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A Method for Checking Missed Eigenvalues in Eigenvalue Analysis with Damping Matrix

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ABSTRACT

In the case of the non-proportionally damped system such as the soil-structure interaction system, the structural control system and composite structures, the eigenproblem with the damping matrix should be necessarily performed to obtain the exact dynamic response. However, most of the eigenvalue analysis methods such as the subspace iteration method and the Lanczos method may miss some eigenvalues in the required ones. Therefore, the eigenvalue analysis method must include a technique to check the missed eigenvalues to become the practical tools. In the case of the undamped or proportionally damped system the missed eigenvalues can easily be checked by using the well-known Sturm sequence property, while in the case of the non-proportionally damped system a checking technique has not been developed yet. In this paper, a technique of checking the missed eigenvalues for the eigenproblem with the damping matrix is proposed by applying the argument principle. To verify the effectiveness of the proposed method, two numerical examples are considered.

Keywords: non-proportional damping, eigenproblem with damping, missed eigenvalues, argument principle

1. Introduction

Most of the eigenvalue analysis methods such as the subspace iteration method and the Lanczos method may miss some eigenpairs in the required ones, because the methods do not calculate the complete eigenvector set of a structure but the lowest small portion of this set. The exact dynamic response cannot be obtained by using the lowest incomplete eigenvectors. For the practical eigenvalue analysis method, a technique to check the missed eigenvalues must be included.

The well-known Sturm sequence property has hitherto been applied to check the missed eigenvalues (Bathe, 1996; Meirovitch, 1980; Petyt, 1990; Huyhes, 1987). The technique using the Sturm sequence property is used in the commercial finite element computer program such as ADINA. However, this technique can only be applied to the eigenproblem without the damping matrix such as the cases of the undamped and proportionally damped system (Newland, 1989).

In the case of the non-proportionally damped system such as the soil-structure interaction system, the structural control system and composite structures, the eigenproblem with the damping matrix should be analyzed to obtain the exact dynamic response. A number of researches[6-10] have been performed to solve the eigenproblem with the damping matrix, whereas there have been few studies on a technique to check the missed eigenvalues in this case in the literature.

If the technique using the Sturm sequence property can be expanded to the eigenproblem with the damping matrix, the missed eigenvalues for structures with non-proportional damping can easily be checked by this technique. However, since the Sturm sequence property was basically

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derived to count the number of roots of a polynomial in real domain, it cannot be directly applied to the eigenproblem having the complex eigenvalues. Tsai and Chen (1993) proposed the extended Sturm sequence property to count the number of roots of a polynomial in complex domain. The technique proposed by Tsai and Chen has to know all the coefficients of the polynomial, which needs a large number of operations. Hence, the technique cannot be applied to the characteristic polynomial of the large eigenproblem.

In this paper, the argument principle (Carrier *et al.*, 1966; Henrici, 1974; Korn and Korn, 1968; Spiegel, 1964; Franklin *et al.*, 1994) which can count the number of the eigenvalues inside a simple closed contour in complex plane, is used to check the missed eigenvalues for eigenproblem with the damping matrix. And, by using the iterative approach in which one discretizes the contour into a set of checking points and calculates the argument at each checking point by the LDL^T factorization process, a technique of checking the missed eigenvalues for large eigenproblem is developed.

This paper organized as follows. A technique using the argument principle is presented and considerations of the proposed method are discussed in Chapter 2. In Chapter 3, numerical examples are analyzed to verify the effectiveness of the proposed method. Finally, the concluding remarks and further studies are expressed in Chapter 4.

2. Technique Of Checking Missed Eigenvalues

2.1 Theory

The eigenpairs of the non-proportionally damped system can be obtained by solving the following eigenproblem with the damping matrix:

$$\lambda_1^2 M\phi_i + \lambda_i C\phi_i + K\phi_i = 0 \tag{1}$$

where M, C, and K are the n by n mass, damping and stiffness matrices, respectively, λ_i the ith eigenvalue and ϕ_i the corresponding eigenvector.

As mentioned earlier, a technique of checking the missed eigenvalues for the above quadratic eigenproblem has not been developed yet. Now, by applying the argument principle, a technique of checking the missed eigenvalues is proposed.

First, let us consider the relationship between the eigenvalues of an eigenproblem and the zeros of the corresponding characteristic polynomial. That is, the eigenvalues of the quadratic eigenproblem as Eq. (1) are equal to the zeros of the following characteristic polynomial:

$$\begin{aligned} f(\lambda) &= det(\lambda^2 M + \lambda C + K) \\ &= a_0 + a_1\lambda^{2n-1} + \cdots + a_{2n-1}\lambda^{2n-1} + a_{2n}\lambda^{2n} \end{aligned} \tag{2}$$

where λ is complex value and $a_i (i = 0, 1, \ldots, 2n)$ the real coefficients.

And, the argument principle (Carrier *et al.*, 1966; Henrici, 1974; Korn and Korn, 1968; Spiegel, 1964; Franklin *et al.*, 1994) can be applied to the above characteristic polynomial as follows: if the polynomial $f(\lambda)$ is analytic inside and on a simple closed contour S, the following equation is introduced

$$N = \frac{1}{2\pi i}\oint_S \frac{f'(\lambda)}{f(\lambda)} d\lambda = \frac{\Delta\theta}{2\pi} \tag{3}$$

where N is the number of zeros of $\phi(\lambda)$ inside the contour S and $\Delta\theta$ the variation of the argument θ of $\phi(\lambda)$ around the contour S.

Eq. (3) means that a polynomial $f(\lambda)$ maps a moving point λ describing the contour S into a moving point $f(\lambda)$ that encircles the origin of the $f(\lambda)$-plane N times if the

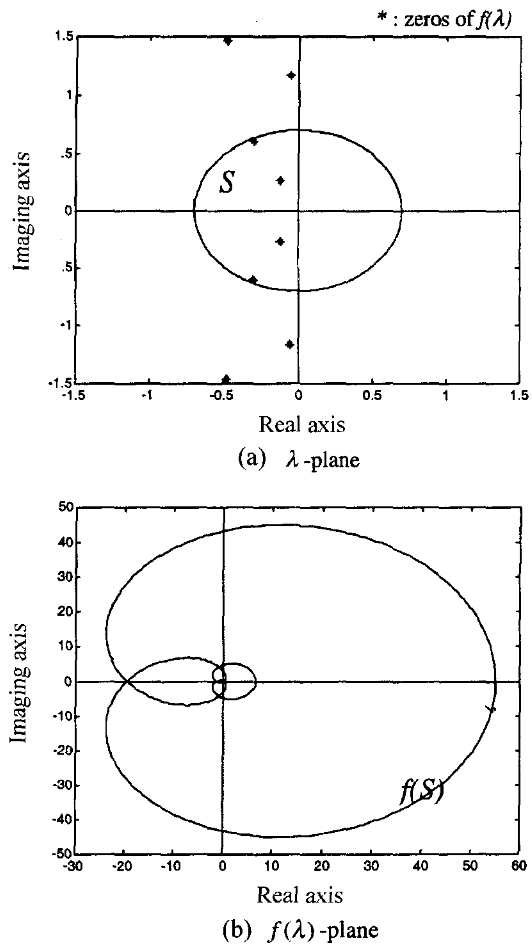


Fig. 1. Argument principle.

polynomial $f(\lambda)$ has N zeros inside the contour S in the λ-plane. As seen from Fig. 1, a moving point $f(\lambda)$ encircles the origin of the $f(\lambda)$-plane four times because the polynomial $f(\lambda)$ has four zeros inside the contour S in the $f(\lambda)$-plane.

However, since it is impossible to directly find the characteristic polynomial of the large eigenproblem by using the symbolic algebraic operations, the numerical, or the iterative, approach is needed to apply the aforementioned argument principle to the large eigenproblem with the damping matrix. The following two strategies are introduced to perform the iterative approach. The first strategy is the discretization of the simple closed contour S, and the second one the relationship between the characteristic polynomial and the factorized matrices by the LDL^T factorization process. That is, the contour S is considered as the set of the checking points as described in Fig. 2. And, the LDL^T factorization process is performed at each checking point. Then, the argument at each checking point can calculate as follows (Korn and Korn, 1968; Pearson, 1974):

$$f(\lambda_j) = det(\lambda_j^2 M + \lambda_j C + K) = detLDL^T = \prod_{i=1}^{n} d_{ii} = r_j \angle \theta_j \tag{4}$$

where d_{ii} is the diagonal elements of the diagonal matrix D, and r_j and θ_j the magnitude and argument of the value $f(\lambda_j)$ in polar form, respectively. The number of the eigenvalues inside the contour S is calculated by summing the variation of the argument of each checking point.

The process of checking the missed eigenvalues using the argument principle is briefly described in Fig. 2. First, we check the upper-half plane along the arc in counterclockwise (1). And then we perform the LDL^T factorization at each checking point. The checking process in the real axis can skip because of no variation of the argument (2). The total variation of the arguments is calculated by summing the variation of the argument of each checking point. Finally, we check the missed eigenvalues to compare the total rotation number (N in Eq. (3)) with the number of the considered eigenvalues. The shape and size of the simple closed contour S and the number of checking points are discussed in detail in the next section.

2.2 Considerations

2.2.1 Shape and Size of the Simple Closed Contour

In applying the proposed method to a practical problem, it is very important to properly choose the shape and the size of the simple closed contour S. First, let us consider the shape of the contour S. The simplest shape of the contour is a circle as shown in Fig. 3(a). Since the eigenvalues are always complex conjugate pairs in the case of the underdamped system, the contour can be considered only in the upper half-plane as in Fig. 3(b). In the case of the stable structures, all the eigenvalues exist in the left half-

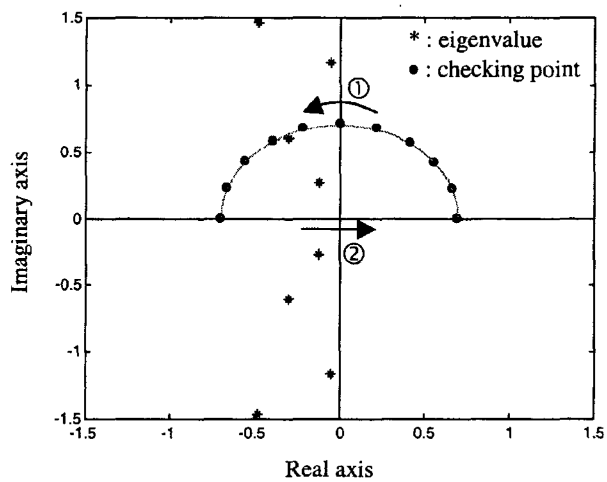


Fig. 2. Process of checking the missed eigenvalues using the argument principle.

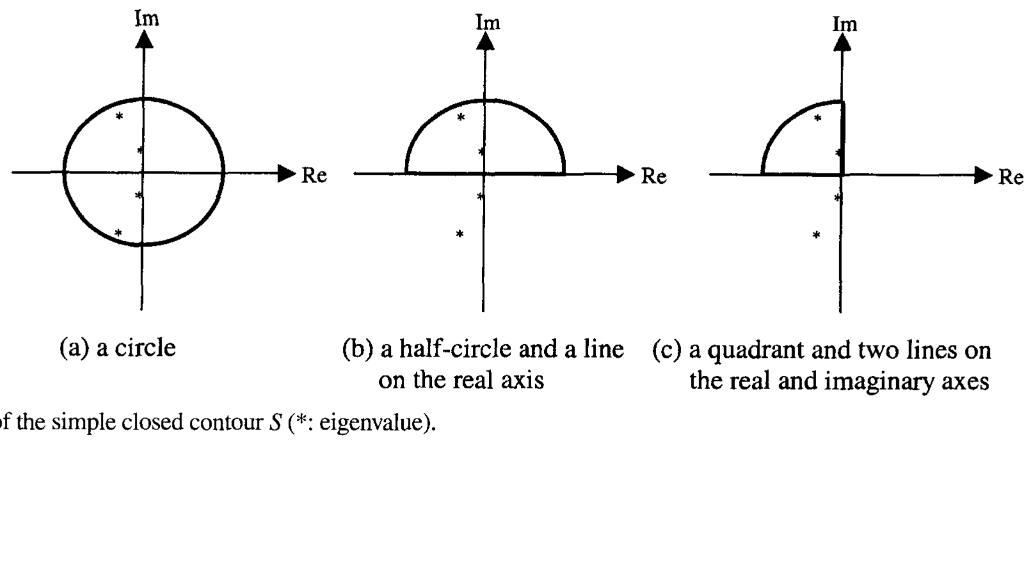


Fig. 3. Shapes of the simple closed contour S (*: eigenvalue).

plane. We, therefore, can consider only in the second quarter-plane as in Fig. 3(c). In the case of the lightly damped system, however, the contour as in Fig. 3(c) causes the difficulties in the argument jump because the part of the contour along the imaginary axis passes very close to the eigenvalues. As seen from the above discussion, we conclude that a half-circle and a line on the real axis as in Fig. 3(b) are the most appropriate contour to check the missed eigenvalues for an eigenproblem with the damping matrix.

The size of the contour, i.e., the radius of a half-circle as in Fig. 3(b) should be only a very little larger than the largest eigenvalue to be considered to ensure that the next largest eigenvalue is not within the contour. However, if the difference between the radius of the half-circle and the magnitude of the largest eigenvalue is too small, the argument jump occurs in the part of the contour that is close to the largest eigenvalue. To exactly check the missed eigenvalues, therefore, it is important to properly choose the size of the contour. The Sturm sequence check proposed by Bathe (1996) has used 1.01 times the magnitude of the largest eigenvalue for the eigenproblem without the damping matrix. As the results from analyzing several eigenproblems with the damping matrix, however, the value used by Bathe (1996) is too large to ensure that the next largest eigenvalue is not within the contour. In the proposed method, therefore, the size of the contour is chosen by 1.005 times the magnitude of the largest eigenvalue, and the part of the contour close to the largest one is subdivided to check the drastic variation of the argument without any difficulty.

2.2.2 Number of Checking Points

If checking points are chosen sufficiently a lot, the missed eigenvalues can exactly be checked by the proposed method. Using more checking points, however, will increase the computational effort. Since the optimal number of checking points cannot be obtained by analytic operations, the optimal value should be found by results of analyzing the numerous numerical examples. Through five numerical examples with 50 to 1018 degrees of freedom, we have found that six times the number of eigenvalues considered is the most effective. Only the two of them are shown as numerical examples. After the contour is equally divided into checking points, the part of the contour close to the largest eigenvalue is subdivided because the argument jump occurs in the part of the contour close to an eigenvalue. And, if the drastic change of the variation of the argument between two adjacent checking points may occur, the extra checking points between two adjacent checking points should be added. The further study on the optimal number of checking points is now in progress. As seen from the above discussion, the algorithm of the proposed method can be expressed in Table 1.

Table 1. Algorithm of the proposed method
(q: number of calculated eigenvalues, p: number of considered eigenvalues ($p = q/2$)).

Step 1: Calculate the size of the contour, ρ.
- Select 1.005 times the magnitude of the qth eigenvalue ($\rho = 1.005|\lambda_q|$).

Step 2: Determine the initial checking points.
- Divide the contour into $6p$ equal parts.
- If necessary, subdivide the part of the contour that is close to an eigenvalue.

Step 3: Perform the checking process.
- Perform the LDL^T factorization at each checking point.
- Calculate the argument θ_j at each checking point.

Step 4: Analyze the variations of the arguments.
- If an aggressive variation of the argument occurs at a checking point, then go to Step 5 and if not, go to Step 6.

Step 5: Add the extra checking points.
- Go to Step 3.

Step 6: Check the missed eigenvalues.
- Calculate the total variation of the argument and the number of rotations.
- Compare the number of rotations (N in Eq. (3)) with the number of considered eigenvalues (p).

3. Numerical Examples

To show the effectiveness of the proposed method, two numerical examples are analyzed. First, a simple spring-mass-damper system that has the correct analytical eigenvalues is considered to verify that the proposed method can exactly check the missed eigenvalues of the eigenproblem with the damping matrix. And the three-dimensional frame structure is considered to verify that the proposed method can be applied to large structural systems in practice.

3.1 Simple Spring-Mass-Damper System

The finite element discretization of the system results in a diagonal mass matrix, a tridiagonal damping and stiffness matrices of the following forms

$$M = mI \tag{5}$$

$$C = \alpha M + \beta K \tag{6}$$

Table 2. The lowest six eigenvalues by analytical solutions.

Mode Number	Eigenvalues	
	Real	Imaginary
1	–0.02524	+0.01817
2	–0.02524	–0.01817
3	–0.02718	+0.08923
4	–0.02718	–0.08923
5	–0.03103	+0.15224
6	–0.03103	–0.15224

$$K = k\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & O & O & \\ & & O & 2 & -1 \\ & & & -1 & 1 \end{bmatrix} \qquad (7)$$

where α and β are the damping coefficients of the Rayleigh damping. The analytical solutions can be relationships

$$\lambda_{2i-1,2i} = -\xi_i\omega_i \pm j\omega_i\sqrt{1-\xi_i^2} \quad \text{for } i = 1,\cdots,n \qquad (8)$$

$$\xi_i = \frac{1}{2}\left(\frac{\alpha}{\omega_i}+\beta\omega_i\right) \qquad (9)$$

$$\omega_i = 2\sqrt{\frac{m}{k}}\sin\frac{2i-1}{2n+1}\frac{\pi}{2} \qquad (10)$$

where ω_i and ξ_i are the undamped natural frequency and modal damping ratio, respectively.

A system with order 50 is used in analysis. k and m are 1, and the coefficients, α and β, of the Rayleigh damping are 0.05 and 0.5, respectively. The lowest six eigenvalues by analytical solutions are expressed in Table 2. No missed eigenvalues exist in the lowest six eigenvalues because these are the analytical solutions.

The process of checking the missed eigenvalues is as follows. The number of the considered eigenvalues is three ($p = 3$) and the size of the contour is calculated by the 1.005 times the magnitude of the sixth eigenvalue ($\rho = 1.005|\lambda_6| = 0.1561$). The half-circle with radius ρ in the upper half-plane are divided into eighteen equal parts. And, since the argument of the largest eigenvalue is 101.5°, the part of contour between 101° and 102° is subdivided into four equal parts. The results of the first checking process are described in Table 3. The additional checking process does not need because there are no drastic change of the variation of the argument at all checking points. Since the total variation of the argument is 1080° as in Table 3, we conclude that the total number of rotations is

Table 3. The arguments and the variations of the arguments.

First checking process				$\sum\Delta\theta_j$
λ_j	θ_j	$\Delta\theta_j$	Y/N	
origin	0.0	–	–	0.0
$\rho\angle 10°$	108.9	108.9	N	108.9
$\rho\angle 20°$	215.4	106.5	N	215.4
$\rho\angle 30°$	316.5	101.1	N	316.5
$\rho\angle 40°$	50.0	93.5	N	410.0
$\rho\angle 50°$	133.2	83.2	N	493.2
$\rho\angle 60°$	203.7	70.5	N	563.7
$\rho\angle 70°$	259.2	55.5	N	619.2
$\rho\angle 80°$	297.6	38.4	N	657.6
$\rho\angle 90°$	317.5	19.9	N	677.5
$\rho\angle 100°$	329.9	12.4	N	689.9
$\rho\angle 101°$	348.0	18.1	N	708.0
$\rho\angle 101.25°$	5.7	17.7	N	725.7
$\rho\angle 101.5°$	44.8	39.1	N	764.8
$\rho\angle 101.75°$	88.0	43.2	N	808.0
$\rho\angle 102°$	108.4	20.4	N	838.4
$\rho\angle 110°$	139.1	30.7	N	859.1
$\rho\angle 120°$	151.8	12.7	N	871.8
$\rho\angle 130°$	173.2	21.4	N	893.2
$\rho\angle 140°$	201.6	28.4	N	921.6
$\rho\angle 150°$	235.7	34.1	N	955.7
$\rho\angle 160°$	274.4	38.7	N	994.4
$\rho\angle 170°$	316.4	42.0	N	1036.4
$\rho\angle 180°$	0.0	43.6	N	1080.0

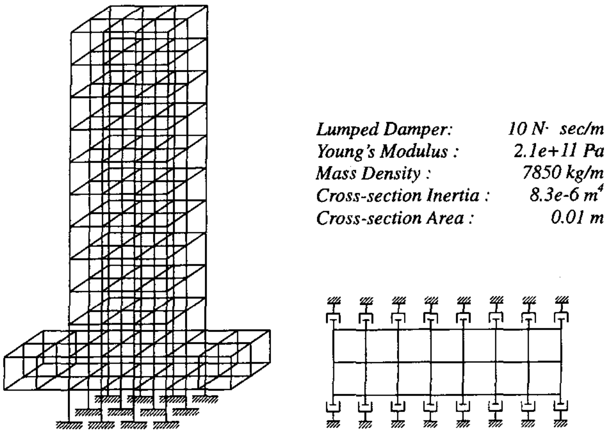
where $0° \le \theta_j < 360°$, $\Delta\theta_j = \theta_j - \theta_{j-1}$ and '*Y*' means that the additional checking points are required and '*N*' the additional ones are not required.

$$N = \frac{\sum\Delta\theta_j}{2\pi} = \frac{1080°}{360°} = 3$$

Finally, we check the missed eigenvalues by comparing the number of rotations with the number of the considered eigenvalues. Since the number of the eigenvalues inside the simple closed contour S and the number of rotations are all three, the missed eigenvalues do not exist in the simple closed contour S. As seen from this result, therefore, we verify that the proposed method is a technique of checking the missed eigenvalues for an eigenproblem with the damping matrix.

3.2 *Three-Dimensional Frame Structure with Concentrated Dampers*

In this example, a three-dimensional frame structure with concentrated dampers is presented. Two layers of the foundation are damped only in the horizontally transla-



(a) Three-dimensional frame structure (b) Damping from two-layer foundation

Fig. 4. Three-dimensional frame structure with concentrated dampers.

tional direction as shown in Fig. 4. The system could be considered as a representative of a control system or a passively damped space structure. The model has 1008 degrees of freedom. The material and cross-sectional properties are shown in Fig. 4. The consistent mass matrix is used to define M. The damping matrix C consists of the Rayleigh damping and concentrated dampers. The Rayleigh coefficient α is 0.0306 and β is 0.1016. The damping for each damper is 10.

The lowest ten eigenvalues are calculated by the Lanczos method developed by Kim and Lee (1999) as in Table 4. Since the missed eigenvalues may exist in the required ones, it is necessary to check the missed ones.

The number of the considered eigenpairs is five ($p = 5$). And, the size of the contour is as follows: $\rho = 1.005|\lambda_{10}| = 8.7914$. To select the initial checking points, the half-circle with radius ρ in the upper half-plane are divided into thirty equal parts. And, since the argument of the largest eigenvalue is 90.9°, the part of contour between 90° and 92° is subdivided into two equal parts. Since there are the aggressive increase or the decrease of the argument in the part of contour between 96° and 180°, the additional checking process in the part of the contour should be performed to calculate the exact variation of the argument. The results of the first and the second checking processes are described in Table 5. Since the total variation of the argument is 1800° as in Table 5, we conclude that the total number of rotations is

$$N = \frac{\sum \Delta\theta_i}{2\pi} = \frac{1800°}{360°} = 5$$

Table 4. The lowest ten eigenvalues of the three-dimensional frame structure with concentrated dampers by the Lanczos method.

Mode Number	Eigenvalues	
	Real	Imaginary
1	−0.0304	+3.0301
2	−0.0304	−3.0301
3	−0.0309	+3.0901
4	−0.0309	−3.0901
5	−0.0374	+3.6581
6	−0.0374	−3.6581
7	−0.1427	+8.6586
8	−0.1427	−8.6586
9	−0.1404	+8.7465
10	−0.1404	−8.7465

Finally, we check the missed eigenvalues by comparing the number of rotations with the number of the calculated eigenvalues. The missed eigenvalues do not exist in the simple closed contour S because the number of rotations and the number of the considered eigenvalues are all five. Therefore, the proposed method can be applied to large structural systems with the nonproportionally damping matrix in practice.

4. Conclusions

This paper presents a technique of checking the missed eigenvalues for the eigenproblem with the damping matrix by using the argument principle. To apply the proposed method to the large eigenproblem, the iterative approach is introduced. By analyzing the numerical examples, it is verified that the proposed method can exactly check the missed eigenvalues and can be applicable to the large eigenproblem. The proposed method is the first technique of checking the missed eigenvalues for the eigenproblem with the damping matrix.

The technique using the Sturm sequence property only requires one factorization process at one checking point. On the other hand, the proposed method requires many factorization processes at many checking points. This cannot be inevitable, because the proposed method is executed in the complex plane whereas the technique using the Sturm sequence property on the real axis. Therefore, the proposed method has a shortcoming that needs a large number of operation counts. To apply more effectively the proposed method to the practical problem, the research to reduce the operation counts of the proposed method should be performed.

Table 5. The arguments and the variations of the arguments.

First checking process				Second checking process				$\sum \Delta\theta_j$
z_j	θ_j	$\Delta\theta_j$	*Y/N*	z_j	θ_j	$\Delta\theta_j$	*Y/N*	
origin	0.00	–	–					0.00
$\rho\angle 6^\circ$	269.6	269.6	*N*					269.6
$\rho\angle 12^\circ$	176.2	266.6	*N*					536.2
$\rho\angle 18^\circ$	76.7	260.5	*N*					796.7
$\rho\angle 24^\circ$	328.1	251.4	*N*					1048.1
$\rho\angle 30^\circ$	207.5	239.4	*N*					1287.5
$\rho\angle 36^\circ$	72.1	224.6	*N*					1512.1
$\rho\angle 42^\circ$	279.2	207.1	*N*					1719.2
$\rho\angle 48^\circ$	106.1	196.9	*N*					1906.1
$\rho\angle 54^\circ$	270.3	104.2	*N*					2070.3
$\rho\angle 60^\circ$	49.5	139.2	*N*					2209.5
$\rho\angle 66^\circ$	161.5	112.0	*N*					2321.5
$\rho\angle 72^\circ$	244.6	83.1	*N*					2404.6
$\rho\angle 78^\circ$	297.9	53.3	*N*					2457.9
$\rho\angle 84^\circ$	323.2	25.3	*N*					2483.2
$\rho\angle 90^\circ$	355.7	32.5	*N*					2515.7
$\rho\angle 91^\circ$	43.0	47.3	*N*					2563.0
$\rho\angle 92^\circ$	83.8	40.8	*N*					2603.8
$\rho\angle 96^\circ$	96.2	12.4	*N*					2616.2
				$\rho\angle 99^\circ$	82.5	−13.7	*N*	2602.5
$\rho\angle 102^\circ$	64.6	**328.4 *or* −31.6**	*Y*	$\rho\angle 102^\circ$	64.6	−17.9	*N*	2584.6
				$\rho\angle 105^\circ$	44.4	−20.2	*N*	2564.4
$\rho\angle 108^\circ$	22.6	**318.0 *or* −42.0**	*Y*	$\rho\angle 108^\circ$	22.6	−21.8	*N*	2542.6
				$\rho\angle 111^\circ$	359.6	−23.0	*N*	2519.6
$\rho\angle 114^\circ$	335.7	**313.1 *or* −46.9**	*Y*	$\rho\angle 114^\circ$	335.7	−23.9	*N*	2495.7
				$\rho\angle 117^\circ$	310.8	−24.9	*N*	2470.8
$\rho\angle 120^\circ$	285.2	**309.5 *or* −50.5**	*Y*	$\rho\angle 120^\circ$	285.2	−25.6	*N*	2445.2
				$\rho\angle 123^\circ$	258.8	−26.4	*N*	2418.8
$\rho\angle 126^\circ$	231.6	**306.4 *or* −53.6**	*Y*	$\rho\angle 126^\circ$	231.6	−27.2	*N*	2391.6
				$\rho\angle 129^\circ$	203.6	−28.0	*N*	2363.6
$\rho\angle 132^\circ$	174.9	**303.3 *or* −56.7**	*Y*	$\rho\angle 132^\circ$	174.9	−28.7	*N*	2334.9
				$\rho\angle 135^\circ$	145.4	−29.5	*N*	2305.4
$\rho\angle 138^\circ$	115.2	**300.3 *or* −59.7**	*Y*	$\rho\angle 138^\circ$	115.2	−30.2	*N*	2275.2
				$\rho\angle 141^\circ$	84.4	−30.8	*N*	2244.4
$\rho\angle 144^\circ$	52.9	**297.7 *or* −62.3**	*Y*	$\rho\angle 144^\circ$	52.9	−31.5	*N*	2212.9
				$\rho\angle 147^\circ$	20.7	−32.2	*N*	2180.7
$\rho\angle 150^\circ$	348.0	**295.1 *or* −64.9**	*Y*	$\rho\angle 150^\circ$	348.0	−32.7	*N*	2148.0
				$\rho\angle 153^\circ$	314.8	−33.2	*N*	2114.8
$\rho\angle 156^\circ$	281.1	**293.1 *or* −66.9**	*Y*	$\rho\angle 156^\circ$	281.1	−33.7	*N*	2081.1
				$\rho\angle 159^\circ$	246.9	−34.2	*N*	2046.9
$\rho\angle 162^\circ$	212.4	**291.3 *or* −68.7**	*Y*	$\rho\angle 162^\circ$	212.4	−34.5	*N*	2012.4
				$\rho\angle 165^\circ$	177.5	−34.9	*N*	1977.5
$\rho\angle 168^\circ$	142.3	**289.9 *or* −70.1**	*Y*	$\rho\angle 168^\circ$	142.3	−35.2	*N*	1942.3
				$\rho\angle 171^\circ$	107.0	−35.3	*N*	1907.0
$\rho\angle 174^\circ$	71.4	**289.1 *or* −70.9**	*Y*	$\rho\angle 174^\circ$	71.4	−35.6	*N*	1871.4
				$\rho\angle 177^\circ$	35.7	−35.7	*N*	1835.7
$\rho\angle 180^\circ$	0.0	**288.6 *or* −71.4**	*Y*	$\rho\angle 180^\circ$	0.0	−35.7	*N*	1800.0

where $0^\circ \le \theta_j < 360^\circ$, $\Delta\theta_j = \theta_j - \theta_{j-1}$ and '*Y*' means that the additional checking points are required and '*N*' means that the additional ones are not required.

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