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A Note on the Ahlfors function on an annulus

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Abstract

In this paper we represent the Ahlfors function on the multiply connected planar domain, especially on an annulus and apply it to the Carathéodory metric.

0. Introduction

The fact that all simply connected domains with more than one boundary point are conformally equivalent is known as the Riemann mapping theorem by Riemann(1826-1866) and is of fundamental importance in the theory of conformal mapping and in its application. So any simply connected planar domain other than the whole plane may be mapped one-to-one and conformally onto the unit disc. The Riemann mapping has made a big step possible on the study on the simply connected domains. Very often a physical problem defined in some complicated region or with a complicated boundary, as for example in fluid mechanics, can be solved by using a complex function to map the region or boundary to a simpler region or boundary on which the problem may be solved. By the help of the Riemann mapping, the property on the simply connected domain can be known by using the property on the unit disc. The physics of Riemann's mapping theorem is well explained in [4, p. 540].

For any multiply connected domain, there is the Ahlfors function instead of the Riemann map, which was discovered by Ahlfors in 1947-1950. Many mathematicians tried to express the Ahlfors function. In 1950, Bergman represented the Riemann mapping via Bergman kernel and later it helps to represent the Ahlfors function using the Szegö kernel.

Ahlfors function is related to the Carathéodory metric because it solves an extremal problem. The Carathéodory metric which was invented by Carathéodory(1873-1950) has the properties that are of interest and has the advantage that it may often be computed, or at least estimated. Historically the Carathéodory and Kobayashi metrics on strongly pseudoconvex domains were understood before the Bergman metric was. We know that the Carathéodory and Kobayashi metrics are, respectively, the smallest and largest metrics under which holomorphic maps are distance decreasing(see [3, p. 371]). On the other hand, none of them is very well under control even at this time.

In this paper, we represent the Ahlfors function on an annulus and apply it to the Carathéodory metric.

1. Preliminaries

Let D be an n-connected, bounded, planar domain with C^{∞} boundary components. We call $f: D \to U$ an Ahlfors function where U is the unit disc if it is a (ramified) covering of order n. Clearly an Ahlfors function is determined up to a multiplicative constant of modulus 1 by its n zeroes.

Given a point $p \in D$ the Ahlfors function associated to the pair (D, p) with $f(p) = 0$ and $f'(p) > 0$ is related to the Szegö Kernel $S(z, p)$ and the Garabedian Kernel $L(z, p)$ via

$$f(z) = \frac{S(z, p)}{L(z, p)} \qquad \text{---------------(1)}$$

(see [3, p. 390]).

For the annulus, the Ahlfors function is a 2-to-one mapping.

Let $D_1 = \{ z \in C : 1/R < |z| < R \}$ and $D = \{ z \in C : \rho < |z| < 1 \}$. Note that two concentric circular rings are of different conformal type unless the ratio of the two radii is the same for both rings(see [3, p. 334]). This ratio, that is, the quantity r_2/r_1 if the ring in question is $\{ z \in C : r_1 < |z| < r_2 \}$, is known as the modulus of the ring. Therefore, there exists a conformal map between D_1 and D only when $\rho = \frac{1}{R^2}$.

2. Ahlfors function

We describe the Ahlfors function for an annulus $D_2=\{ z\in C : \frac{1}{R^2} < |z| < 1 \}$. First of all, we need the following lemmas.

Lemma 1 [7]. Let $D=\{z\in C: \rho<|z|<1\}$ be an annulus. For each fixed $p\in D$, the Szegö Kernel $S(z,p)$ has one zero at $z=-\frac{\rho}{\bar{p}}$.

It implies that given a point $p\in D_2$ the Ahlfors function associated to (D_2, p) with $f(p)=0$ and $f'(p)>0$ has zeroes at p and $-\frac{1}{R^2\,\bar{p}}$.

Lemma 2 [6]. Let $D_1=\{ z\in C : \frac{1}{R} < |z| < R \}$ and let $p_1, q_1\in D_1$. Then there exists an Ahlfors function with zeroes at p_1 and q_1 if and only if $|p_1 q_1|=1$.

Lemma 3 [2, pp. 335-336]. For $p_1\in D_1,\ p_1>0$, set

$$f_1(p_1,z)=(1-\frac{z}{p_1})\frac{\prod_1^{\infty}(1-z/R^{4n}p_1)(1-p_1/R^{4n}z)}{\prod_1^{\infty}(1-p_1 z/R^{4n-2})(1-1/R^{4n-2}p_1 z)}.$$

Then $f_1(p_1,\cdot)$ is holomorphic in D_1, has constant modulus R/p_1 on $|z|=R$ and modulus 1 on $|z|=1/R$, and p_1 is its only zero.

Theorem 4. For any p_2, q_2 in D_2 with $|p_2 q_2|=\frac{1}{R^2}$, the Ahlfors function f_2 on D_2 with zeroes at p_2 and q_2 is given by

$$f_2(w)=\frac{1}{R^2 w}f_1(Rp_2, Rw)f_1(Rq_2, Rw) \qquad \text{-----------(2)}$$

where, for $t=re^{i\theta}$, $f_1(t,z)=f_1(r,e^{-i\theta}z)$.

Proof. Since D_1 and D_2 are of same conformal type, there exists a conformal map $h(z)=\frac{z}{R}$ from D_1 onto D_2. For any p_2, q_2 in D_2 with $|p_2 q_2|=\frac{1}{R^2}$,

$|Rp_2 \cdot Rq_2| = 1$ and Rp_2, Rq_2 are in D_1. By Lemma 2, there exists an Ahlfors function f_1 on D_1 with zeroes at Rp_2 and Rq_2.

For $t = re^{i\theta}$, let $f_1(t,z) = f_1(r, e^{-i\theta}z)$. Then $f_1(t, \cdot)$ has only zero at t. Note that $f_1(h^{-1}(p_2), h^{-1}(w)) = f_1(Rp_2, Rw) = f_1(R\,|p_2|, e^{-i\theta}Rw)$ where $Rp_2 = R\,|p_2|e^{i\theta}$.

By Lemma 3, $f_1(R\,|p_2|, \cdot)$ is holomorphic in D_1, has constant modulus $\frac{1}{|p_2|}$ on $|z| = R$ and modulus 1 on $|z| = \frac{1}{R}$, and $R\,|p_2|$ is its only zero. Hence $f_1(Rp_2, \cdot)$ has the same property as $f_1(R\,|p_2|, \cdot)$ except the zero at Rp_2. Therefore $f_1(h^{-1}(p_2), h^{-1}(\cdot))$ is holomorphic in D_2 and has constant modulus $\frac{1}{|p_2|}$ on $|w| = 1$ and modulus 1 on $|w| = \frac{1}{R^2}$, and p_2 is its only zero. Similar property holds for $f_1(h^{-1}(q_2), h^{-1}(\cdot))$.

Thus $f_1(Rp_2, \cdot)f_1(Rq_2, \cdot)$ is holomorphic in D_1, has constant modulus $\frac{1}{|p_2q_2|}$ on $|z| = R$ and modulus 1 on $|z| = \frac{1}{R}$. Hence $\frac{1}{Rz} f_1(Rp_2, z) f_1(Rq_2, z)$ I s holomorphic in D_1, has constant modulus 1 on $|z| = R$ and $|z| = \frac{1}{R}$, and Rp_2 and Rq_2 are two zeroes. It means that the Ahlfors function on D_1 with zeroes at Rp_2 and Rq_2 is given by

$$f_1(z) = \frac{1}{Rz} f_1(Rp_2, z) f_1(Rq_2, z).$$

Therefore by using biholomorphic map h from D_1 onto D_2 , the Ahlfors function on D_2 with zeroes at p_2 and q_2 is given by

$$\begin{aligned} f_2(w) &= \frac{1}{R^2 w} f_1(h^{-1}(p_2), h^{-1}(w)) f_1(h^{-1}(q_2), h^{-1}(w)) \\ &= \frac{1}{R^2 w} f_1(Rp_2, Rw) f_1(Rq_2, Rw). \end{aligned}$$

Remark. Both (1) and (2) represent the formula for the Ahlfors function on annulus. It is interesting that the formula in (1) uses the infinite sum and the formula in (2) uses infinite product.

3. Application to Carathéodory metric

The Carathéodory pseudodistance between two points z_1, z_2 of an annulus $D_2 = \{ z \in C : \frac{1}{R^2} < |z| < 1 \}$ is defined by

$$c_2(z_1, z_2) = \sup\{ \rho(g(z_1), g(z_2)) : g: D_2 \to U \text{ holomorphic}\}$$

where

$$\rho(w_1, w_2) = \log \frac{|1 - w_1 \overline{w_2}| + |w_1 - w_2|}{|1 - w_1 \overline{w_2}| - |w_1 - w_2|}$$

is the invariant distance on the unit disc U. Clearly $c_1(h^{-1}(z_1), h^{-1}(z_2)) = c_2(z_1, z_2)$ where c_1 is the Carathéodory pseudodistance between two points of $D_1 = \{ z \in C : \frac{1}{R} < |z| < R \}$ and $h: D_1 \to D_2$ is given by $h(z) = z/R$.

The infinitesimal Carathéodory metric is defined by

$$F_{c_2}(z) = \sup\{ |g'(z)| : g : D_2 \to U \text{ holomorphic with } g(z) = 0\}.$$

The maximizing function is unique up to multiplicative constants of modulus one and defines ramified coverings of the unit disc of order two (see [1]). In this section, we determine the formula for $F_{c_2}(p_2)$.

Theorem 5. For $p_2 > 0$, we have

$$f_2'(p_2) = \frac{1}{Rp_2} f_1(\frac{1}{Rp_2}, -Rp_2) f_1'(Rp_2, Rp_2)$$

where $f_1(Rp_2, z) = (1 - \frac{z}{Rp_2}) \prod(Rp_2, z)$.

Proof. By Theorem 4, $f_2(w) = \frac{1}{R^2 w} f_1(Rp_2, Rw) f_1(Rq_2, Rw)$. Hence

$$f_2'(w) = -\frac{1}{R^2 w^2} f_1(Rp_2, Rw) f_1(Rq_2, Rw)$$

$$+ \frac{1}{R^2 w} [f_1'(Rp_2, Rw) f_1(Rq_2, Rw) + f_1(Rp_2, Rw) f_1'(Rq_2, Rw)] R .$$

Since $f_1(Rp_2, Rp_2)=0$ by Lemma 3 and $q_2 = -\frac{1}{R^2 p_2}$ by Lemma 1,

$$f_2'(p_2) = \frac{1}{R^2 p_2} f_1'(Rp_2, Rp_2) f_1(-\frac{1}{Rp_2}, Rp_2) \cdot R$$
$$= \frac{1}{Rp_2} f_1'(Rp_2, Rp_2) f_1(\frac{1}{Rp_2}, -Rp_2).$$

Corollary 6. For $p_2 > 0$, we have

$$F_{c_2}(p_2) = \frac{1}{Rp_2} f_1'(Rp_2, Rp_2) f_1(\frac{1}{Rp_2}, -Rp_2).$$

Proof. If f_2 denotes the Ahlfors function with $f_2(p_2)=0$ and $f_2'(p_2)>0$, by the uniqueness of f_2, we get $f_2'(p_2) = F_{c_2}(p_2)$ and hence we get the desired result by Theorem 5.

Recently, the zeroes of the Ahlfors function on 3-connected planar domain were inspected by Tegtmeyer in 2001 and numerical evidence for 4-connected and 5-connected planar domains leads to a conjecture for the zeroes of the Ahlfors function on n-connected planar domain. In the near future, we hope to get a precise representation of the zeroes of the Ahlfors function on n-connected planar domains.

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