J. KSIAM **Vol.5**, No.2, 1-6, 2001

MULTIPLICITY OF PERIODIC SOLUTIONS FOR SECOND ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS*

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Abstract. Multiplicity of nonlinear second order nonlinear ordinary differential equations will be discussed

1. Introduction

Let R be the set of all real numbers. By $C^k[0, 2\pi]$ we denote the Banach space of 2π-periodic continuous functions $x : [0, 2\pi] \to R$ whose derivatives up to order k are continuous. The norm is given by

$$\|x\|_{C^k} = \sum_{i=1}^{k} \|x^{(i)}\|_\infty,$$

where $\|y\|_\infty = \sup_{t\in[0,2\pi]} |y(t)|$, the norm in $C^0[0, 2\pi]$.

For multiplicity results of periodic solutions of Lienard equations, we may see in Hirano and kim[2], and Kim[3]. In this note, we will study the multiple existence of solutions to the problem

$$(E) \qquad x''(t) + h(t, x(t), x'(t)) + g(t, x(t)) = e(t),$$

$$(B) \qquad x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0,$$

where $h : [0, 2\pi] \times R \times R \to R$ is continuous and satisfies Nagumo-type condition, and $g : [0, 2\pi] \times R \to R$ and $e : [0, 2\pi] \to R$ are continuous functions.

The proof of our result is based on upper-lower solution method and coincidence degree theory.

1991 *Mathematics Subject Classification.* 34A34, 34b15, 34c25.

Key words and phrases. existence, multiplicity,nonlinear, ordinary differential equation.

This work was supported by Hanyang University, Korea, made in the program year of 2000

Assume

$$h(t, x, 0) = 0$$

for every $(t, x) \in [0, 2\pi] \times R$ and that there exists some $T > 0$ such that

$$g(t, x + T) = g(t, x)$$

for every $(t, x) \in [0, 2\pi] \times R$.

We will say that h in problem (E)(B) satisfies Nagumo-type condition on $[r, s]$ if there exists a constant $C > 0$ such that for each $\lambda \in [0, 1]$ and each possible solution of

$$(E') \qquad x''(t) + \lambda h(t, x(t), x'(t)) + \lambda g(t, x(t)) = \lambda e(t),$$

$$(B) \qquad x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0$$

satisfying $r \le x(t) \le s,\ t \in [0, 2\pi]$, we have

$$\|x'\|_\infty < C.$$

Examples of admissible h are the following ones:

1) h depends only on $x'(see[4])$;

2) $|h(t, x, y)| \le \gamma(|y|)$ for $(t, x, y) \in [0, 2\pi] \times [r, s] \times R$ where γ is positive, continuous and such that

$$\int_0^\infty \frac{sds}{\gamma(s)} = +\infty,$$

(see [1]).

Our result contains more general result than that of [6]. Now we have the following

Main Result

THEOREM. *Assume, besides the above conditions on h and g there exists there exists real numbers r_1, r_2, s_1, s_2 with $r_1 < s_2 < r_2 < s_1$ and $0 < s_1 - r_1 < T$ such that*

$$g(s_1) \le g(s_2), \quad g(r_2) \le g(r_1)$$

and h satisfies Nagumo type condition on $[s_1 - T, s]$. Then (E)(B) has at least one solution if, for all $t \in [0, 2\pi]$,

$$(I_1) \qquad g(t, s_1) \le e(t) \le g(t, r_1),$$

and (E)(B) has at least two solutions not differing by a multiple of T if, for all $t \in [0, 2\pi]$,

$$(I_2) \qquad g(t, s_1) < g(t, s_2) \leq e(t) \leq g(t, r_2) < g(t, r_1),$$

and (E)(B) has at least four solutions not differing by a multiple of T if strict inequalities holds in (I_2).

Proof. Suppose (I_1). Then, for all $t \in [0, 2\pi]$, we have

$$e(t) - g(t, r_1 + kT) - h(t, r_1 + kT, 0) = e(t) - g(t, r_1) \leq 0,$$

$$e(t) - g(t, s_1 + jT) - h(t, s + jT, 0) = e(t) - g(t, s_1) \geq 0$$

with strict inequalities if they hold in (I_1). Hence, by Mawhin's classical results(see [1]), there exists, by taking $k = j = 0$, at least one solution $x_1(t)$ of (E)(B) such that $r_1 \leq x(t) \leq s_1$.

Now, suppose the strict inequalities holds; i.e., for all $t \in [0, 2\pi]$,

$$(I_1') \qquad g(t, s_1) < e(t) < g(t, r_1).$$

If we define

$$L : D(L) \subseteq C^1[0, 2\pi] \longrightarrow C^0[0, 2\pi], \quad x \longmapsto x''$$

where $D(L) = C^2[0, 2\pi]$ and

$$N : C^1[0,2\pi] \longrightarrow C^0[0, 2\pi]. \quad x \longmapsto x''$$

Then L is a Fredholm mapping of index zero and N is L-completely continuous.

Let

$$\Omega_{k,j} = \{x \in C^1[0, 2\pi] | \, r_1 + kT < x(t) < s_1 + jT \text{ for } t \in [0, 2\pi] \text{ and } \|x'\|_\infty < C\}.$$

Then the boundary value problem (E')(B) becomes

$$Lx - \lambda Nx = 0, \qquad \lambda \in [0, 1]$$

and when the strict inequalities hold in (I_1), the following coincidence degree exist and have the corresponding values, where d_B denotes the Brouwer degree, and

$$D_L(L - N, \Omega_{0,0}) = d_B(\Gamma, (r_1, s_1), 0) = +1,$$

$$D_L(L - N, \Omega_{-1,-1}) = d_B(\Gamma, (r_1 - T, s_1 - T), 0) = +1,$$

$$D_L(L-N,\Omega_{-1,0}) = d_B(\Gamma,(r_1-T,s_1),0) = +1,$$

where $(\Gamma u)(t) = \frac{1}{2\pi}\int_0^{2\pi}[e(t)-g(t,u(t))]dt$. But

$$\Omega_{0,0}\cap\Omega_{-1,-1} = \emptyset$$

and

$$\Omega_{0,0}\subseteq\Omega_{-1,0}, \qquad \Omega_{-1,-1}\subseteq\Omega_{-1,0}.$$

So that the excision property of degree implies

$$\begin{aligned} 1 = D_L(L-N,\Omega_{-1,0}) =& D_L(L-N,\Omega_{-1,-1},0) \\ &+ D_L(L-N,\Omega_{0,0},0) \\ &+ D_L(L-N,\Omega_{-1,0}\backslash(\bar{\Omega}_{-1,-1}\cup\bar{\Omega}_{0,0})) \\ =& 2 + D_L(L-N,\Omega_{-1,0}\backslash(\bar{\Omega}_{-1,-1}\cup\bar{\Omega}_{0,0})). \end{aligned}$$

Hence,

$$D_L(L-N,\Omega_{-1,0}\backslash(\bar{\Omega}_{-1,-1}\cup\bar{\Omega}_{0,0})) = -1.$$

Hence, there exists a solution x_2 such that, for all $t\in[0,2\pi]$, $r_1-T<x_2(t)<s_1$, $x_2(\tau)>s_1-T$ for some $\tau\in[0,2\pi]$ and $x_2(\tau')<r_1$ for some $\tau'\in[0,2\pi]$.

Consequently, this solution cannot differ from the one in $\Omega_{0,0}$ by a multiple of T. Hence (E)(B) has at least two solutions not differing by a multiple of T if (I_1') holds.

Now, suppose (I_2). Then, for all $t\in[0,2\pi]$, we have

$$e(t)-g(t,s_1)-h(t,s_1,0) = e(t)-g(t,s_1) > 0,$$

$$e(t)-g(t,r_2)-h(t,r_2,0) = e(t)-g(t,r_2) \leq 0.$$

Hence, there exists at least one solution $x_1(t)$ of (E)(B) such that $r_2\leq x_1(t)\leq s_1$ for all $t\in[0,2\pi]$. Again, for all $t\in[0,2\pi]$, we have

$$e(t)-g(t,s_2)-h(t,s_2,0) = e(t)-g(t,s_2) \geq 0,$$

$$e(t)-g(t,r_1)-h(t,r_1,0) = e(t)-g(t,r_1) < 0.$$

Therefore, there exists at least one solution $x_2(t)$ of (E)(B) such that $r_1\leq x_2(t)\leq s_2$ for all $t\in[0,2\pi]$. Since $r_1<s_2<r_2<s_1$, two solutions are different and moreover two solutions can not differ from by a multiple of T because $0<s_1-r_1<T$. Since $g(t,s_1)<e(t)<g(t,r_1)$,as we did by the coincidence degree, we have a solution x_3 such that, for all $t\in[0,2\pi]$, $r_1-T<x_3(t)<s_1$, $x_3(\tau)>s_1-T$ for some $\tau\in[0,2\pi]$ and hence $x_3(\tau)>s_2-T$, and $x_3(\tau')<r_1$ for some $\tau'\in[0,2\pi]$ and hence $x_3(\tau')<r_2$. Therefore the third solution can not differ from x_1,x_2 in $\Omega_{0,0}$ by a multiple of T.

Consequently, there exist at least three solutions of (E)(B) not differing by a multiple of T.

Now, suppose the strict inequalities hold;i.e., for all $t \in [0, 2\pi]$,

$$(I_2') \qquad g(t, s_1) < g(t, s_2) < e(t) < g(t, r_2) < g(t, r_1).$$

Note that, for all $t \in [0, 2\pi]$, we have

$$e(t) - g(t, s_i + kT) - h(t, s_i + kT, 0) = e(t) - g(t, s_i) > 0,$$

$$e(t) - g(t, r_i + jT) - h(t, r_i + jT, 0) = e(t) - g(t, r_i) < 0, \quad i = 1, 2.$$

Then clearly (E)(B) has three solutions $x_1(t), x_2(t)$ and $x_3(t)$ such that $r_1 \leq x_1(t) \leq s_2$, $s_2 \leq x_2(t) \leq r_2$ and $r_2 \leq x_3(t) \leq s_1$, for all $t \in [0, 2\pi]$, and they are distinct and each of them are not differing by a multiple of T. For our fourth solution. Let

$$\Omega_{k,J}^{<i,j>} = \{x \in C^1[0, 2\pi] |\, r_i + kT < x(t) < s_j + jT,\, t \in [0, 2\pi],\, \|x'\|_\infty < C\},$$

$$\Omega_{k,J}^{[i,j]} = \{x \in C^1[0, 2\pi] | s_i + kT < x(t) < r_j + jT,\, t \in [0, 2\pi],\, \|x'\|_\infty < C\}$$

$(k \leq 1)$, where C is constant given by Nagumo condition. But $\Omega_1 = \Omega_{-1,-1}^{<1,2>}$, $\Omega_2 = \Omega_{-1,-1}^{[2,2]}$, $\Omega_3 = \Omega_{-1,-1}^{<2,1>}$, $\Omega_4 = \Omega_{0,0}^{<1,2>}$, $\Omega_5 = \Omega_{0,0}^{[2,2]}$, $\Omega_6 =_{0,0}^{<2,1>}$ are mutually disjoint subset of $\Omega_{-1,0}^{<1,1>}$ and

$$D_L(L - N, \Omega_{-1,0}^{<1,1>}) = d_B(\Gamma, (r_1 - T, s_1), 0) = +1,$$

$$D_L(L - N, \Omega_1) = d_B(\Gamma, (r_1 - T, s_2 - T), 0) = +1,$$
$$D_L(L - N, \Omega_2) = d_B(\Gamma, (s_2 - T, r_2 - T), 0) = -1,$$
$$D_L(L - N, \Omega_3) = d_B(\Gamma, (r_2 - T, s_1 - T), 0) = +1,$$
$$D_L(L - N, \Omega_4) = d_B(\Gamma, (r_1, s_2), 0) = +1,$$
$$D_L(L - N, \Omega_5) = d_B(\Gamma, (s_2, r_2), 0) = -1,$$
$$D_L(L - N, \Omega_6) = d_B(\Gamma, (r_2, s_1), 0) = +1.$$

Hence, by the excision property of degree,

$$1 = D_L(L - N, \Omega_{-1,0}^{<1,1>}) = 2 + D_L(L - N, \Omega_{-1,0}^{<1,1>} \setminus \cup_{1 \leq i \leq 6} \bar{\Omega}_i).$$

Therefore

$$D_L(L - N, \Omega_{-1,0}^{<1,1>} \setminus \cup_{1 \leq i \leq 6} \bar{\Omega}_i) = -1.$$

Consequently, (E)(B) has a solution x_4 in $\Omega_{-1,0}^{<1,1>} \setminus \cup_{1\le i\le 6} \bar{\Omega}_i$;i.e., a solution such that $r_1 - T < x(t) < s_1$ for all $t \in [0, 2\pi]$, $x_4(\tau_1) > s_2 - T$, $x_4(\tau_2) < s_2 - T$, $x_4(\tau_3) > r_2 - T$, $x_4(\tau_4) < r_2 - T$, $x_4(\tau_5) > s_1 - T$, $x_4(\tau_6) < r_1$, $x_4(\tau_7) > s_2$, $x_4(\tau_8) < s_2$, $x_4(\tau_9) < r_2$, $x_4(\tau_{10}) > r_2$ for some $\tau_1, \tau_2, \cdots, \tau_{10} \in [0, 2\pi]$. Thus this solution x_4 can not differ from x_1, x_2, x_3 by a multiple of T.

EXAMPLE. Suppose h is a function satisfying the assumption above and Nagumo condition on $[r_1 - 2\pi, 2\pi - r_1]$ where r_1 is the point at which $a\,sin\,x + b\,sin\,2x$ has its maximum value. Let $r_2 \in [0, 2\pi]$ be a point at which $a\,sin\,x + b\,sin\,2x$ has it lelative maximum such that $g(r_2) < g(r_1)$. Then the boundary value problem

$$x''(t) + h(t, x(t), x'(t)) + [a\,sin\,x + b\,sin\,2x] = e(t),$$

$$x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0$$

has at least one solution if $\|e\|_\infty \le a\,sin\,r_1 + b\,sin\,2r_1$, at least two solutions not differing by a multiple of 2π if $\|e\|_\infty < a\,sin\,r_1 + b\,sin\,2r_1$, at least three solutions not differing by a multiple of 2π if $\|e\|_\infty \le a\,sin\,r_2 + b\,sin\,2r_2$ and at least four solutions not differing by a multiple of 2π if $\|e\|_\infty < a\,sin\,r_2 + b\,sin\,2r_2$.

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