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Topology Optimization of Continuum Structures Using a Nodal Volume Fraction Method

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ABSTRACT

The general topology optimization can be considered as optimal material distribution. Such an approach can be unstable, unless composite materials are introduced. In this research, a nodal volume fraction method is used to obtain the optimum topology of continuum structures. This method is conducted from a composite material model composed of isotropic matter and spherical void. Because the appearance of the chessboard patterns makes the interpretation of the optimal material layout very difficult, this method contains a chessboard prevention strategy. In this research, several topology optimization problems are presented to demonstrate the validity of the present method and the recursive quadratic programming algorithm is used to solve the topology optimization problems.

Keywords: topology optimization, nodal volume fraction method, continuum structures, chessboard patterns, recursive quadratic programming algorithm

1. Introduction

The topology optimization using the design domain concept can be considered as the material layout optimization. In the topology optimization based on only the indicator function, which is one if the material occupies the point and zero otherwise, the existence of the solution is not guaranteed because the indicator function is not smooth. So, the concept of the composite material model is needed to relax this problem into the well-defined problem. Since the late of 1980's, there are many methods which introduce composite material model composed of matter and void into the topology optimization based on the concept of the design domain (Bendsøe and Kikuchi, 1988; Gea, 1996; Mlejnek and Schirrmacher, 1993; Swan and Kosaka, 1997; Yang and Chuang, 1994; Youn and Park, 1997). Of all these methods, there are mainly two approaches for topology optimization of continuum structures, namely, homogenization and density function method.

In the homogenization method (Bendsøe and Kikuchi, 1988), in order to find the optimal material distribution of a continuum structure, it is assumed that the material of the structure is not homogeneous and has a variable solid-cavity microstructure. This method places infinitely many microscale rectangular holes in design cells forming perforated materials. The sizes and orientation angle of the rectangular micro-cavity in each design cell are treated as design variables. It is the significant drawback of this method that the number of design variables is very large. And, this method requires much computational efforts for calculating the equivalent material properties of each design cell, because the displacement of each design cell is necessary to find the equivalent material properties. On the other hand, the density function method (Yang and Chuang, 1994) replaces the foam-like material with an equivalent homogeneous substitute and uses the material density of the substitute as the design variable. In this method, a fictitious relationship between the equivalent Young's modulus and the density is proposed. The rela-

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tionship is very simple and attractive because it is introduced from the engineer's intuition. However, the shortcoming of this method is the lack of theoretical support of the fictitious relationship.

Most structural topology optimizations using the displacement-based finite element method have an undesirable feature that material is distributed in the chessboard patterns. The cause of chessboard patterns is numerical rather than physical in nature (Jog and Haber, 1996). That is, the chessboard patterns are generated from the reason that the numerical stability of each design cell, which is a finite element, is not guaranteed in optimal iteration process. This numerical instability is caused from the assumption that the inner density of each element is constant. Because the presence of the patterns makes the interpretation of the optimal material layout difficult, it is necessary to suppress the formation of the anomalies. In order to overcome these anomalies, the use of higher order elements (Jog *et al.*, 1994), image processing method (Sigmund, 1994), density redistribution algorithm (Youn and Park, 1997), etc. have been studied in literature. The method using higher order elements has a drawback that the size of problem becomes too large to be practical since a large number of elements are needed to manage the necessary resolution. The other methods inspect the formation of chessboard patterns over design domain and reduce the porous regions in the optimal distribution during iterative process. These chessboard prevention methods may have an effect on the stability of the optimization algorithm.

The main objective of this research is the development of a topology optimization method with chessboard prevention strategy. In this research, the developed topology optimization method is named the nodal volume fraction method. In order to achieve the main objective, the present research uses 1) a composite material model made up of isotropic matter and spherical void, 2) volume fraction of each node as design variable and 3) the shape function of a linear finite element as a interpolation function.

The composite material model is selected to solve the significant shortcomings of the homogenization and density function method. The equivalent material properties of the composite material model are conducted using the Mori-Tanaka mean field theory (Mori and Tanaka, 1973) in conjunction with Eshelby's equivalence principle (Eshelby, 1957). This approach gives a simple closed-form relationship between the equivalent Young's modulus and volume fraction of each element. This relationship is identical to that proposed by Gea (1996). However, the optimal material layout of a continuum structure based on this relationship has many elements in which volume fractions are intermediate values between void and solid material (Lim and Lee, 1998). Therefore, in order to obtain a rigorous composite material model, a penalty factor is introduced into the relationship. The numerical role of a penalty factor is to ensure that the design domain is primarily occupied by either void or solid material. The bilinear function utilizing nodal volume fractions and shape functions of linear elements is the key to prevent the chessboard patterns entirely. Since the change of the penalty factor does not entirely prevent the chessboard patterns, a strategy is necessary to obtain the optimal material layout without chessboard patterns. This strategy has no effect on the stability of the optimization algorithm because the strategy does not need to investigate the formation of chessboard patterns during optimal iterative process. In this research, several topology optimization problems are presented to demonstrate the validity of the nodal volume fraction method and the recursive quadratic programming (RQP) algorithm, PLBA (Pshenichny-Lim-Belegundu-Arora) algorithm (Lim and Arora, 1986), is used to solve the topol-

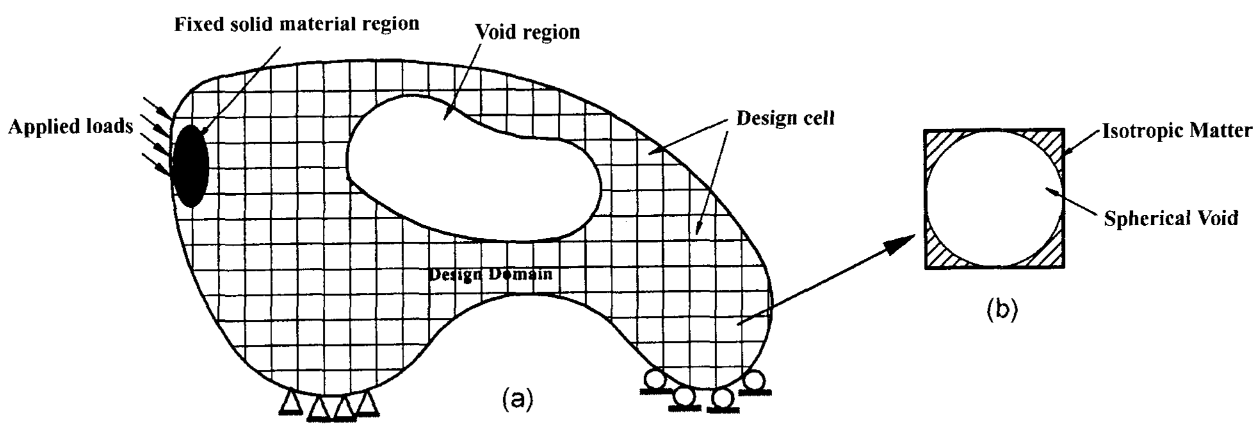


Fig. 1. (a) The generalized structure for optimal material distribution and (b) the composite material model.

ogy optimization problems.

2. Nodal Volume Fraction Method

As shown in Fig. 1, we consider a complete undeformed design domain which is divided into suitable design cells in order to find the optimal material distribution for a continuum structure. The suitable design cells intimate that the design domain can be discretized in a number of different ways. That is, it can be modeled by full three-dimensional elements or plate elements with in or out of plane forces, etc. In Fig. 1, the design domain is subjected to the applied loading and boundary conditions. Initially, the design domain can have void region and fixed solid material region.

To analyze the optimal material distribution problem, as shown in Fig. 1, assume that each design cell is made of a composite material model which consists of isotropic solid matter and isotropic void. In order to estimate the equivalent Young's modulus of this model, we use the Mori-Tanaka approximation and Eshelby's equivalence principle. The relationship between the equivalent Young's modulus and volume fraction of each design cell is defined as follows:

$$E_H = \frac{c_o}{2-c_o} E_o \quad (1)$$

where E_H represents the equivalent Young's modulus of the composite model, c_o and E_o are the volume fraction of each design cell and the true Young's modulus for the structure, respectively.

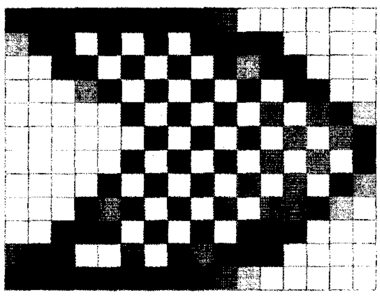
In this research, we introduce a penalty factor to Eq. (1) as follows because the optimal material layout based on Eq. (1) has many intermediate volume fractions and the composite material model can not present a perfect void of each design cell. The modification form of Eq. (1) is:

$$E_H = \frac{c_o^p}{2-c_o} E_o \quad (2)$$

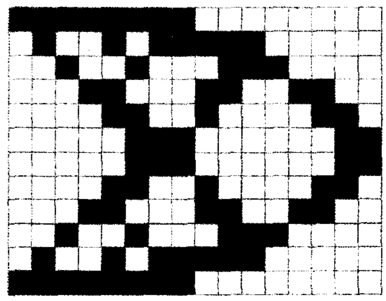
where p is the penalty factor which has a value more than '1'.

In order to investigate the effect of the penalty factor on the result of optimal topology problem, we consider a realistic problem that a short cantilever beam is subjected to a vertical load at the middle of the right side and its entire left side is clamped. This problem is well-known topology optimization problem and the result has been reported in many literatures. In this example, the design objective is to minimize the mean compliance while satisfying the weight constraint is set to be less than 35% of the total weight. The effects of the penalty factor on the optimal material distribution of a short beam are presented in Fig. 2.

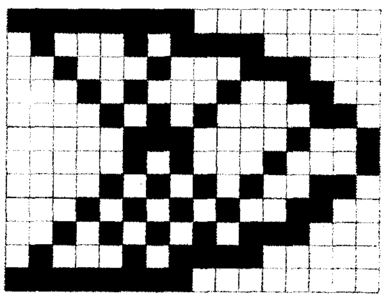
In Fig. 2, the white design cells represent the voids. The more black a design cell is, the larger a volume fraction is. Thus, the most black design cells represent that the volume fractions of the design cells are '1'. As shown in



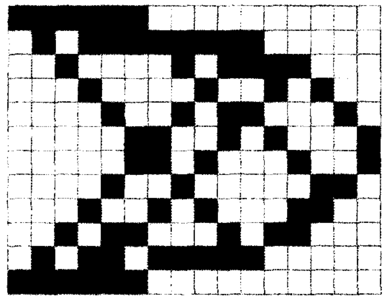
(penalty factor p=1)



(penalty factor p=3)



(penalty factor p=4)



(penalty factor p=5)

Fig. 2. The effects of the penalty factor on the optimal material distribution.

Fig. 2, using a penalty factor $p = 1$, the chessboard patterns are appeared remarkably in this result. The results, using a penalty factor more than '3', are represented by the rough optimal material distributions. However, in the case of a penalty factor $p = 3$, the layout has the volume fractions neighboring void or solid and shows the smooth optimal material distributions. Therefore, in this research, we use the penalty factor $p = 3$ in Eq. (2).

As shown in Fig. 2, it is known that the change of the penalty factor does not entirely prevent the chessboard patterns. The chessboard patterns are generated from the assumption that the inner volume fraction of each cell is constant. In this research, a strategy is developed which efficiently prevents the chessboard patterns and has no effects on the stability of the optimization algorithm. The basic idea of this strategy is to relax the assumption that the inner volume fraction of each design cell is constant. That is, we assume that the inner volume fraction of each design cell can be expressed as a continuous function of the shape function and the nodal volume fraction. The physical meaning of 'volume fraction of a certain node' is equivalent to the volume fraction of infinitely small cell in the design domain. Using this assumption, the inner volume fraction of each design cell is defined as:

$$c_o = L_1 c_1 + L_2 c_2 + \cdots + L_n c_n = \sum_{i=1}^{n} L_i c_i \tag{3}$$

where L_i is the shape function, c_i is the nodal volume fraction which is used for the design variable in this research and n denotes the total number of nodes in design cell. Therefore, when Eq. (3) is substituted into Eq. (2), the relationship between the equivalent Young's modulus and the design variable is derived as follows:

$$E_H = \frac{\left(\sum_{i=1}^{n} (L_i c_i)\right)^p}{2 - \sum_{i=1}^{n} (L_i c_i)} E_o \tag{4}$$

Next, we assume also that the inner volume fraction of each design cell is same as the center volume fraction of each design cell. From Eq. (4), the unique value of the equivalent Young's modulus is calculated for each design cell. Using this value, the stiffness matrix of each design cell can be symmetrically organized.

Finally, the optimum values of nodal volume fractions obtained from topology optimization can be interpolated into the center value of each design cell by the shape function. This center value of each design cell shows the optimal material distribution without the chessboard patterns. Using this strategy, the optimal material layout of the short cantilever beam is as follows. This optimal layout is quite similar to the results obtained from many researches.

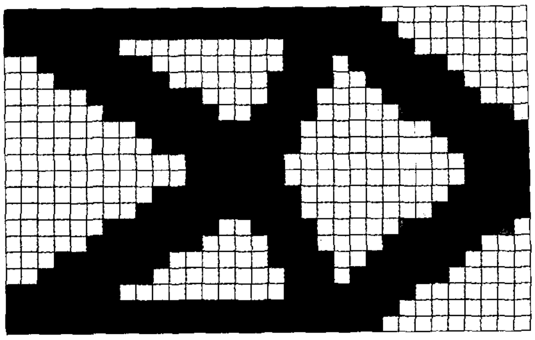


Fig. 3. The optimal material distribution of the short cantilever beam.

3. Problem Formulation and Sensitivity Analysis

In this research, we consider two objective functions. The first is to minimize the mean compliance and the second is to maximize the natural frequency of a designated mode. For each case, same constraint, i.e. a total material usage constraint, is imposed. And, the volume fraction of each node is considered as the design variable. Design variables are continuous varying between '0' and '1'. Therefore, the volume fraction of the void phase is theoretically 0. But, in numerical analysis, the volume fraction of the void phase is generally a small value compared to that of the solid phase in order to maintain the state to avoid the singularity of the finite element analysis. Thus, the volume fraction of the void material is taken as $c_{void} = 10^{-3} c_{solid}$. The following problem formations are analogous to well known formulations for sizing optimization.

3.1 Minimizing the mean compliance of a structure

3.1.1 Problem formulation

Find : c_i

Minimize : $W = \{D(c_i)\}^T [K(c_i)] \{D(c_i)\}$

Subject to : $[K(c_i)]\{D(c_i)\} = \{F\}$

$g = m_T(c_i) - m_A \le 0$

$$10^{-3} \le c_i \le 1, \; i = 1, \cdots, N \tag{5}$$

where $\{D(c_i)\}$ and $[K(c_i)]$ represent the global displacement and the global stiffness matrix respectively, $\{F\}$ represents the applied load, m_T is the total usage mass, m_A is the allowable mass, c_i is the design variable, and N is the total number of design variables which is equivalent to the total number of nodes.

In order to evaluate the global stiffness matrix as a function of design variables, it is necessary to know the stress-strain matrix of each element. This matrix is simply derived by using Eq. (4), for example, the matrix for the plane-stress element is as follows:

$$[E_e] = \frac{\left(\sum_{i=1}^{n}(L_i c_i)\right)^p}{2-\sum_{i=1}^{n}(L_i c_i)} \frac{E_o}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \tag{6}$$

where $[E_e]$ represents the stress-strain matrix and v denotes the Poisson's ratio of the true material of a structure.

3.1.2 Sensitivity Analysis

The first order derivative of the objective function Eq. (5) with respect to design variables ci can be easily calculated as follows:

$$\frac{dW}{dc_i} = \{D\}^T\frac{\partial[K]}{\partial c_i}\{D\} + 2\{D\}^T[K]\frac{\partial\{D\}}{\partial c_i} \tag{7}$$

And then, substitute the following expression into Eq. (7),

$$\frac{\partial\{D\}}{\partial c_i} = [K]^{-1}\left(\frac{\partial\{F\}}{\partial c_i} - \frac{\partial[K]}{\partial c_i}\{D\}\right) \tag{8}$$

where, in general, $\partial\{F\}/\partial c_i = 0$ because applied load is independent of the design variables. Thus, we can derive the simplified form of the first order design sensitivity analysis of the objective function as:

$$\frac{dW}{dc_i} = -\{D\}^T\frac{\partial[K]}{\partial c_i}\{D\} \tag{9}$$

In Eq. (9), the derivative of the global stiffness matrix can be presented as follows:

$$\frac{\partial[K]}{\partial c_i} = \sum_{j=1}^{NBE}\int_{v_j}[B_j]^T\frac{\partial[E_j]}{\partial c_i}[B_j]dv_j \tag{10}$$

where NBE is the total number of neighboring element of i th node, v_j is the volume of j th element, $[B_j]$ and $[E_j]$ represent the strain-displacement matrix and elasticity matrix of j th element, respectively.

3.2 Maximizing a certain natural frequency of a structure

3.2.1 Problem formulation

Find : c_i

Maximize : $f_j(c_j)$

Subject to : $[K(c_i)]\{y\}_j = \lambda_j[M(c_i)]\{y\}_j$

$$g = m_T(c_i) - m_A \le 0$$

$$10^{-3} \le c_i \le 1, i = 1, \cdots, N \tag{11}$$

where, f_j, λ_j and $\{y\}_j$ are the j th natural frequency, eigenvalue and eigenvector, $[M(c_i)]$ is the global mass matrix. In order to evaluate the global mass matrix as a function of design variables, the equivalent density is defined from expression as follows:

$$\rho_H = \rho_o\sum_{i=1}^{n}(L_i c_i) \tag{12}$$

where ρ_H and ρ_o denote the equivalent density and the true material density.

3.2.2 Sensitivity Analysis

The j th natural frequency is related to the corresponding eigenvalue, as shown in the following equation:

$$f_j = \frac{\sqrt{\lambda_j}}{2p} \tag{13}$$

In order to evaluate the sensitivity of the natural frequency, first, we evaluate the sensitivity of the corresponding eigenvalue. And then, using the following equation, the sensitivity of the j th natural frequency can be derived as:

$$\frac{df_j}{dc_i} = \frac{1}{4\pi\sqrt{\lambda_j}}\frac{d\lambda_j}{dc_i} \tag{14}$$

Thus the following sensitivity equation of the j th eigenvalue with respect to design variables is used in this research, which is well known and for further details on this equation, please refer to Ref. (Haug, Choi and Komkov, 1986):

$$\frac{d\lambda_j}{dc_i} = \{y\}_j^T\frac{\partial[K]}{\partial c_i}\{y\}_j - \lambda_j\{y\}_j^T\frac{\partial[M]}{\partial c_i}\{y\}_j \tag{15}$$

3.3 Sensitivity analysis of constraint function

Eq. $g = m_T(c_i) - m_A \le 0$ of Eqs. (5) and (11) can be expressed as:

$$g = \sum_{j=1}^{EN} v_j r_o\left(\sum_{i=1}^{n} L_i c_i\right) - m_A \le 0 \tag{16}$$

where EN is the total number of elements. As shown in Eq. (16), the constraint function is linear in the design variable. Thus, the sensitivity analysis of the constraint function can be directly evaluated with respect to design variables as follows:

$$\frac{dg}{dc_i} = \sum_{j=1}^{NBE} v_j L_j \tag{17}$$

where *NBE* is the total number of neighboring element of *i* th node.

4. Numerical Examples

In this section, in order to attest the propriety of the described topology optimization approach in this research, we consider optimal material layouts of three examples. The first example is to minimize the mean compliance while the second is to maximize a certain natural frequency of a structure in plane stress conditions. The third example contains maximizing a certain natural frequency of a thin square plate according to changing the 4-corner boundary conditions. In the third example, it is assumed that the thin square plate is governed by Kirchoff's plate theory (Dym and Shames, 1973). In second and third examples, the subspace method (Bathe, 1996) is used to evaluate eigenvalues and their corresponding eigenvectors of the structure for the designated free vibration mode. Of all numerical examples, the constraint functions are the same that the used quantity of material must not exceed an allowable quantity. For all numerical examples, initial values of the design variables are set to '1' so that the initial design domain is identical to the geometry of the structure which do not have holes. And, the material properties of the isotropic matrix of the composite material model are given by $E_0 = 207$ Gpa, $\rho_o = 7700$ Kg/m³ and $\nu = 1/3$.

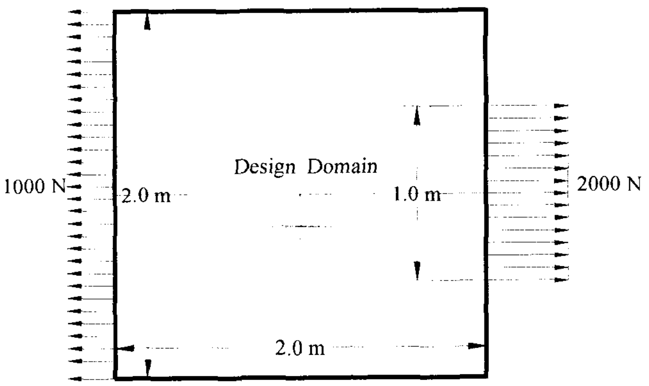


Fig. 4. A square structure is subjected to uniform loads.

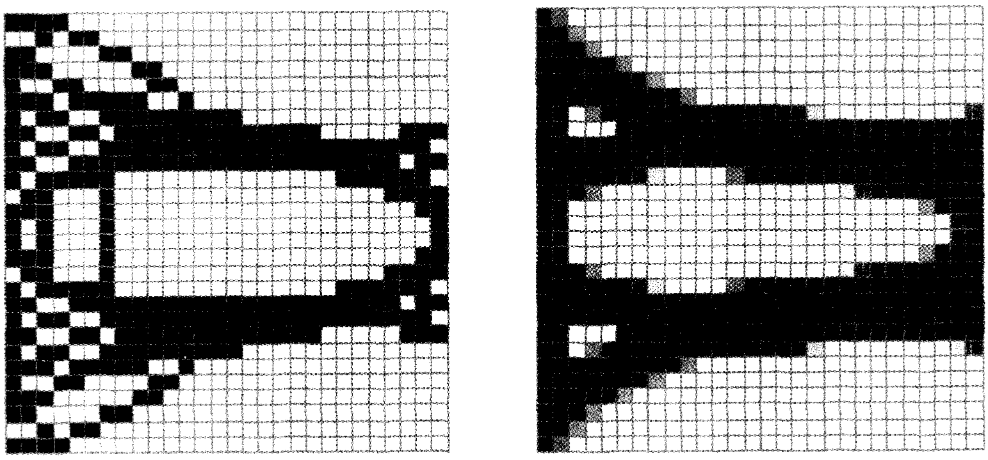
Example 1: A square structure subjected to uniform loads

In this example, we consider a square structure in Fig. 4. To avoid the singularity of the finite element static analysis, this structure is imposed on boundary conditions in which horizontal and vertical motions of center point are fixed. Here, for the finite element analysis, 784 (28 x 28) four-node plane stress elements and 841 nodes are used. The constraint function is set not to exceed 35% of the entire material.

Fig. 5 shows that the optimal material distribution obtained by using the homogenized material model with $p = 3$ based on Eq. (2) and Eq. (4). From Fig. 5, it is known that chessboard patterns are successfully suppressed by the nodal volume fraction method. This example has been analyzed by Youn and Park (Youn and Park, 1997). They used an artificial material model based on Hashin-Shtrikman theory as a homogenized material and the density redistribution algorithm as the chessboard prevention strategy. The optimal material layout in Fig. 5(b) is quite similar to the optimal configuration suggested by them. The objective function is changed from 0.547510^{-2} $N \cdot m$ to 0.189510^{-2} $N \cdot m$ for the optimal topology of Fig. 5(a) and from 0.547510^{-2} $N \cdot m$ to 0.247310^{-2} $N \cdot m$ for the that of Fig. 5(b)

Example 2: A clamped 3-D beam for the axisymmetric mode about z-axis

In this example, as shown in Fig. 6, we consider a 3-



(a) (b)

Fig. 5. Optimal material distribution (a) without and (b) with chessboard prevention strategy.

dimensional beam (the height is 1 m, the thickness is 1 m, and the length is 7 m) which is clamped at the left and right middle points. And then, we idealize the beam to a plane structure and analyze the optimal material layout of the idealized beam for the natural frequency of a axi-symmetric mode as shown in Fig. 7. Finally, using the optimal material layout of the idealized beam, we assume that the change of volume fraction is equal to the change of thickness of the beam. That is, if the volume fraction of a element is '0.3', the thickness of the element is '0.3'. In that case, the optimal layout and shape of 3-dimensional is obtained as shown in Fig. 8.

In following Fig. 8, using 8-node solid elements of ANSYS package, the bending mode of 3-D beam is presented. The initial frequency of the beam was 169.78 Hz and the result frequency is 203.95 Hz.

Example 3: Optimal topology of a thin square plate

In this example, we consider the optimal material distribution of the thin square plate shown in Fig. 9 according to changing the 4-corner boundary conditions of this plate. The optimal topology of the 4-corner pin supported thin square plate for the 2nd eigenmode has been discussed in Ref. (Kosaka and Swan, 1999) with a symmetry reduction method. As shown in Fig. 9, we divide the thin square plate into 400 triangular elements and 221

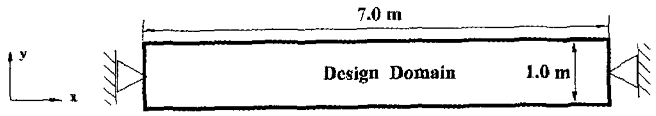


Fig. 6. Clamped beam at middle points of the left and right sides.

Fig. 7. Optimal material layout of the idealized beam.

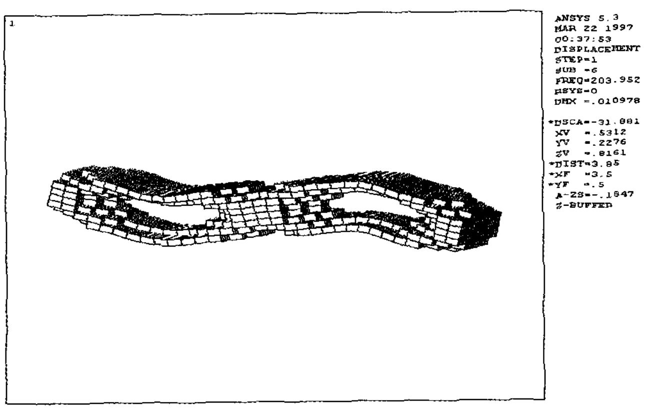


Fig. 8. The axisymmetric mode of the 3-D optimal beam.

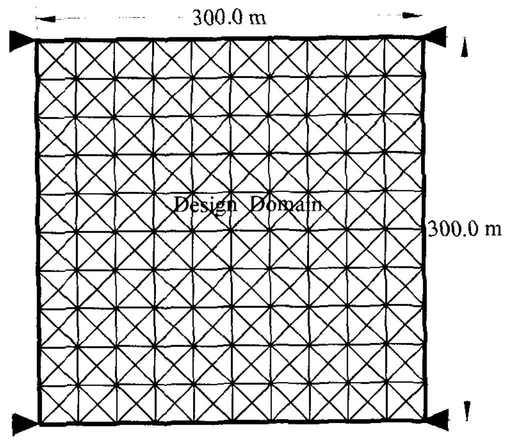


Fig. 9. Thin square plate for the topology optimization.

Table 1. The first five natural frequencies of each case constraint of initial models.

Model / Mode	Natural frequency of each case of initial models (Unit : Hz)			
	Case 1	Case 3	Case 4	Case 5
1^{st} mode	0.01977	0.02371	0.02442	0.02177
2^{nd} mode	0.04382	0.05027	0.04712	0.04504
3^{rd} mode	0.04382	0.05030	0.05410	0.04905
4^{th} mode	0.05435	0.06241	0.06145	0.05827
5^{th} mode	0.10699	0.11808	0.11439	0.11154

nodes. The thickness of this plate is 1 m and the total number of design variables is equal to the total node numbers. The objective function is to maximize the natural frequency of 2nd free vibration mode and the used quantity of the material must not exceed 60% of the entire design domain. The optimal material layouts of this plate are shown in Fig. 10.

In Fig.10, the symbol '▲' represents the pin-supported boundary condition and the symbol '◆' represents the clamped boundary condition. As shown in Fig.10, white elements represent voids, black elements represent that the volume fraction of the element is '1'. In these optimal results, it is also investigated that the chessboard patterns must be removed. The optimal layout of 'case 1' of Fig.10 is quite similar to the result obtained using symmetry reduction method in Ref. (Kosaka and Swan, 1999). From results of Fig.10, two obvious phenomena can be inspected. First, the optimal material layout is symmetric for the symmetric boundary condition. Second, the material in and around the clamped boundary condition is surely maintained after optimal material distribution process. Except case 2 and case 6 constraints, the first five natural

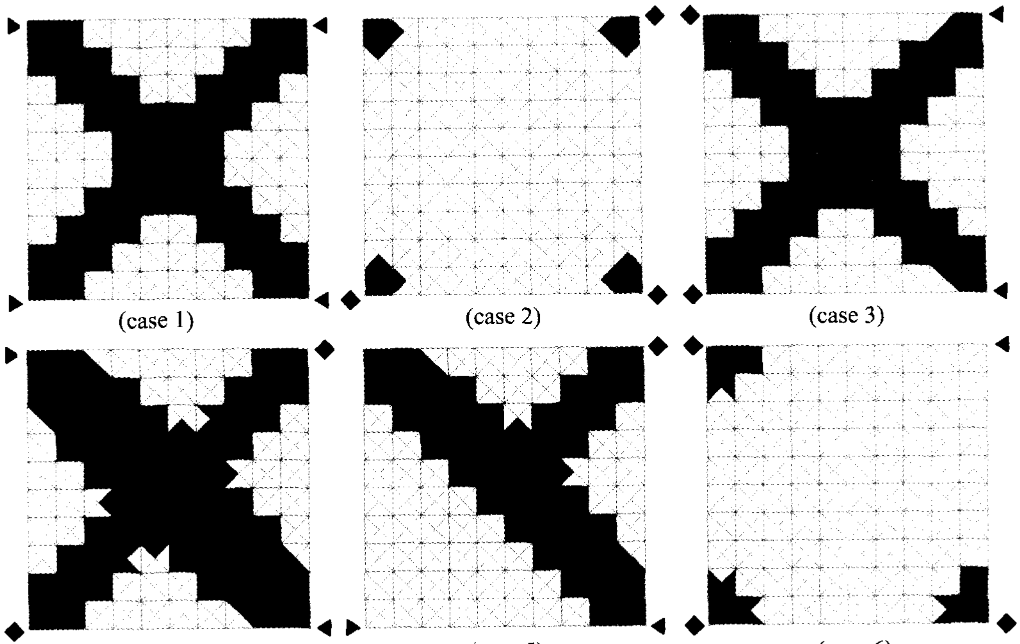


Fig. 10. Optimal material layouts according to changing the 4-corner boundary conditions.

frequencies of initial models are represented in Table 1 and the first five natural frequencies of optimal layouts are represented in Table 2. A commercial software, ANSYS package, is used to check up frequencies of numerical topology optimization results. From comparison Table 1 with Table 2, it is certain that the objective function to increase the natural frequency of a designated mode is satisfied. And, this causes frequencies of 3rd and 4th modes to increase.

5. Conclusions

In this research, a new topology optimization method, i.e. the nodal volume fraction method, based on the design domain method has been proposed. A rigorous composite material model made up of isotropic matter and spherical void was used in this method to solve the significant shortcomings of the homogenizati-on and density function methods. And, to prevent the chessboard patterns entirely, a bilinear function utilizing the nodal volume fraction and shape functions of linear elements was used in this method. Using this method, a mathematical programming problem for topology optimization, which has the explicit relationship between the equivalent material properties and design variables, can be simply formulated. Thanks to the explicit relationship, the sensitivity analysis of topology optimization problem can be easily obtained also by applying the direct differentiation method and the chessboard patterns can be removed in the condition which has no effects on the stability of the optimization algorithm.

Table 2. The first five natural frequencies of each case constraint of optimal material layouts.

Model / Mode	Natural frequency of each case of optimal layouts (Unit : Hz)			
	Case 1	Case 3	Case 4	Case 5
1st mode	0.01552	0.02086	0.02124	0.01451
2nd mode	0.04784	0.05837	0.05328	0.05208
3rd mode	0.04784	0.05855	0.05446	0.05350
4th mode	0.09039	0.10304	0.09926	0.10663
5th mode	0.11143	0.13566	0.10231	0.10965

Several numerical examples have confirmed that the present method can be used accurately for minimizing the mean compliance and maximizing a certain natural frequency of the designated mode of a structure. And, it is expected that the present method can be applied not just for simple linear elastic application but a wide variety of other structural topology applications.

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