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Partially conformal geodesic transformation on the Kahler manifolds*

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Abstract

Using Fermi coordinate and Jacobi operator on Kahler manifold, we characterize partially conformal geodesic transformation in Kahler geometry.

1. Historical background and introduction

In 1972, S. Tochibana introduced the notion of a geodesic conformal transformations around submanifolds in a Riemannian manifold. These transformations are extensions of geodesic symmetries and local reflections with respect to submanifolds. The notion of a reflections generalize that of reflections with respect to linear subspaces in Euclidean space. Recently, E. Garcia-Rio, L. Vanhecke and B.Y. Chen begun a systematic study of geodesic conformal transformation. They show that conformality is a strong condition and motivated the study of the notion of a partially conformal geodesic transformation.

In this paper, we deal with partially conformal geodesic transformations in Kahler geometry by using Fermi coordinates when the submanifold is a point. We derive the necessary and sufficient condition for the existence of such transformation in terms of the Jacobi operator and its derivative.

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2. Kahler manifolds and Fermi coordinates

Let (M, g) be a connected smooth Riemannian manifold and ∇ its Levi Civita connection. Denote by R its associated Riemannian curvature tensor defined by

$$R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$$

for all vector fields $X, Y \in \chi(M)$. We put

$$R_{XYZW} = g(R_{XY}Z, W).$$

Let M be a n-dimensional Kahler manifold with structure (M, g, J):

$$J^2 = -I,$$
$$g(JX, JY) = g(X, Y),$$
$$\nabla_X(J)Y = 0$$

for all vector fields $X, Y \in \chi(M)$.

Then

$$R(X, Y)J = JR(X, Y),$$
$$R(JX, JY) = R(X, Y).$$

A plane section of the tangent space T_pM at a point $p \in M$ is called a holomorphic section if it is spanned by vectors X and JX in T_pM. The sectional curvature of a holomorphic section is called a holomorphic sectional curvature. A Kahler manifold of constant holomorphic sectional curvature c is called a complex space form and its curvature tensor is given by

$$R_{XY}Z = \frac{c}{4}\{g(X, Z)Y - g(Y, Z)X + g(JX, Z)JY - g(JY, Z)JX + 2g(JX, Y)JZ\}. \quad (*)$$

Theorem 1([6]). A Kahler manifold M of dimension ≥ 4 is a complex space form if and only if, for every vector field X on M, $R_{XJX}X$ is colinear with JX.

Let B be a embedded submanifold of M with $\dim B = q$ and $\exp_\nu$ the exponential map of the normal bundle $\nu = T^\perp B$ of B and $m \in B$ and $\{E_1, \ldots, E_n\}$ a local orthonormal frame field of M defined along B in a neighborhood of m. We specialize the fields such that $E_1, \ldots, E_q$ are tangent to B and $E_{q+1}, \ldots, E_n$ normal vector fields of B. For a system of coordinates $(y^1, \ldots, y^q)$ of B in a neighborhood of m such that $(\partial/\partial y^i)(m) = E_i(m)$, $i=1, \ldots, q$, the Fermi coordinates $(x^1, \ldots, x^n)$ with respect to m, $(y^1, \ldots, y^q)$ and $(E_{q+1}, \ldots, E_n)$ are defined by

$$x^i(\exp_\nu(\sum_{q+1}^n t^a E_a)) = y^i, \qquad i=1, \ldots, q,$$

$$x^a(\exp_\nu(\sum_{q+1}^n t^a E_a)) = t^a, \qquad a=q+1, \ldots, n$$

in an open neighborhood U_m of $m \in M$.

Put $s(r) = \rho(r) r$, where r is the normal distance. Then $r^2 = \sum_{a=q+1}^n (x^a)^2$.

Let $u \in T_m^\perp B \subset T_m M$ and $\gamma(r) = \exp_m(ru)$ the normal geodesic with $\gamma(0) = m$, $\gamma'(0) = u = E_n(m)$. Denote by $\{F_1, \ldots, F_n\}$ the frame field along γ obtained by parallel translating $\{E_1(m), \ldots, E_n(m)\}$ along γ. Consider the $n-1$ Jacobi vector fields Y_α, $\alpha = 1, \ldots, n-1$ along γ, determined by the initial conditions

$$Y_i(0) = E_i(m), \quad Y_i'(0) = (\nabla_u \partial/\partial x^i)(m), \quad i=1, \ldots, q,$$
$$Y_a(0) = 0, \qquad Y_a'(0) = E_a(m), \qquad a = q+1, \ldots, n-1.$$

Then $Y_i(r) = \frac{\partial}{\partial x^i}(\gamma(r))$, $\quad Y_a(r) = r\frac{\partial}{\partial x^a}(\gamma(r))$.

Put $Y_\alpha(r) = D_u(r) F_\alpha$, $\alpha = 1, \ldots, n-1$. Then D_u satisfies the Jacobi equation

$$D_u'' + R \circ D_u = 0,$$

where $R(x)X = R_{\gamma'(r)X}\, \gamma'(r)$.

Using the initial conditions for Y_α,

$$D_u(0) = \begin{pmatrix} I_q & 0 \\ 0 & 0 \end{pmatrix}, \qquad D_u'(0) = \begin{pmatrix} T(u) & 0 \\ -{}^t\perp(u) & I_{n-q-1} \end{pmatrix}$$

where $T(u)_{ij} = g(T(u)E_i, E_j)(m)$, $\perp(u)_{ia} = g(\perp_{E_i} E_a, E_n)(m)$ and $(\perp_X N)(m) = (\nabla_X^\perp N)(m)$.

Then $\quad g_{ij}(p) = ({}^tD_u D_u)_{ij}(r)$, $\; g_{ia}(p) = \frac{1}{\gamma}({}^tD_u D_u)_{ia}(r)$,

$$g_{ab}(p)=\frac{1}{r^2}({}^tD_uD_u)_{ab}(r),\quad g_{in}=g_{an}=0,\quad g_{nn}=1,$$

$i, j=1, \ldots, q$, and $a, b=q+1, \ldots, n-1$.

3. Main results

We consider the local diffeomorphism

$$\phi_B: p=\exp_\nu(ru) \mapsto \phi_B(p)=\exp_\nu(s(r)u)$$

for $u\in T_m^{\perp}B$, $\|u\|=1$. ϕ_B is called the geodesic transformation with respect to B, which is locally given by

$$\phi_B: (x^1, \ldots, x^n) \mapsto (x^1, \ldots, x^q, \rho(r)x^{q+1}, \ldots, \rho(r)x^n).$$

Let η be the one form defined by $\eta(X)=g(X, JN)$.

If $\phi_B^* g=e^{2\sigma}g+f\eta\otimes\eta$ for some function f which is depends only on the normal distance function r, ϕ_B is said to be partially conformal.

Lemma 2. A geodesic transformation ϕ_B with respect to B is partially conformal if and only if

$$g_{ij}(\phi_B(p))=(e^{2\sigma}g+f\eta\otimes\eta)_{ij}(p),\qquad \rho g_{ia}(\phi_B(p))=(e^{2\sigma}g+f\eta\otimes\eta)_{ia}(p),$$
$$\rho^2 g_{ab}(\phi_B(p))=(e^{2\sigma}g+f\eta\otimes\eta)_{ab}(p),\quad e^{2\sigma}=(\rho' r+\rho)^2=(s')^2.$$

where $i, j=1, \ldots, q$, and $a, b=q+1, \ldots, n-1$.

Lemma 3. Let (M, g, J) be a Kahler manifold and ϕ_B a patially conformal with respect to a point B of M. Then

$$s'(0)^2(1-s'(0)^2)R_{uaub}(B)=2(s'(0)s'''(0)\delta_{ab}+3/4\, f''(0)\delta_{qa}\delta_{qb})\ \text{along}\ \gamma.$$

proof. Since ϕ_B is partially conformal, $\rho^2 g_{ab}(\phi_B(p))=(e^{2\sigma}g+f\eta\otimes\eta)_{ab}(p)$.
When $r=0$, $f(0)=0$.
Since $\rho(r)^2=s'(0)^2+s'(0)s''(0)r+1/2(s''(0)/2+2/3s'(0)s'''(0))r^2+\circ(r^3)$ and

$g_{ab}(p)=\delta_{ab}-1/3\ r^2R_{uaub}+\circ(r^3)$, expanding the both side of above equation,

$$s'(0)^2\delta_{ab}+s'(0)s''(0)\delta_{ab}r+\circ(r^2)$$
$$=s'(0)^2\delta_{ab}+(2s'(0)s''(0)\delta_{ab}+f'(0)\delta_{qa}\delta_{qb})r+\circ(r^2).$$

Hence $f'(0)=0,\ s''(0)=0$.

Comparing each r^2-term in power series and using the above results,

$$s'(0)s'''(0)\delta_{ab}-s'(0)^4R_{uaub}=3(s'(0)s'''(0)\delta_{ab}-1/3\,s'(0)^2R_{uaub}+1/2\,f''(0)\delta_{qa}\delta_{qb}).$$

There required result follows.

Theorem 4. (M, g, J) is an n-dimensional Kahler manifold of complex space form $M_n(c)$, $c\neq 0$ if and only if for each point $B\in M$ there exists a non-conformal partially conformal. In this case, the geodesic transformation is non-Euclidean similarity

$$\tan\frac{s\sqrt{c}}{4}=C\tan\frac{r\sqrt{c}}{4},\qquad C^2\neq 0,\ 1.$$

Proof. Since $M=M_n(c)$, using (*)

$$R=\begin{pmatrix} c & 0 \\ 0 & c/4\ I_{n-2}\end{pmatrix}$$

Hance

$$A=\begin{pmatrix} 1/\sqrt{c}\ \sin r\sqrt{c} & 0 \\ 0 & 2/\sqrt{c}\ \sin\sqrt{c}/2\ I_{n-2}\end{pmatrix}.$$

Thus

$$g_{11}(\gamma(r))=(1/\sqrt{c}\ \sin r\sqrt{c})^2,\quad g_{ab}(\gamma(r))=(2/r\sqrt{c}\ \sin r\sqrt{c}/2)^2 \text{ for } a,\ b=2,\ \ldots,\ n-1.$$

Since $\eta(\partial/\partial x^1)=1/r\sqrt{c}\ \sin r\sqrt{c}$, by Lemma 2

$$(1/s\sqrt{c}\ \sin s\sqrt{c})^2=(e^{2\sigma}+f)(1/\sqrt{c}\ \sin r\sqrt{c})^2,$$
$$\rho^2(2/s\sqrt{c}\ \sin s\sqrt{c}/2)^2=e^{2\sigma}(2/r\sqrt{c}\ \sin r\sqrt{c}/2)^2.$$

We have

$$(\sin s\sqrt{c})^2=(e^{2\sigma}+f)(\sin r\sqrt{c})^2,$$
$$(\sin s\sqrt{c}/2)^2=e^{2\sigma}(\sin r\sqrt{c}/2)^2=(ds/dr)^2(\sin r\sqrt{c}/2)^2.$$

Therefore,

$$\tan\frac{s\sqrt{c}}{4} = C\tan\frac{r\sqrt{c}}{4} \quad \text{and} \quad f = -\left(\frac{\sin s\sqrt{c}/2}{\sin r\sqrt{c}/2}\right)^2 + \left(\frac{\sin s\sqrt{c}}{\sin r\sqrt{c}}\right)^2.$$

Conversely, put $\bar{t} = \tan s\frac{\sqrt{c}}{4}$ and $t = \tan r\frac{\sqrt{c}}{4}$. Then $\bar{t} = Ct$. Thus $s = \frac{4}{\sqrt{c}}\tan^{-1}Ct$.

By power series expansion

$$s = Cr - \frac{c}{48}C(C^2-1)r^3 + o(r^4).$$

Since $s'(0) = C$ and $s''(0) = 0$, $C^2(1-C^2)R_{u1ub}(B) = 0$ by Lemma 3.

Hence $R_{uJu}u$ is proportional to Ju. Using Theorem 1, $M = M_n(c)$.

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