Economic Design of a Moving Average Control Chart with Multiple Assignable Causes when Two Failures Occur

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Abstract

The economic design of control charts has been researched for over four decades since Duncan proposed the concept in 1956. Few studies, however, have focused attention on the economic design of a moving average (MA) control chart. An MA control chart is more effective than the Shewhart chart in detecting small process shifts [9]. This paper provides an economic model for determining the optimal parameters of an MA control chart with multiple assignable causes and two failures in the production process. These parameters consist of the sample size, the spread of the control limit and the sampling interval. A numerical example is shown and the sensitivity analysis shows that the magnitude of shift, rate of occurrence of assignable causes and increasing cost when the process is out of control have a more significant effect on the loss cost, meaning that one should more carefully estimate these values when conducting an economic analysis.

Keywords: Economic design, Moving average control charts, Continuous flow process, Exponential distribution, Multiple-assignable-causes

1. Introduction

The purpose of a control chart is to differentiate the random causes and the assignable causes in a process. If assignable causes occur in a process, they should be detected and removed. How can a control chart be designed on the basis of an economic viewpoint? Duncan (1956) first proposed the economic design of $\bar{x}$ control charts to control normal process means and ensure that the economic design control chart actually has a lower cost, compared with a Shewhart control chart. The control limits of $\bar{x}$ charts are set at ±k standard deviations from the target mean. A sample size n is taken from the output of the process at an interval of every h hours. The sample mean is then plotted on the chart. These three parameters are then selected optimally to minimize the average cost of a process in the model. Duncan's (1956) model assumed that the process has only one single assignable cause and that the occurrence time of an assignable cause from an in-control state to out-of-control is exponentially distributed. This assumption has been widely used in subsequent work on the subject. Duncan (1971) extended his research from single to multiple assignable causes in which the occurrence time was also assumed to be an exponential distribution. Saniga (1977) proposed a joint economic design of $\bar{x}$ and R control charts for which two assignable causes were employed in the process. The first assignable cause results in a shift of the process mean; the second results in a shift of the process variance. The occurrence time from in-control to out-of-control is also exponentially distributed. Lorenzen and Vance (1986) proposed a general method for determining the economic design of $\bar{x}$ control $\bar{x}$ charts. This method can be applied regardless of the statistic used. It is necessary to calculate only the average run-length of the statistics, assuming the process is in-control and also assuming the process is out-of-control in some specified manner. Collani (1986) proposed a different procedure to determine the economic design of $\bar{x}$ control charts. In this procedure, in addition to the possibility of employing a regular $\bar{x}$ chart, the alternative of a periodic inspection of the process without performing a sampling inspection is also included. Banerjee and Rahim (1988) pointed out that the use of an increasing failure rate was a more realistic approach, thereby modifying Duncan's (1956) model for the economic design of $\bar{x}$ to extend the occurrence time from exponential to the Weibull distribution. Collani and Sheil (1989) developed an economic model for the s chart. Their model minimized the expected loss-per-item produced. The above research was all conducted in a piece-part production environment. Koo and Case (1990) reconsidered the production process from discrete to continuous flow and developed an economic model. It is evident from the above research that the emphasis on the optimal economic design of control charts for variables has been on the process mean and its variances. Chen, Yeh and Yu (2000) considered a continuous-flow and provided an MA control chart economic design with multiple assignable cause and single failure. The purpose of this study is to extend

Chen, Yeh and Yu's (2000) single-failure model to one with two failures to develop a model to analyze the loss cost when an MA control chart is used in the production process. This study incorporates Koo and Case's (1990) sampling scheme into Lorenzen and Vance's (1986) approach to construct an MA economic model of optimum n, h and k. Such a model will be helpful in reducing the quality cost in a continuous flow process or a small lot-size production type.

2. ASSUMPTIONS AND GLOSSARY OF SYMBOLS

Suppose that the individual observations have been collected; then let x1, x2,...denote these observations. The moving average of span n at time t is defined as $\overline{X_t} = (X_t + X_{t-1} + ... + X_{t-n+1})/n$. That is, at time period t, the oldest observation in the moving average set is dropped and the newest one is added to the set. The features of the model considered in this article are as follows:

(1) A second occurrence of an assignable cause may follow the first occurrence in the process.

(2) The distribution of X and $\overline{X_t}$ is normal.

(3) The process will produce a shift $\delta_m\sigma$ (m=1,2,3,...), if it has been distributed by the occurrence of the *m*th assignable cause where σis the standard deviation of X. This standard deviation is assumed to remain invariant and to be known. The δ_m will be the shift parameter of the *m*th assignable cause.

(4) The process-mean will shift to $\Delta\sigma$ due to the joint effect of the second assignable cause occurring after the first assignable cause A_m.

(5) The process is at any time in one of two states. Either it is in-control or it has been distributed by the occurrence of assignable cause A_m. The in-control time caused by the *m*th assignable cause is assumed to be independently exponentially distributed with a mean time of $1/\lambda_m$.

(6) A second assignable cause that will occur between *0* and h is taken equal to $1-e^{-\lambda' h}$ where λ' may or may not be equal to λ where $\lambda = \sum \lambda_m$.

(7) The process is a continuous flow.

(8) Drawing random samples of size 1, for which the time interval between sampling is h, monitors the process.

(9) The time to sample and chart one item is negligible.

(10) Production is continuous during the search and brings the process back to an in-control state.

The glossary of symbols is as follows:

(1) Parameters related to the assignable causes.

A_m = assignable cause m;

λ_m = the average rate of occurrence per unit time of cause Am when the process is in control;

$\lambda = \sum \lambda_m$;

λ' = the rate of occurrence of a second assignable cause;

$\delta_m \sigma$ = the shift in the mean of the process resulting from the occurrence of cause A_m where σ is the standard deviation of the product characteristic X;

$\Delta\sigma$ = the shift in the mean of the process resulting from a double occurrence of assignable causes;

(2) The design variables

n = the size of moving samples;

h = the interval between samples;

k = the control limits measured in times of standard deviation;

(3) Cost and time parameters

fc = the fixed cost per sample of sampling, testing and plotting that is independent of sample size;

vc = the variable cost per item of sampling, testing and plotting;

v = the cost per occasion of looking for an assignable cause when none exists;

W_m = the average cost of finding assignable cause Am when it occurs;

W' = the average cost of finding the combined assignable causes, assumed to be independent of the assignable causes;

Y = the average time of isolating assignable cause A_m when it has caused a point to fall outside the control limit;

D_m = the average time of repairing assignable cause A_m after it has been identified;

D' = the average time of repairing the combined assignable causes after a point has fallen outside control limits;

U_m = the increasing loss due to a greater percentage of items being outside the limits when assignable cause Am occurs;

U' = the increasing loss due to a greater percentage of items being outside the limits when combined failure occurs;

(4) Variables related to computing

u_m = the mean of moving subgroup when the first assignable cause Am occurs;

μ'_m = the mean of moving subgroup when the second cause occurs after the first assignable cause Am;

$P_{m,j,i}$ = the detected probability when the *m*th assignable cause occurs between the *j*th and *(j+1)*th sample and is detected in subsequent *i* moving subgroup size;

$Q_{m,j,i} = 1 - P_{m,j,i}$;

$P'_{m,j,i}$ = the probability of detecting the presence of the assignable cause when the *m*th assignable cause occurs between the jth and *(j+1)*th samples and is detected in the subsequent *i* sample;

$P'_{m,j}$ = the probability of detecting the presence when the *m*th assignable cause occurs between the *j*th and *(j+1)*th samples;

P'_m = the probability of detecting the presence when the *m*th assignable cause occurs;

$P''_{m,j,i}$ = the probability of the occurrence of the second assignable cause before the first assignable cause being detected when the first assignable A_m cause occurs at the *j*th sample and the second assignable cause occurs at the subsequent sample *i* ;

$P''_{m,j}$ = the probability of the occurrence of the second assignable cause before the first assignable cause being detected when the first assignable cause A_m occurs between the *j*th and *(j+1)*th sample;

P''_m = the probability of the occurrence of the second assignable cause before the first assignable cause being detected when the first assignable cause A_m occurs;

$P_{m,j,i,l}$ = the detected probability of joint effect when the *m*th assignable cause occurs between the *j*th and *(j+1)*th sample and the second assignable cause occurs between the ith and *(i+1)*th sample and is then detected in the subsequent *l* moving subgroup size;

$Q_{m,j,i,l} = 1 - P_{m,j,i,l}$;

$P'_{m,j,i,l}$ = the probability of detecting the joint effect presence when the *m*th assignable cause occurs between the *j*th and *(j+1)*th sample and the second assignable cause occurs between the subsequent ith and *(i+1)*th sample, the joint effect is detected in the subsequent *l* moving subgroup size;

α = the probability of a point falling outside the control limit when the process is in an in-control state;

$E_{m,j,i}$ = the expected sampling numbers of detecting the joint effect presence when the *m*th assignable cause occurs between the *j*th and *(j+1)*th sample and the second assignable cause occurs between the subsequent ith and *(i+1)*th sample;

$E'_{m,j}$ = the expected number of samplings when the *m*th assignable cause occurs between the *j*th and *(j+1)*th;

$E''_{m,j}$ = the expected sampling number before the occurrence of the second assignable cause when the first assignable cause occurs at the *j*th sample;

$AVGE'_{m,j}$ = the average sampling number that will be taken when the first assignable cause occurs between the *j*th and *(j+1)*th samples and until the joint effect is detected;

$AVGSN_m(1)$ = the average sampling number that will be taken when the *m*th assignable cause occurs;

$AVGSN_m(2)$ = the average sampling number that will be taken before the occurrence of the second assignable cause when the first assignable cause occurs;

$AVGSN_m(3)$ = the average sampling number that will be taken when the first assignable cause occurs until the joint effect is detected;

NFA = the expected number of false alarms

τ_m = the average time between the sample taken just prior to the occurrence of assignable cause A_m and the occurrence itself;

τ' = the average time between the sample taken just prior to the occurrence of the second failure and the occurrence itself;

$E_m(1)$ = total time that the second assignable cause does not occur until after the first assignable cause has been detected;

$E_m(2)$ = the total time before the occurrence of the second assignable cause when the first assignable cause occurs;

$E_m(2)$ = the average time until the joint effect is detected;

$TOTCL$ = the total cycle time;

L_1 = the expected additional loss from an out-of-control state;

L_2 = the loss cost to bring an out-of-state process back to an in-control state;

L_3 = the expected cost for discovering the false alarms;

L_4 = the expected sampling cost;

L = the hourly total loss in an average cycle process

On the basis of the above assumptions, Lorenzen and Vance's (1986) method was employed to construct a cost model to which Rahim's (1993) iteration was applied to find the optimum parameters of n, h and k to minimize the loss-cost-per-unit time. Now there are two major elements to be constructed, namely, the expected cycle length and the expected loss cost generated in a cycle length. After these are determined, the operating loss cost per unit time can then be obtained. The cycle length is defined as the total time from which the process starts in-control, after which first or second assignable cause occurs, is detected, located and brought back to an in-control state. So the process is assumed to be in one of three states: (1) a state of control, (2) a state disturbed by the occurrence of an assignable cause A_m which produces a shift in the process mean of $\delta_m\sigma$, or (3) a state disturbed by the occurrence of a second assignable cause following the first. The assignable causes are also assumed to occur independently when the process goes out of control, i.e., from state 1 to state 2, there is a negative exponential distribution with a mean time of $1/\lambda$ where $\lambda = \sum\lambda_m$. When the process is in state 2, two different situations exist. One is that the second assignable cause does not occur until after the first assignable cause has been detected; the other is that the second assignable cause occurs before the first cause has been detected. The detailed cycle time is analyzed in section 3. The expected loss cost generated in a cycle length includes the loss cost when the process is out-of-control, the location and repair of assignable causes, a false alarm and the sampling cost. The detailed loss cost is analyzed in section 4.

3. EXPECTED CYCLE TIME

Consider an economic design of a moving average control chart when there are multiple assignable causes. It is assumed that the control chart is in-control and maintained to detect multiple assignable causes at the beginning. These various assignable causes also occur independently at random, so that the probability is $e^{-t\sum\lambda_m}$ if no assignable cause has occurred at the end of time t. Therefore, equation (1) expresses the mean time during which the process is in an in-control state.

$$\int_0^\infty t \sum_{m=1}^{\infty} \lambda_m e^{-t\sum_{m=1}^{\infty}\lambda_m} dt = \frac{1}{\lambda} \qquad (1)$$

When the process is in state 2, it is assumed that no further disturbance occurs until after the first sample is taken. After that, so long as the first assignable cause continues undetected, a second assignable cause (possibly a repetition of the first) is assumed to occur at random with a mean time of $1/\lambda'$, where λ' may or may not equal λ. Suppose the process mean shifts to $\mu_0+\delta_m\sigma$ when the process goes out-of-control before it is detected due to only the occurrence of the *m*th assignable cause. The mean of this moving subgroup is denoted as u_m. Let $P_{m,j,i}$ $(j=0,1,2..,.n-1,\ i=1,2,...,n,...)$ be the detected probability when the mth assignable cause occurs between the *j*th and $(j+1)$th sample and is detected in subsequent *i* moving subgroup size. Thus $Q_{m,j,i}(1-P_{m,j,i})$ denotes the probability that this mth assignable cause will not be detected on the following i moving subgroups when it occurs between the jth and $(j+1)$th samples. The three different situations for the u_m and $P_{m,j,i}$ are listed in equations (2) and (3),

$$P_{m,j,i}=\begin{cases}1-\Phi(k-\frac{i\,\delta_m}{\sqrt{i+j}})+\Phi(-k-\frac{i\,\delta_m}{\sqrt{i+j}}) \\ \text{if } j+i<n \\ 1-\Phi(k-\frac{i\,\delta_m}{\sqrt{n}})+\Phi(-k-\frac{i\,\delta_m}{\sqrt{n}}) \\ \text{if } j+i\geq n \text{ an } i<n \\ 1-\Phi(k-\frac{n\,\delta_m}{\sqrt{n}})+\Phi(-k-\frac{n\,\delta_m}{\sqrt{n}}) \\ \text{if } j+i\geq n \text{ an } i\geq n\end{cases} \quad (2)$$

$$P'_{m,j,i}=\begin{cases}P_{m,j,i} & \text{if } i=1 \\ P_{m,j,i}\left(e^{-\lambda' h}\right)^{i-1}\prod_{s=1}^{i-1}Q_{m,j,s} & \text{if } 2\leq i\leq n-1 \\ P_m\cdot\left(Q'_m\right)^{i-n}Q_{m,j,r} & \text{if } i\geq n\end{cases} \quad (3)$$

Where $\Phi(x)$ is a cumulative density function (CDF) of the standard normal distribution and k is the coefficient of the control limit. In the third situation, $(j+i)\geq n$ and $i\geq n$, $P_{m,j,i}$ are independent on j or i; thus we can let $P_{m,j,i}=P_{m,j,n}=P_m$, $Q_{m,j,i}=1-P_m=Q_m$. From the above three situations, when the *m*th assignable cause occurs between the *j*th and $(j+i)$th samples and is detected in the subsequent i sample, the probability of detecting the presence of the assignable cause is expressed as equation (4),

$$P'_{m,j,i}=\begin{cases}P_{m,j,i} & \text{if } i=1 \\ P_{m,j,i}\left(e^{-\lambda' h}\right)^{i-1}\prod_{s=1}^{i-1}Q_{m,j,s} & \text{if } 2\leq i\leq n-1 \\ P_m\cdot\left(Q'_m\right)^{i-n}Q_{m,j,r} & \text{if } i\geq n\end{cases}$$

(4)

Where $Q_{m,j,r}=\prod_{s=1}^{n-1}Q_{m,j,s}$ and $Q'_m=Q_m e^{-\lambda' h}$.

This means that if the *m*th assignable cause occurs between the *j*th and $(j+1)$th samples, the detected probability, denoted as $P'_{m,j}$, and the expected number of samplings, denoted as $E'_{m,j}$, $(j=0,1,2...)$, are expressed as equations (5) and (6):

$$P'_{m,j}=\begin{cases}\left\{P_{m,j,1}+\sum_{i=2}^{n-1}P_{m,j,i}\left(e^{-\lambda' h}\right)^{i-1}\left[\prod_{s=1}^{i-1}Q_{m,j,s}\right]\right\} \\ +Q'_{m,j,r}P_m\left(\frac{1}{1-Q'_m}\right) \\ \text{if } j\leq n-1 \\ P'_{m,n-1} \quad \text{if } j\geq n\end{cases} \quad (5)$$

$$E'_{m,j}=\begin{cases} P_{m,j,1}+\sum_{i=2}^{n-1} i\,P_{m,j,i}\left[\prod_{s=1}^{i-1} Q_{m,j,s}\right]\left(e^{-\lambda' h}\right)^{i-1} \\ +Q'_{m,j,r}P_m\left[\dfrac{n}{1-Q'_m}+\dfrac{Q'_m}{(1-Q'_m)^2}\right] & if\ \ j\le n-1 \\ E'_{m,n-1} \\ if\ j\ge n \end{cases} \tag{6}$$

So, the detected probability (P'_m) and the average sampling number, denoted as $AVGSN_m(1)$, that will be taken when the mth assignable cause has occurred are:

$$P'_m=\left(1-e^{-\lambda_m h}\right)\left(\sum_{j=0}^{n-2} e^{-\lambda_m jh}P'_{m,j}\right)+e^{-\lambda_m(n-1)h}P'_{m,n-1} \tag{7}$$

$$AVGSN_m(1)=\left(1-e^{-\lambda_m h}\right)\sum_{j=0}^{n-2} e^{-j\lambda_m h}E'_{m,j}+e^{-(n-1)\lambda_m h}E'_{m,n-1} \tag{8}$$

The average time before a later sample falls outside the limits will be $h*AVGSN_m(1)$ if the mth assignable cause occurs immediately after the taking of a sample. The mth assignable cause may occur, however, at anytime between two samples if these are the jth and $(j+1)$th. The average time of occurrence of the mth assignable cause (denoted as τ_m) within an interval is

$$\tau_m=\frac{\int_{jh}^{(j+1)h}\lambda_m(t-jh)\,e^{-\lambda_m t}dt}{\int_{jh}^{(j+1)h}\lambda_m e^{-\lambda_m t}dt}=\frac{e^{\lambda_m h}-(1+\lambda_m h)}{\lambda_m\left(e^{\lambda_m h}-1\right)} \tag{9}$$

Thus, there is another delay time for testing the sample, analyzing the results and plotting. It is assumed that this is a constant time (denoted as Y) for different assignable causes. After the assignable cause has been discovered, the process must be brought back to an in-control state. It is assumed that the average time differs from other causes (denoted as D_m), thus necessitating its addition to the time cycle of state 2. So the total time in this situation is expressed as equation (10):

$$E_m(1)=h*AVGSN_m(1)+(-\tau_m+Y+D_m)P'_m \tag{10}$$

In state 2, if we assume that the first assignable cause occurs at the jth sample and the second assignable cause occurs at the subsequent sample i, the probability of the occurrence of the second assignable cause is expressed as equation (11):

$$P''_{m,j,i}=\begin{cases}\left(1-e^{-\lambda' h}\right)\prod_{s=1}^{i} Q_{m,j,s}\left(e^{-\lambda' h}\right)^{i-1} & \text{if } i\le n-1\\ \left(1-e^{-\lambda' h}\right)Q'_{m,j,r}Q_m\left(Q'_m\right)^{i-n} & \text{if } i\ge n\end{cases} \tag{11}$$

Where $and\ Q_m^{'} = Q_m e^{-\lambda' h}$ and

$$Q_{m,j,r}^{'} = e^{-\lambda'(n-1)h} \prod_{s=1}^{n-1} Q_{m,j,s}$$

So the probability of the occurrence of the second assignable cause, denoted as $P_{m,j}^{''}$,and the expected sampling number, denoted as $E_{m,j}^{''}$, are shown in equations (12) and (13):

$$P_{m,j}^{''} = \begin{cases} \left[\left(1-\left(e^{-\lambda' h}\right)^{n-1}\right) \sum_{i=1}^{n-1} \prod_{s=1}^{i} Q_{m,j,s}\right] \\ + Q_{m,j,r}^{'} Q_m \left(1-e^{-\lambda' h}\right)\left(\frac{1}{1-Q_m^{'}}\right) & if\ j \le n-1 \\ P_{m,n-1}^{''} \\ if\ j \ge n \end{cases} \tag{12}$$

$$E_{m,j}^{''} = \begin{cases} \left\{\left(1-e^{-\lambda' h}\right)\sum_{i=1}^{n-1} i\left(e^{-\lambda' h}\right)^{i-1}\left[\prod_{s=1}^{i} Q_{m,j,s}\right]\right\} \\ + Q_{m,j,r}^{'} Q_m \left(1-e^{-\lambda' h}\right)\left(\frac{n}{1-Q_m^{'}} + \frac{Q_m^{'}}{\left(1-Q_m^{'}\right)^2}\right) & if\ j \le n-1 \\ E_{m,n-1}^{''} \\ if\ j \ge n \end{cases} \tag{13}$$

So the occurrence probability, $P_m^{''}$, and the average sampling number, $AVGSN_m(2)$, that will be taken in a state 2 situation are:

$$P_m^{''} = \left(1-e^{-\lambda_m h}\right)\left(\sum_{j=0}^{n-2} P_{m,j}^{''} e^{-j\lambda_m h}\right) + e^{-\lambda_m (n-1)h} P_{m,n-1}^{''} \tag{14}$$

$$AVGSN_m(2) = \left(1-e^{-\lambda_m h}\right)\sum_{j=0}^{n-2} e^{-j\lambda_m h} E_{j,m}^{''} + e^{-(n-1)\lambda_m h} E_{n-1,m}^{''} \tag{15}$$

The average time between the samples taken just prior to the occurrence of the second assignable cause, denoted as τ', is

$$\tau' = \frac{\int_{jh}^{(j+1)h} \lambda'(t-jh)\, e^{-\lambda' t} dt}{\int_{jh}^{(j+1)h} \lambda' e^{-\lambda' t} dt} = \frac{e^{\lambda' h} - (1+\lambda' h)}{\lambda'\left(e^{\lambda' h}-1\right)} \tag{16}$$

So the total time in this situation is expressed as equation (17):

$$E_m(2) = h * AVGSN_m(2) + \left(-\tau_m + \tau'\right) P_m^{''} \tag{17}$$

If we assume that the joint effect produces a process shift to $\mu_0 + \Delta\sigma$ when the second cause occurs after the first assignable cause A_m, then the mean of this moving subgroup is denoted as $\mu_m^{'}$. Let $P_{m,j,i,l}\ (j=0,1,2...,n-1,\quad i=1,2,...,n,...,l=1,2,...n,...)$ be the detected probability of joint effect when the mth assignable cause occurs between the jth and $(j+1)$th sample and the second assignable cause occurs between the ith and $(i+1)$th sample and is then detected in the subsequent l moving subgroup size.

Thus $Q_{m,j,i,l} = 1 - P_{m,j,i,l}$ enotes the undetected probability of joint effect. The four different situations for the μ_m' and $P_{m,j,i,l}$ are listed in equations (18) and (19):

$$\mu_m' = \begin{cases} \mu_0 + \dfrac{(i\delta_m + l\Delta)\sigma}{j+i+l} \\ \text{when } (j+i+l) < n \\ \mu_0 + \dfrac{(i\delta_m + l\Delta)\sigma}{n} \\ \text{when } (j+i+l) \ge n, (i+l) < n \\ \mu_0 + \dfrac{((n-l)\delta_m + l\Delta)\sigma}{n} \\ \text{when } (j+i+l) \ge n, (i+l) \ge n, l < n \\ \mu_0 + \Delta\sigma \\ \text{when } (j+i+l) \ge n, (i+l) \ge n, l \ge n \end{cases} \tag{18}$$

$$P_{m,j,i,l} = \begin{cases} 1 - \Phi\left(k - \dfrac{i\delta_m + l\Delta}{\sqrt{j+i+l}}\right) + \Phi\left(-k - \dfrac{i\delta_m + l\Delta}{\sqrt{j+i+l}}\right) \\ 1 - \Phi\left(k - \dfrac{i\delta_m + l\Delta}{\sqrt{n}}\right) + \Phi\left(-k - \dfrac{i\delta_m + l\Delta}{\sqrt{n}}\right) \\ 1 - \Phi\left(k - \dfrac{(n-l)\delta_m + l\Delta}{\sqrt{n}}\right) + \Phi\left(-k - \dfrac{(n-l)\delta_m + l\Delta}{\sqrt{n}}\right) \\ 1 - \Phi\left(k - \dfrac{n\Delta}{\sqrt{n}}\right) + \Phi\left(-k - \dfrac{n\Delta}{\sqrt{n}}\right) \end{cases} \tag{19}$$

In the fourth situation, $(j+i+l) \ge n$, $(i+l) \ge n, l \ge n$ $P_{m,i,j,l}$ are independent on j, i, or l; thus we can let $P_{m,j,i,l} = PI_m$, $Q_{m,j,i,l} = 1 - PI_m = QI_m$. In the above four situations, when the mth assignable cause occurs between the jth and $(j+1)$th sample and the second assignable cause occurs between the subsequent ith and $(i+1)$th sample, the joint effect is detected in the subsequent l moving subgroup size. The probability of detecting the joint effect presence is expressed as equation (20),

$$P'_{m,j,i,l} = \begin{cases} P''_{m,j,i} P_{m,j,i,l} & \text{if } l = 1 \\ P''_{m,j,i} P_{m,j,i,l} \prod_{s=1}^{l-1} Q_{m,j,i,s} & \text{if } 2 \le l \le n-1 \\ P''_{m,j,i} Q_{m,j,i,r} PI_m (QI_m)^{l-n} & \text{if } l \ge n \end{cases} \tag{20}$$

where $Q_{m,j,i,r} = \prod_{s=1}^{n-1} Q_{m,j,i,s}$

This means that the expected sampling numbers, $E_{m,j,i}$, are

If $i \le (n-2)$;

$$E_{m,j,i} = P^{*}_{m,j,i} \left\{ \begin{aligned} &\left[P_{m,j,i,1} + \sum_{l=2}^{n-1} l P_{m,j,i,l} \left(\prod_{s=1}^{l-1} Q_{m,j,i,s} \right) \right] \\ &+ PI_m Q_{m,j,i,r} \left(\frac{n}{1-QI_m} + \frac{QI_m}{(1-QI_m)^2} \right) \end{aligned} \right\} \tag{21}$$

If $tt = \sum_{i=n-1}^{\infty} i$

$$E_{m,j,tt} = \left[P^{*}_{m,1,n-1} + Q'_{m,j,r} Q_m \left(1 - e^{-\lambda' h}\right)\left(\frac{1}{1-Q'_m}\right) \right] PE_{m,j,n-1} \tag{22}$$

where

$$PE_{m,j,n-1}=\begin{Bmatrix}\left[\left[P_{m,j,n-1,1}+\sum_{l=2}^{n-1}lP_{m,j,n-1,l}\left(\prod_{s=1}^{l-1}Q_{m,j,n-1,s}\right)\right]\right]\\ +\mathrm{PI}_m\mathrm{Q}_{m,j,n-1,r}\left(\frac{n}{1-QI_m}+\frac{QI_m}{\left(1-QI_m\right)^2}\right)\end{Bmatrix}$$

So the average sampling number, denoted as $AVGE'_{m,j}$, that will be taken when the first assignable cause occurs between the *j*th and *(j+1)*th samples and until the joint effect is detected, is shown in equation (23):

$$AVGE'_{m,j}=\begin{cases}\sum_{i=1}^{n}E_{m,j,i} & if\, j\le n-1\\ AVGE'_{m,n-1} & if\, j\ge n\end{cases} \tag{23}$$

The expected average sampling number, denoted as $AVGSN_m(3)$, that will be taken when the process is in state 3 is shown in equation (24):

$$AVGSN_m(3)=\left(1-e^{-\lambda_m h}\right)\sum_{j=0}^{n-2}e^{-j\lambda_m h}\;AVGE'_{m,j}+e^{-\lambda_m(n-1)h}\;AVGE'_{m,n-1} \tag{24}$$

The average time in state 3, denoted as $E_m(3)$, is expressed as equation (25):

$$E_m(3)=h*AVGSN_m(3)+\left(-\tau'+Y+D'\right)P''_m \tag{25}$$

So the total cycle time is

$$TOTCL=\frac{1}{\lambda}+\sum_{m=1}^{\infty}\frac{\lambda_m}{\lambda}\left(E_m(1)+E_m(2)+E_m(3)\right) \tag{26}$$

4. EXPECTED COST GENERATION

The expected total-loss-cost of the process during a cycle consists of the following cost components:

(1) If cause A_m occurs, it will be assumed to produce an increasing loss U_m in state 2 and U'in state 3 due to a greater percentage of items being outside the specification limits. The expected additional loss from an out-of-control state will be

$$L_1=\left(\sum_{m=1}^{\infty}\frac{\lambda_m*\left(E_m(1)+E_m(2)\right)*U_m}{\lambda}\right)+\left(\sum_{m=1}^{\infty}\frac{\lambda'_m*E_m(3)*U'}{\lambda'}\right) \tag{27}$$

(2) If cause A_m occurs, it will cost W_m in state 2 and W'in state 3 to locate and repair this assignable cause. The probability of a second assignable-cause occurrence is P''_m.

Therefore, the loss cost to bring an

out-of-state process back to an in-control state is

$$L_2=\left(\sum_{m=1}^{\infty}\frac{\lambda_m*W_m}{\lambda}\left(1-P_m^*\right)\right)+\left(\sum_{m=1}^{\infty}\frac{\lambda_m'*W'}{\lambda'}P_m^*\right) \quad (28)$$

(3) Let α be the probability of a point falling outside the control limit when the process is in an in-control state. The expected number of false alarms (denoted as *NFA*) per cycle depends on α and the expected sampling number taken in the in-control period. The *NFA* is shown in equation (29):

$$NFA = a\cdot\sum_{j=0}^{\infty} j\int_{jh}^{(j+1)h}\lambda e^{-\lambda t}\,dt = a\frac{1}{e^{\lambda h}-1},$$

where $\alpha=2[\Phi(-k)]$. (29)

The average cost-per-false-alarm is *V* when the process is in-control; hence the expected cost for discovering the false alarms per cycle is

$$L_3=NFA*V \quad (30)$$

(4) Since *fc* is the cost of taking a sample that is independent of the sample size and *vc* is the variable cost per item for sampling, testing and plotting, each sampling will have one point to plot on the control chart. The expected sampling cost is

$$L_4=\frac{fc+vc}{h} \quad (31)$$

(5) Therefore, the hourly total loss in an average cycle process will be the sum of the above costs,
which is

$$L=(L_1+L_2+L_3)/TOTCL+L_4 \quad (32)$$

We can arbitrary assign the initial value of the design parameters, moving the subgroup size (n), sampling interval (h), and control limit coefficient (k), to obtain the total loss cost from equation (32).

5. DETERMINATION OF OPTIMAL PARAMETERS

The goal of the economic design of MA control charts is to find the design parameters, n, h and k, to minimize the loss-cost function in equation (32). Since L is a very complicated function of the decision variables, n, h and k, Rahim's search technique (1993) is employed for this calculation algorithm and is rewritten using Matlab 5.3 software to solve the optimal design parameters for reaching the minimum cost. The sample size (n) is always a discrete variable in the sampling. It is possible to treat n as an integer and h or k as a continuous value to obtain a different loss cost; then, select the minimum loss cost and its h and k for the proposed design parameters. The following numerical example illustrates the use of the model. Suppose that there is seven assignable causes in the process. The parameter λ_m is the average rate of occurrence per unit time of assignable cause A_m which, when it occurs, produces a shift in the process mean of $\delta_m \sigma$. Hence, given the occurrence of the assignable cause, the ratio λ_m/λ is the conditional probability of A_m . If the rate of occurrence of assignable causes for the total system(λ) is given as 0.02, λ_m is referred to as the negative exponential prior distribution mentioned by Duncan (1971) and is chosen proportional to NE_m ($NE_m = 0.5\ exp(-0.5\ \delta_m)$). Suppose that the magnitudes of the shift are 1σ, 1.5σ, 1.8σ, 2σ, 2.2σ, 2.5σ and 3σ. When the magnitude is $2\sigma (m = 4)$, the parameters of cost and time are selected on the basis of the example from Koo and Case (1990), which comes from the chemical industry, i.e. δ_m=2, U_m=4000, Y=1.25, D_m=2, V=2000, W_m=1000, fc=20, vc=20. The magnitudes of the shift δ_m also have an effect on other cost factors, namely λ , D, W, and U, which will vary with the magnitude of the shift δ. These Z_m and W_m values are functions of δ_m, meaning that D_m is (NE_m/NE_4)* 2 and W_m is (NE_m/NE_4)* 1000. For example: D_1 is $(0.5 * exp(-0.5 * 1)) / (0.5 * exp(-0.5 * 2)) * 2 =$ 3.295. The value of U_m is determined by the assumption that the distribution of $\bar{x}$ is normal with the shift to $\delta_m \sigma$, meaning that U_m is $(P_m/P_4)*4000$ where P_m is $1 - \Phi(3 - \delta_m)$.

The related parameters pertaing to the second failure are $\lambda' = 0.02$, $\Delta = 2$, $U' = 4000$, $D' = 2$, $W' = 1000$. The detailed input data are listed in Table 1.

The optimal economic design parameters and minimum loss cost solved by Matlab are shown in Table 2 when the subgroup size (n) is changed from 2 to 13. The loss cost will obviously increase when n increases on this table; hence, we conclude that:

(1) The minimum loss-cost obtained here is \$417.711099 within the optimal economic design parameters n=2, h=0.6375 and k=2.6547.

(2) If all the parameters related to the second assignable cause are set to zero, no second assignable cause can occur, The loss cost is \$411.573159, meaning that the loss cost of two failures will be 1.50% more than only a single failure in the process.

(3) If the values n=2, k=3, and h=1 are incorporated into equation (32) for a conventional MA control chart, the loss cost will be \$468.315328. Thus, the

economic design cost is reduced to 89.19 % of that on a conventional chart. The loss cost is reduced to 97.71% if the values n=2, k=3 and h=0.5 are used, and to 78.37% if the values n=2, k=3 and h=1.5 are used in the conventional chart.

6. SENSITIVITY ANALYSIS

This section discusses the robustness of the model when the time, cost, shift and failure-rate parameters vary. The values of the parameters given in the example are assumed to be the basic case, the joint effect-cost item and parameters remaining the same, and the unique cost item or input parameters such as fc/vc, δ, λ , U, W, D or Y being changed by ±10%, ±25% and ± 50% of the original data under subgroup size n=2 to determine the trend in the minimum loss cost. For each of the 6×7 cases run, the optimal values of n, h and k are determined, the results of which are shown in Table 3. Such an analysis also gives an indication of the sensitivity to each of the input parameters. When the sampling cost (fc/vc) varies from 0.5, 0.75, 0.9, 1, 1.1, 1.25, to 1.5 times the original value, the estimated minimum cost will be 90.90% to 106.60% of the original and h will change in the same direction in which fc/vc changes. Also the k value will decrease when the fc/vc value increases. Other interesting observations from Table 3 are the following:

(1) The values of h and k will change in the same direction with the magnitude of shift (δ). The loss cost will decrease from 136.47% to 89.71% when δ increases.

(2) When the rate of occurrence of assignable causes (λ_m) increases, the values of h decrease and the loss cost increases, but the value of k has no significant effect. That is, the higher the rate at which assignable causes occur, the shorter the time of sampling intervals. The loss cost changes from 61.35% to 133.05% of the original.

(3) The increasing cost (U_m) when the process is out-of-control has the same effect as λ_m . The smaller the U_m, the larger the h. The loss cost changes from 62.81% to 135.14% of the original.

(4) The location and repair cost (W_m) have only a negligible effect on h or k, whereby the loss cost changes from 97.68% to 102.32% of the original.

(5) The repair time (D_m) has only a negligible effect on h or k, whereby the loss cost changes from 86.77% to 112.70%.

(6) The time for locating the assignable cause (Y) also has a negligible effect on h, k, whereby the loss cost changes

from 90.06 to 109.71% of the original.

(7) It is necessary to pay more attention to obtaining the parameters of the sampling cost (fc/vc), the magnitude of shift (δ_m), the rate of occurrence of assignable causes (λ_m) and the increasing cost (U_m) when the process is out-of-control. Detailed changing rates of loss cost are shown in Table 4.

If only the minimum or maximum value of δ, λ, U, (i.e. assignable cause m =1 or 7) is changed with the rate of 50% and 150%, we can obtain the loss cost rates shown in Table 5 with the following results:

(1) Decreasing the magnitude of shift (δ) is more sensitive to the loss cost than increasing the magnitude of shift; however, there is only a negligible difference between the small magnitude of shift (δ=1) and the large magnitude of shift (δ=3), meaning that estimating the small magnitude of shift can be done more carefully.

(2) There is almost the same effect when only the rate of occurrence of the assignable causes λ_1 or λ_7 changes.

(3) The increasing cost when the process is out-of-control, U_1 or U_7, also has the same effect on the loss cost.

7. CONCLUSIONS

This report has shown the detailed development of an MA control chart with two failures under multiple assignable causes for continuous-flow processes from an economic viewpoint by providing economically optimum values of n, h and k in consideration of the shift, time and relative cost parameters involved. A numerical example showing that the minimum loss cost obtained when n=2, h=0.6375 and k=2.6547 has been presented. This result is different from the conventional chart set with k=3, n=2 or 3. Also when two failures in the process instead of only one are considered, the loss cost increases only 1.50%. If the values n=2, k=3, and h=1 are incorporated into a conventional MA control chart, the economic design cost is reduced to 89.19 % of that of a conventional chart, and the loss cost is reduced to 97.71% if the values n=2, k=3 and h=0.5 are used and to 78.37% if the values n=2, k=3 and h=1.5 are used. A sensitivity analysis has also shown that the parameters of magnitude of shift (δ), increasing cost (U) when the process is out-of-control, and the rate of occurrence of assignable causes (λ) should receive more attention for estimating the data for loss cost calculation.

Table 1 Input Data for MA Control Chart

A_m	δ_m	NE_m	P_m	λ_m	U_m	W_m	Z_m
1	1.0	.303	.0228	.004502	575	1647	3.295
2	1.5	.236	.0668	.003503	1684	1283	2.566
3	1.8	.203	.1151	.003013	2902	1104	2.207
4	2.0	.184	.1587	.002732	4000	1000	2.000
5	2.2	.166	.2119	.002464	5342	902	1.805
6	2.5	.143	.3085	.002123	7778	777	1.555
7	3.0	.112	.5000	.001663	12606	609	1.218

Table 2 Optimal Design Parameters and Loss Cost

n	h	k	loss cost
2	0.6375	2.6547	471.711099
3	0.5680	2.8227	417.878681
4	0.5258	2.9078	423.778232
5	0.4977	2.9484	430.578396
6	0.4773	2.9664	437.127516
7	0.4617	2.9727	443.163672
8	0.4492	2.9742	448.673739
9	0.4391	2.9727	453.709441
10	0.4305	2.9695	458.332112
11	0.4227	2.9672	462.597976
12	0.4156	2.9641	466.555946
13	0.4094	2.9609	470.247479

Table 3 Loss Costs in Sensitivity Analysis

	50% 100%				75% 125%				90% 150%			
	n	h	k	loss cost	n	h	k	loss cost	n	h	k	loss cost
fc/vc change	2	0.4187	2.8484	379.700426	2	0.5352	2.7375	400.660400	2	0.5977	2.6852	411.240648
	2	0.6758	2.6258	423.799998	2	0.7297	2.5875	432.332324	2	0.8164	2.5305	445.262344
δ change	2	0.5594	2.2617	553.538222	2	0.5930	2.4867	464.005003	2	0.6195	2.5930	432.903134
	2	0.6555	2.7086	405.612634	2	0.6844	2.7750	391.537829	2	0.7375	2.8508	374.710981
λ change	2	0.8555	2.6547	256.240959	2	0.7180	2.6539	340.813909	2	0.6648	2.6547	387.711766
	2	0.6133	2.6555	446.819690	2	0.5836	2.6562	488.98448	2	0.5445	2.6586	555.757113
U change	2	0.8727	2.6148	262.353418	2	0.7289	2.6398	341.492617	2	0.6703	2.6492	387.507637
	2	0.6086	2.6586	447.589807	2	0.5727	2.6633	491.876771	2	0.5242	2.6695	564.498701
W change	2	0.6352	2.6516	408.002902	2	0.6367	2.6523	412.857795	2	0.6367	2.6539	415.769946
	2	0.6367	2.6562	419.652072	2	0.6383	2.6562	422.562826	2	0.6383	2.6586	427.4130326
D change	2	0.6211	2.6523	362.447580	2	0.6289	2.6539	390.366490	2	0.6336	2.6547	406.841006
	2	0.6406	2.6547	428.492189	2	0.6461	2.6547	444.499509	2	0.6531	2.6570	470.748910
Y change	2	0.6297	2.6586	376.201399	2	0.6328	2.6570	397.075085	2	0.6359	2.6555	409.484945
	2	0.6391	2.6539	425.899880	2	0.6414	2.6523	438.113512	2	0.6445	2.6508	458.286366

Table 4 Sensitivity Analysis Summary

item	span	fc/vc	δ	λ	U	W	D	Y
1	-0.50	0.909002	1.364697	0.613441	0.628074	0.976759	0.867699	0.900626
2	-0.25	0.959181	1.143961	0.815908	0.817533	0.988381	0.934537	0.950597
3	-0.10	0.984510	1.067282	0.928182	0.927693	0.995353	0.973977	0.980307
4	original	1	1	1	1	1	1	1
5	0.10	1.014577	0.965300	1.069686	1.071530	1.004647	1.025810	1.019604
6	0.25	1.035003	0.923815	1.170628	1.177553	1.011615	1.064131	1.048843
7	0.50	1.065958	0.897058	1.330482	1.351409	1.023226	1.126972	1.097137

Table 5 Loss cost rates of original when only one datum of seven assignable causes changes

Item of Assignable cause	changing rate	δ	Changing λ	parameter U	Item of Assignable cause	changing rate	δ	Changing λ	parameter U
1	-50%	104.77%	97.33%	97.31%	7	-50%	105.97%	92.76%	92.70%
1	+50%	98.30%	102.49%	102.63%	7	+50%	98.78%	107.14%	107.23%

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