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J-equivalence of representations of finite group G

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Abstract

In this paper we consider the topological properties of θ_k and show that the induced map $\widetilde{\theta_k}$ is well defined and renders the diagram commutative.

0. Introduction

K-theory was introduced by M.F. Atiyah and F. Hirzebruch, after the original idea was suggested by A. Grothendieck. The Bott periodicity theorem is essential for the development of the theory. There are important applications of K-theory to differential topology, such as the Riemann-Roch theorems for differentiable manifolds (due to Atiyah and Hirzebruch) [3], [4], the solution of the vector field problem on spheres (due to J. Adams) [1], and applications to immersion and embedding problems. The concept of differentiable manifolds and the study of their topological structures or global properties have their origins in Poincare's work. However, the progress of research has been slow, except that important results have been obtained concerning homological properties of manifolds and topological structures of lower-dimensional manifolds. Development of research has been remarkable, and in addition to classical results such as H. Whitney's embedding theorem [9], the triangulability of S. Cairns [5], and Morse theory [8], we have the theories of fiber bundles, characteristic classes, and homotopy.

1. Preliminaries

Two unitary representations E, F of an arbitrary finite group G are said to be J-equivalent if there are G-maps from $S(E)$ to $S(F)$ and from $S(F)$ to $S(E)$ both of degree prime to the order of G. If E is a representation, denoted by $[E]$ its class in $R(G)$ and if $T(G)$ is the subgroup consisting of elements $[E]-[F]$ where E and F are J-equivalent, define $J(G) = R(G)/T(G)$.

Define a λ-ring to be a commutative ring R with identity and a countable set of maps $\lambda^n: R \to R$ such that for all $x, y \in R$

(a) $\lambda^0(x) = 1$

(b) $\lambda^1(x) = x$

(c) $\lambda^n(x+y) = \sum_{r=0}^{n} \lambda^r(x)\lambda^{n-r}(y)$

If t is an indeterminate, for $x \in R$ define:

(d) $\lambda_t(x) = \sum_{n \geq 0} \lambda^n(x)t^n$

(e) $\lambda_t(x+y) = \lambda_t(x)\lambda_t(y)$

The ring Z of integers may be given a λ-structure by defining $\lambda_t(1) = 1 + \sum m_n t^n$ where $m_1 = 1$.

The bott cannibalistic class θ_k is a natural exponential map given for a one dimensional element x by

$$\theta_k(x) = 1 + x + \cdots + x^{k-1}.$$

Definition 1.1. I is said to be a special γ-ring if it is a commutative ring (without identity) with operations $\{\gamma^i\}$ such that there is an augmented λ-ring R with I as kernel of the augmentation.

Definition 1.2. Two unitary representations E, F of a finite group G are said to be J-equivalent if there are G-maps $\Phi: S(E) \to S(F)$, $\theta: S(F) \to S(E)$ each of degree prime to the order of G.

If E, F are J-equivalent, write $E \sim F$.

Let E be a unitary representation space of G, E^+ its one-point compactification. Then using K_G-theory with compact supports we can introduce $K_G(E) = K_G(E^+, +)$. It is a module over $K_G(point) = R(G)$. The main theorem of the subject, as proved in [2] asserts that $K_G(E)$ in a free module over $R(G)$ with a canonical generator μ_E. Moreover the method of proof in [2] shows also that the map $j^* = K_G(E) \to K_G(P(E \oplus 1))$ is injective, where $P(E \oplus 1)$ is the projective space associated to $E \oplus 1$ (1 denoting the trivial representation C) and j^* is induced by the open inclusion $j: E \to P(E \oplus 1)$ given by $j(u) = (u, 1)$. If h is the class of the standard line bundle H over $P(E \oplus 1)$, the image of μ_E is [2]

$$j^*(\mu_E) = \sum(-1)^r h^r \lambda^r(E).$$

If $i: P(E) \to P(E \oplus 1)$ is the natural inclusion then $i^* j^* = 0$ and so $\sum(-1)^r (i^*(h))^r \lambda^r(E) = 0$ in $K_G(P(E))$.

Replacing E by $E \oplus 1$ we deduce the equation

$$\sum(-1)^r h^r \lambda^r(E \oplus 1) = 0$$

or equivalently

$$(1-h)\sum(-1)^r h^r \lambda^r(E) = 0 \qquad \text{------------------} \quad (1)$$

From these facts it follows that we can identify $K_G(E)$ (as λ-ring) with the $R(G)$ -module. To see this we map an indeterminate ξ to $h^{-1} \in K_G(P(E \oplus 1))$. This induces a homomorphism

$$\alpha: R(G)_E \to j^* K_G(E)$$

in which

$$\begin{aligned} \alpha(\sum(-1)^r \eta^{n-r} \lambda^r(E)) &= h^{-n}(\sum(-1)^r h^r \lambda^r(E)) \\ &= \sum(-1)^r h^r \lambda^r(E) \qquad (by\ (1)) \\ &= j^*(\mu_E). \end{aligned}$$

$I(x)$ is the ideal in $R[\xi]$ generated by

$$L(\xi, x) = \xi^n - \lambda^1(x)\xi^{n-1} + \cdots + (-1)^n \lambda^n(x)$$

Proposition 1.3. Let $R_x = I(x) / I(x+1)$ then R_x is a free R-module on one generator μ_x, where μ_x is the image of $L(\xi, x)$ in R_x, R_x is a special γ-ring (a special λ-ring without identity) and for $z \in R$

(a) $\Psi^k(z \cdot \mu_x) = \Psi^k(z) \cdot \Psi^k(\mu_x)$

(b) $\Psi^k(\mu_x) = \theta_k(x)\mu_x$.

proof. $L(\xi, x+1) = (\xi - 1)L(\xi, x)$ and so if μ_x is the image of $L(\xi, x)$ is R_x and η is the image of ξ, we see that $(\eta - 1)\mu_x = 0$ i.e. $\eta \cdot \mu_x$. Any element in $I(x)$ is uniquely of the form $f(\xi)L(\xi, x)$ for $f(\xi) \in R[\xi]$ and the image of $f(\xi)L(\xi, x)$ in R_x is $f(\eta)\mu_x = f(1)\mu_x$ since $\eta\mu_x = \mu_x$. So R_x is an R-module on the generator μ_x. If $a \in R$ and $a \cdot \mu_x = 0$, then $a \cdot L(\xi, x) \in I(x+1)$ and this implies $a = 0$, so R_x is a free R-module.

We can now read off the action Ψ^k on $K_G(E)$. Namely we have

$$\Psi^k(z\mu_E) = \Psi^k(z) \cdot \Psi^k(\mu_E) \quad \text{---------------------} \quad (2)$$

$$\Psi^k(\mu_E) = \theta_k(E) \cdot \mu_E \quad \text{-----------------------} \quad (3)$$

The structure of $K_G(P(E))$ is also known - it can be easily deduced from the main theorem of [2] by various methods - and one has

$$K_G(E) \cong R(G)[\xi] / I(E).$$

Suppose now that E, F are two unitary representations of G, and suppose we have a G-map $\Phi: S(E) \to S(F)$. Extending radially to the balls Φ induces the suspension $\Omega_\Phi: B(E)/S(E) \to B(F)/S(F)$. Since we have the obvious identification of $B(E)/S(E)$ with E^+ we obtain an $R(G)$-homomorphism of λ-rings

$$\Phi^! : K_G(F) \to K_G(E).$$

Since these are free module there is a unique $z \in R(G)$ such that $\Phi^!(\mu_E) = z\mu_E$. Applying Ψ^k to this and using the formula (1) for E and also for F we obtain

$$\theta_k(F) z\mu_E = \theta_k(E)\Psi^k(z)\mu_E.$$

Since μ_E is a free generator we deduce

$$\theta_k(F)z = \theta_k(E)\Psi^k(z).$$

It remains to show that $\varepsilon(z) = \deg \Phi$. This is straightforward.

The inclusion of the trivial group 1 in G induces the maps

$$f_E: K_G(E) \to K(E), \quad \varepsilon: R(G) \to R(1) = Z$$

which forget the G-structure. Since $K_G(E)$ is an $R(G)$-module in a natural way, f_E is equivariant, that is to say

$$f_E(z\mu_E) = \varepsilon(z)\mu_E, \quad z \in R(G). \qquad \text{------------------ (4)}$$

$\Phi: S(E) \to S(F)$ induces $\Phi^!: K(F) \to K(E)$ where

$$\Phi^!(\mu_F) = \deg \Phi \cdot \mu_E. \qquad \text{---------------------- (5)}$$

Naturally, the diagram

$$\begin{array}{ccc} K_G(F) & \xrightarrow{\Phi^!} & K_G(E) \\ \downarrow f_F & & \downarrow f_E \\ K(F) & \xrightarrow{\Phi^!} & K(E) \end{array}$$

commutes.

From (4), (5), $\deg \Phi = \varepsilon(z)$.

We summarize the results above in:

Proposition 1.4. Let E, F be unitary representations of a finite group G. If $\Phi: S(E) \to S(F)$ is a G-map of degree r, there is an element $z \in R(G)$ of augmentation r such that

$$\theta_k(f)z = \theta_k(E)\Psi^k(z).$$

2. Main results

Given a G-map $\Phi: S(E) \to S(F)$ of degree r, there is an element $z \in R(G)$ such that $\varepsilon(z) = r$ and $\theta_k(F)z = \theta_k(E)\Psi^k(z)$. Suppose G is a p-group. If r is prime to p, since $\varepsilon(\Psi^k(z)) = r$, $r^{-1}z$ and $\Psi^k(r^{-1}z) \in 1 + \widehat{I(G)} \subset Z_p \otimes R(G)$. For $(k, p) = 1$, we then have, in $1 + \widehat{I(G)}$, the equation:

$$\theta_k([E]-[F]) = r^{-1}z\cdot[\Psi^k(r^{-1}z)]^{-1}.$$

If $c: 1+\widehat{I(G)} \to (1+\widehat{I(G)})_{\Gamma}$ is the canonical map, we therefore have:

Proposition 2.1. Let G be a p-group and K be prime to p. If there is a G-map $\Phi: S(E)\to S(F)$ of degree prime to p, then $[E]-[F]$ is in the kernel of the map $c\theta_k: I(G)\to(1+\widehat{I(G)})_{\Gamma}$.

Let $\varepsilon: R(G)\to Z$ be the augmentation induced by the dimension of a representation and let $I(G) = Ker\,\varepsilon$, then $R(G) = Z\oplus I(G)$ as abelian groups.

If E and F are J-equivalent, then there is a map $\Phi: S(E)\to S(F)$ of degree prime to the order of the group, and for the degree to be defined, the dimensions of E and F must be equal. This shows $T(G)\subset I(G)$.

Define $\tilde{J}(G) = I(G)/T(G)$, then $J(G) = Z\oplus\tilde{J}(G)$ as abelian groups.

It is also easy to see that $W(G)\subset I(G)$ and that, in the usual notation,

$$I(G)_{\Gamma_N} = I(G)/W(G)$$

and $R(G)_{\Gamma_N} = Z\oplus I(G)_{\Gamma_N}$

This implies:

Proposition 2.2. If G is an M-group (in particular if G is a p-group), there is a canonical epimorphism $\upsilon: R(G)_{\Gamma_N}\to J(G)$ and this induces an epimorphism $\tilde{\upsilon}: I(G)_{\Gamma_N}\to\tilde{J}(G)$.

Corollary 2.3. If E and F are J-equivalent, then $[E]-[F]$ is in the kernel of $c\theta_k$.

$T(G)$ is the subgroup of $I(G)$ consisting of elements $[E]-[F]$ where E, F are J-equivalent, we have:

Corollary 2.4. $T(G)\subset Ker\,c\theta_k$.

But the following diagram commutes:

$$\begin{array}{ccc} I(G) & \xrightarrow{\theta_k} & 1+\widehat{I(G)} \\ \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle c} \\ I(G)_\Gamma & \xrightarrow{(\theta_k)_\Gamma} & (1+\widehat{I(G)})_\Gamma \end{array} \qquad \text{--------------(1)}$$

where d is the canonical map. So $T(G) \subset Ker(\theta_k)_\Gamma d$. Immediately we see that $(\theta_k)_\Gamma$ factors through the group $\mathcal{J}(G) = I(G) / T(G)$. Define $\widetilde{\theta_k}: \mathcal{J}(G) \to (1+\widehat{I(G)})_\Gamma$ by $\widetilde{\theta_k}(x + T(G)) = \theta_k(x)$ for $x \in I(G)$ then:

Proposition 2.5. The following diagram commutes for a p-group G, where $\tilde{\upsilon}$ is the canonical epimorphism of (Proposition 2.2).

$$\begin{array}{ccc} I(G)_\Gamma & \xrightarrow{\tilde{\upsilon}} & \mathcal{J}(G) \\ & {\scriptstyle (\theta_k)_\Gamma} \searrow & \downarrow{\scriptstyle \widetilde{\theta_k}} \\ & & (1+\widehat{I(G)})_\Gamma \end{array} \qquad \text{--------------(2)}$$

We have the commuting diagram:

$$\begin{array}{ccc} I(G)_\Gamma & & \\ \downarrow{\scriptstyle i} & \searrow{\scriptstyle (\theta_k)_\Gamma} & \\ \widehat{I(G)}_\Gamma & \xrightarrow{(\rho_k)_\Gamma} & (1+\widehat{I(G)})_\Gamma \end{array} \qquad \text{--------------(3)}$$

Fitting together diagram (2), (3) with $k=h$ (the generator of Γ), we find the required diagram for a p-group of odd order:

$$\begin{array}{ccccccc} & & 0 & & & & \\ & & \downarrow & & & & \\ & & I(G)_\Gamma & \xrightarrow{\tilde{\upsilon}} & \mathcal{J}(G) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \widehat{I(G)}_\Gamma & \xrightarrow{(\rho_n)_\Gamma} & (1+\widehat{I(G)})_\Gamma & \longrightarrow & 0 \end{array} \qquad \text{--------------(4)}$$

Lemma 2.6. For a p-group of odd-order, $\tilde{\upsilon}$ is an isomorphism.

proof. Chase an element round diagram (4).

$R(G)_\Gamma = Z + I(G)_\Gamma$, $J(G) = Z + \tilde{J}(G)$ and $\upsilon: R(G)_\Gamma \to J(G)$ is given by $\upsilon(n + a) = n + \tilde{\upsilon}(a)$ for $a \in I(G)_\Gamma$, lemma 2.6 implies the main theorem.

Theorem 2.7. For a p-group of odd order $N = p^e$, $\upsilon: R(G)_{\Gamma_N} \to J(G)$ is an isomorphism.

If G is a p-group of odd order, if $[E] - [F] = [E \oplus c] - [F \oplus c] \in T(G)$, then $[E] - [F] \in W(G)$. For conjugate representations, we may construct J-equivalences and so E, F are truly J-equivalent.

Returning to the diagram (4), trivial diagram chasing shows:

Proposition 2.8. $\tilde{\theta}_k: \tilde{J}(G) \to (1 + \widetilde{I(G)})_\Gamma$ is a monomorphism.

From Proposition 2.1, if there is a G-map $\Phi: S(E) \to S(F)$ of degree prime to p, then the image of $[E] - [F]$ in $\tilde{J}(G)$ is in the kernel of $\tilde{\theta}_h$. Proposition 2.8 gives:

Theorem 2.9. If there is a G-map $\Phi: S(E) \to S(F)$ of degree prime to p, then E and F are J-equivalent.

We now consider $W(G)$, the subgroup of $I(G)$ generated by $\{x - \alpha x\}$, $x \in R(G)$, $\alpha \in \Gamma$. If E is irreducible, it is trivial to see that αE is irreducible also. Thus if $\xi_1, \cdots, \xi_m$ are the classes of irreducible representations, they split up into equivalence classes where each class consists of an irreducible representation ξ_r and the elements of the form $\alpha \xi_r$ for $\alpha \in \Gamma$.

We order the classes or irreducible representations $\xi_1, \xi_2, \cdots, \xi_s, \cdots, \xi_m$ so that no two of $\xi_1, \cdots, \xi_s$ are conjugate under the action of Γ and their conjugates exhaust $\xi_1, \cdots, \xi_m$. Then $W(G)$ is generated by elements of the form $\{\xi_i - \alpha \xi_i\}$ where

$1 \le i \le s,\ \alpha \in \Gamma$.

Suppose now E, F and J-equivalent and E is irreducible. We may suppose $[E] = \xi_s$. If $[F] = \sum_{i=1}^{m} n_i \xi_i$, then $[E] - [F] = \xi - \sum n_i \xi_i \in W(G)$ by theorem 2.7. Since $n_i \ge 0$ for $i = 1, \cdots, m$, immediately we see that $[F] = \alpha \xi_s$ for some $\alpha \in \Gamma$. We have:

Theorem 2.10. If G is a p-group of odd order and E is an irreducible unitary representation of G, then for any unitary representation F, there is a G-map $\Phi: S(E) \to S(F)$ of degree prime to p if, and only if, $F = \alpha E$ for some $\alpha \in \Gamma$.

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