq 2.\end{aligned} \tag{3.5}$$

Proof.

¿From (2.22) and the previous arguments, it suffices to show that

$$||\xi_t||_{L_\infty(L_\infty(\Omega))} \leq Ch^{k+1}|\ln h|^{\bar{r}}. \tag{3.6}$$

Similar to (3.2), we have

$$\begin{aligned}\xi_t(z,t) &= (a(U)\nabla \xi_t, \nabla G_z^*) \\ &= (f(U) - f(u) - \xi_t - \eta_t, G_z^*) - ((a(U) - a(u))\nabla R_h u_t, \nabla G_z^*) \\ &\quad -((b(U) - b(u))\nabla R_h u, \nabla G_z^*) - (b(U)\nabla \xi, \nabla G_z^*)\end{aligned}$$

and hence

$$\begin{aligned}
||\xi_t(t)||_{0,\infty} &\leq C(||\xi|| + ||\eta|| + ||\xi_t|| + ||\eta_t||)\,||G_z^*||_{1,1} \\
&\quad +(||\nabla R_h u_t||_{L_\infty(L_\infty(\Omega))} + ||\nabla R_h u||_{L_\infty(L_\infty(\Omega))})(||\xi||_{0,\infty} + ||\eta||_{0,\infty})||G_z^*||_{1,1} \\
&\quad +|(a(U)\nabla(\frac{b(U)}{a(U)}\xi), \nabla G_z^*) - (a(U)\xi\nabla(\frac{b(U)}{a(U)}), \nabla G_z^*)| \\
&\leq C[h^{k+1}|\ln h|^{\bar{r}} + ||\xi||_{0,\infty} + |P_h(\frac{b(U)}{a(U)}\xi)| \\
&\quad +||\xi||_{0,\infty}||G_z^*||_{1,1}] \\
&\leq C[h^{k+1}|\ln h|^{\bar{r}} + \int_0^t ||\xi_t(\tau)||_{0,\infty} d\tau].
\end{aligned}$$

Now applying the Gronwall's inequality completes the proof. □

Finally we turn to the superconvergence.

Theorem 3.3. *Suppose the hypotheses of Thm. 3.1 hold. Then the following superconvergence results hold:*

$$||U - R_h u||_{L_\infty(W^{1,\infty}(\Omega))} + ||(U - R_h u)_t||_{L_\infty(W^{1,\infty}(\Omega))} \leq Ch^{k+1}|\ln h|^{\bar{r}+1}, \tag{3.7}$$

$$\bar{r} = 1 \text{ if } k = 1, \bar{r} = 0 \text{ if } k \geq 2.$$

Proof. By (2.15), (2.16), and (3.4), we derive from (3.2) that

$$\begin{aligned}
||\xi_t(t)||_{1,\infty} &\leq C[(||\xi|| + ||\eta|| + ||\xi_t|| + ||\eta_t||)\,||\partial_z G_z^*||_{1,1} + ||U-u||_{0,\infty}||\partial_z G_z^*||_{1,1}] \\
&\quad + \left|\left(\nabla\cdot(a(U)\xi\nabla\frac{b(U)}{a(U)}), \partial_z G_z^*\right)\right| \\
&\leq C[h^{k+1}|\ln h|^{\bar{r}+1} + |P_h\partial_z(\frac{b(U)}{a(U)}\xi)| + ||\xi||_{1,\infty}|(1 + |\partial_z G_z^*)||_{1,1}] \\
&\leq C[h^{k+1}|\ln h|^{\bar{r}+1} + ||\xi(t)||_{1,\infty} + ||\nabla\xi||\,|\ln h|] \\
&\leq C[h^{k+1}|\ln h|^{\bar{r}+1} + \int_0^t ||\xi_t(\tau)||_{1,\infty} d\tau].
\end{aligned}$$

where in the last step we have used Lemma 2.3. Applying the Gronwall's inequality completes the proof. □

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