International Journal of Automotive Technology, Vol. 2, No. 3, pp. 117–122 (2001)

1229–9138/2001/005–05

A HYBRID SCHEME USING LU DECOMPOSITION AND PROJECTION MATRIX FOR DYNAMIC ANALYSIS OF CONSTRAINED MULTIBODY SYSTEMS

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(Received 24 March 2001)

Abstract–For a dynamic analysis of a constrained multibody system, it is necessary to have a routine for satisfying kinematic constraints. LU decomposition scheme, which is used to divide coordinates into dependent and independent coordinates, is efficient but has great difficulty near the singular configuration. Other method such as the projection matrix, which is more stable near a singular configuration, takes longer simulation time due to the large amount of calculation for decomposition. In this paper, the row space and the null space of the Jacobian matrix are proposed by using the pseudo-inverse method and the projection matrix. The equations of the motion of a system are replaced with independent acceleration components using the null space of the Jacobian matrix. Also a new hybrid method is proposed, combining the LU decomposition and the projection matrix. The proposed hybrid method has following advantages. (1) The simulation efficiency is preserved by the LU method during the simulation. (2) The accuracy of the solution is also achieved by the projection method near the singular configuration.

KEY WORDS : DAE (Differential algebraic equation), Hybrid scheme, LU decomposition, Projection matrix

1. INTRODUCTION

Multibody dynamics, which analyzes the motion of a dynamic system which is consisted of bodies and joints, is widely used in the industry to reduce cost and time in a designing process. Researches in multibody dynamics are focused on several areas, *i.e.*, enhancing the accuracy and the efficiency of computation or generality of modeling process for various types of mechanical systems.

To get more accurate results, a more detailed modeling of the system is needed. In some cases, flexibility of the components should also be counted for. To enhance the computational efficiency, however, a simpler modeling and an efficient numerical algorithm is required in the solution process. For the general applications of commercial programs, the modeling process should be easy and applicable to the wide range of the problems.

In deriving the equations of motion for a multibody system, Cartesian coordinate system and joint coordinate system can be compared in view of efficiency and generality. Although the former is easy to derive equations of motion, it is inefficient because it takes longer simulation time due to the larger number of coordinates and constraint equations. On the other hand, the latter can formulate the equations of motion with a minimum number of coordinates, however it also has drawbacks in generalization of the constraint equations and deriving the equations of motion.

In a constrained multibody system, the equations of motion have a form of DAEs (Differential Algebraic Equations) that combine differential equations of motion and algebraic constraint equations. To solve the DAEs, a solution scheme of ODE cannot be applied directly. To use a well established numerical integration scheme, like a predictor-corrector method, it is inevitable to include a process that guarantees the satisfaction of constraint equations in dynamic analysis routine. One way is to use the constraint stabilization method (Baumgarte, 1972), however it is too difficult to find proper values of constants that guarantees convergence around singular configurations. The coordinate partitioning method, which divides coordinates into independent and dependent coordinates, is widely used. In this method, the values of dependent coordinates are iteratively determined to satisfy the constraint equations.

Typical coordinate partitioning methods are LU de-

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composition (Wehage, 1981), QR decomposition (Kim, 1986), SVD (singular value decomposition, Mani, 1986), Gram-Schmidt orthogonalization method (Liang. 1987), and projection matrix (Kim, 1998). Up to this point, only one of the above decomposition schemes is used, except a hybrid scheme, where LU and QR decomposition schemes are used (Kim, 1997). Other solution techniques are also suggested for the Euler-Lagrange equations (Potra, 1991) and DAEs (Asher, 1992, Petzold, 1992, Bae, 1999).

None of these papers is concerned in a combination of decomposing schemes. This paper aims to suggest a hybrid method of combining LU decomposition and projection method. To verify the efficiency and the accuracy of the suggested method, a spatial four bar mechanism is analyzed.

2. DAEs, EQUATIONS OF MOTION OF A CONSTRAINED SYSTEM

The constraint equations with kinematic and driving constraints can be written as:

$$\Phi(q,t)=[\Phi_1(q,t),...,\Phi_m(q,t)]=0 \tag{1}$$

where q is vector of generalized coordinates and m is the number of constraints. The velocity and the acceleration equation can be obtained by differentiating equation (1) with respect to time.

$$\Phi_q\dot{q}=-\Phi_t \tag{2}$$

$$\Phi_q\ddot{q}=-2\Phi_{qt}\dot{q}-(\Phi_q\dot{q})_q\dot{q}-\Phi_{tt} \tag{3}$$

where Φ_q is the Jacobian matrix, which is a differentiation of constraint equation (1) with respect to generalized coordinate vector q. When the vector q has n components, the dimension of the Jacobian matrix becomes $n\times m$.

The Lagrange multiplier form of the equations of motion can be written as:

$$M\ddot{q}+\Phi_q^T\lambda=g \tag{4}$$

where M is the mass matrix of the system, λ is the Lagrange multiplier vector, and g is the generalized force vector. By combining equation (3) and (4), the equations of motion of a constrained system can be expressed as follows:

$$\begin{bmatrix} M & \Phi_q^T \\ \Phi_q & 0 \end{bmatrix}\begin{bmatrix} \ddot{q} \\ \lambda \end{bmatrix}=\begin{bmatrix} g \\ -2\Phi_{qt}\dot{q}-(\Phi_q\dot{q})_q-\Phi_{tt} \end{bmatrix} \tag{5}$$

This type of equation is called DAEs (differential algebraic equation), which are combination of differential equation of motion and algebraic equations of constraints.

3. SOLUTION OF DAE WITH LU DECOMPOSITION

Decomposing the Jacobian matrix with the lower and upper triangular matrix, the following matrix can be constructed if the Jacobian matrix has full rank.

$$\Phi_q = \quad \begin{array}{|cc|c|} & & \\ & U & \\ L & & R \\ & & \end{array} \tag{6}$$

If the generalized coordinates q are partitioned into independent coordinates v and dependent coordinates u from LU decomposition of the Jacobian matrix, then independent coordinates v are related to the matrix R, which represents the remainder of the rank calculation. Therefore, Jacobian matrix can be partitioned as:

$$\Phi_q=[\Phi_u \;\; \Phi_v] \tag{7}$$

where Φ_u=LU and Φ_v=LR. If the values of the independent coordinates v are known, then the values of independent coordinates u can be obtained by solving the constraint equation (1) using Newton-Raphson method.

$$\begin{bmatrix} \Phi_u & \Phi_v \\ 0 & I \end{bmatrix}_i\begin{bmatrix} \Delta u^i \\ \Delta v^i \end{bmatrix}=\begin{bmatrix} -\Phi \\ 0 \end{bmatrix}_i \tag{8}$$

$$u^{i+1}=u^i+\Delta u^i$$

where i is the iteration number. Velocity vector $\dot{q}$ can also be divided into independent velocity vector $\dot{v}$ and dependent velocity vectordot $\dot{u}$. Velocity equations can be rearranged from equation (2) as:

$$\begin{bmatrix} \Phi_u & \Phi_v \\ 0 & I \end{bmatrix}\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix}=\begin{bmatrix} -\Phi_t \\ \dot{v} \end{bmatrix} \tag{9}$$

If position and velocity are known, acceleration and Lagrange multiplier can be obtained from equation (5). Then, numerical integration is carried out to predict $v(t+dt)$ and $\dot{v}(t+dt)$ using $\dot{v}(t)$ and $\ddot{v}(t)$. Substituting the calculated components into equation (8) and equation (9), dependent terms u and $\dot{u}$ at time $t+dt$ can be calculated.

This process proceeds until the end time for the simulation is reached.

4. SOLUTION OF DAE WITH A PROJECTION MATRIX

Since the Jacobian matrix has more rows than columns, its inverse matrix cannot be defined. If each column of the Jacobian matrix is linearly independent, then the inverse matrix of $\Phi_q\ \Phi_q^T$ exists. The right hand side of inverse matrix for Jacobian can be found as the following equation (Strang. G., 1988).

$$I_R=\Phi_q^T(\Phi_q\ \Phi_q^T)^{-1} \tag{10}$$

By using above equations, the null space of Jacobian matrix for a constrained multibody system can be derived as following (Kim, 1998).

$$P_N=[I-\Phi_q^T(\Phi_q\ \Phi_q^T)^{-1}\Phi_q] \tag{11}$$

Since $\Phi_q P_N=0$, P_N can be used as a null space of the Jacobian matrix. However, since the dimension of the null space is equal to the number of degrees of freedom of the system, only a partial portion of the P_N matrix is enough. Therefore, in this study, the following P_T matrix will be used as the final null space of the Jacobian matrix.

$$P_T=P_N B \tag{12}$$

where the matrix B is a $n\times k$ matrix, which takes out independent rows from P_N matrix and n and k are the number of coordinates and the number of degrees of freedom, respectively (Kim, 1998).

With the derived matrices I_R and P_T, velocity vector can be written as:

$$\dot{q}=[I_R,\ P_{NT}]\begin{bmatrix} b \\ \dot{z} \end{bmatrix} \tag{13}$$

where $P_{NT}=P_N P_T$. The vector b is the right hand side term of equation (2) and $\dot{z}$ is a free vector parallel to the tangential component of the constraint surface.

For the acceleration equations, equation (13) is differentiated with respect to time and the following equation is obtained.

$$\ddot{q}=[I_R,\ P_{NT}]\begin{bmatrix} \dot{b} \\ \ddot{z} \end{bmatrix}+\frac{d}{dt}[I_R,\ P_{NT}]\begin{bmatrix} b \\ \dot{z} \end{bmatrix}$$

Since $\frac{d}{dt}[I_R,\ P_{NT}]=-I_R\dot{\Phi}_q[I_R,\ P_{NT}]$, $\ddot{q}$ is expressed as:

$$\ddot{q}=[I_R,\ P_{NT}]\begin{bmatrix} \dot{b} \\ \ddot{z} \end{bmatrix}-I_R\dot{\Phi}_q\dot{q} \tag{14}$$

$$\ddot{q}=[I_R,\ P_{NT}]\begin{bmatrix} a \\ \ddot{z} \end{bmatrix}$$

where $a=-\dot{\Phi}_q\dot{q}-\dot{\Phi}_t$.

Multiplying the equation of motion for the entire system by $[I_R,\ P_{NT}]^T$ matrix, a differential equation of motion can be written as the following.

$$\begin{bmatrix} I_R^T \\ P_{NT}^T \end{bmatrix}M\ddot{q}=\begin{bmatrix} I_R^T\ g \\ P_{NT}^T\ g \end{bmatrix}-\begin{bmatrix} \lambda \\ 0 \end{bmatrix} \tag{15}$$

Separating the above equation into two terms, the following two equations are obtained.

$$P_{NT}^T\ M\ddot{q}=P_{NT}^T\ g \tag{16}$$

$$\lambda=I_R^T(M\ddot{q}-g) \tag{17}$$

By substituting the acceleration equation (16) into equation (14), the following equation can be obtained.

$$M_{11}\ddot{z}=P_{NT}^T(g-M\ I_R^T a) \tag{18}$$

where $M_{11}=P_{NT}^T\ M\ P_T$.

Therefore, independent acceleration vector $\ddot{z}$ is expressed as the following equations and its dimension is equal to the degree of freedom of the system.

$$\ddot{z}=M_{11}^{-1}\ P_{NT}^T(g-M\ I_R^T a) \tag{19}$$

When the velocity and acceleration vectors are obtained, position and velocity vectors at the next time step can be obtained by numerical integration. When the calculated displacement vector from integration does not satisfy the kinematic constraint equations, correction of dependent coordinates are carried out with Newton-Raphson method.

$$\begin{bmatrix} \Phi_q \\ P_{NT}^T \end{bmatrix}_i \Delta i=\begin{bmatrix} -\Phi \\ 0 \end{bmatrix}_i \tag{20}$$

$$q^{i+1}=q^i+\Delta q^i,\ \ i=0,\ 1,\ 2,\ 3,\ .....$$

Also by integrating independent acceleration vector $\ddot{z}$, independent velocity vector $\dot{z}$ can be obtained for the next time step and $\dot{q}$ can be calculated from the equation (13). And using values of $\ddot{z}$, acceleration vector $\ddot{q}$ can be obtained from equation (14) and Lagrange multiplier vector λ can be calculated from equation (17).

5. COMBINATION OF LU DECOMPOSITION AND PROJECTION MATRIX

As long as a system is not encountered with a singular configuration, the LU method is satisfactory since it

preserves numerical efficiency. However, when a system reaches near a singular configuration, the condition number of the mass matrix in the equations of motion increases rapidly. In this case, the LU decomposition method is used to continuously carry out coordinate partitioning to find a new suitable independent coordinates. Although successive coordinate partitioning is performed, this method sometimes leads to an undesirable solution.

Since the projection method, on the other hand, performs a numerical integration tracing the null space of constraint surface at each time step, convergence property at singular points is much better than the LU method. Projection method, however, is less efficient because it requires more computational burden, associated with inverse operation of the total Jacobian matrix to find the null space at each time.

Therefore, a hybrid coordinate partitioning method, which combines both desirable features of the above two methods, is proposed. The main idea presented in this paper is to use LU decomposition method as a basic tool except at singular configurations, and the projection method is employed to keep high accuracy around the singular points.

The following thirteen steps briefly outline the proposed algorithms.

Step 1: Check the initial conditions whether they satisfy constraint equations.

Step 2: Evaluate the Jacobian matrix and choose independent coordinate and independent velocity vectors using the LU method.

Step 3: Determine the independent coordinate from equation (7) and iterate to determine the dependent coordinate with Newton-Raphson method. Check whether to change independent coordinate in corrector step when integration step size decreases or sustains more than 5 times with the same interval. If independent coordinate is changed, then go back to step 2.
Also check whether to switch coordinate partitioning method based on the followings:
1) frequent changes of independent coordinate for a short time period
2) distinct variation of total energy of the system
3) sudden decrease in the integration step size relative to the prior time step
When the partitioning needs to change, switch the partitioning scheme and go to step 8.

Step 4: Evaluate the dependent velocity vector from the independent velocity vector with equation (8).

Step 5: Evaluate the acceleration vector and Lagrange multiplier from equation (5).

Step 6: Using the explicit/implicit PECE (predictor evaluation and corrector evaluation) algorithms, calculate $\ddot{v}$ and $\dot{v}$.

Step 7: Return to step 3 and proceed this process to the final simulation time.

Step 8: Evalute I_R, P_{NT} and the order of Jacobian matrix.

Step 9: Determine the independent coordinate z and evaluate the position vector with Newton-Raphson method using the equation (20).

$$\begin{bmatrix} \Phi_q \\ P_{NT}^T \end{bmatrix}_i \Delta q^i = \begin{bmatrix} -\Phi \\ 0 \end{bmatrix}_i$$

$$q^{i+1} = q^i + \Delta q^i, \quad i=0, 1, 2, 3, \ldots\ldots$$

Step 10: Evaluate the generalized velocity from equation (13).

Step 11: Evaluate the independent acceleration $\ddot{z}$ from equation (19). Using $\ddot{z}$, calculate $\ddot{q}$ from equation (14) and evaluate Lagrange multiplier λ from equation (17).

Step 12: If an integration is continuously performed more than 5 times with the maximum allowable integration step size, then the system is judged to be stable. Switch to the LU decomposition, choose independent coordinates and return to step 2.

Step 13: Using $\dot{q}$, $\ddot{z}$ and explicit/implicit Adams Bashforth PECE algorithms like DE program (Shampine, 1975), perform the integration and return to step 9 until the final simulation time.

6. NUMERICAL EXAMPLE AND SIMULATION

A parallel four-bar linkage, shown in Figure 1, which has 10° of initial rotation angle is chosen as an example because it has a singular configuration at horizontal position. When this mechanism starts from the initial configuration under the gravity, it reaches to the first singular point at 0.18 second and the second singular point at 1.0 second.

The constraint equations for the 4-bar mechanism can be written as:

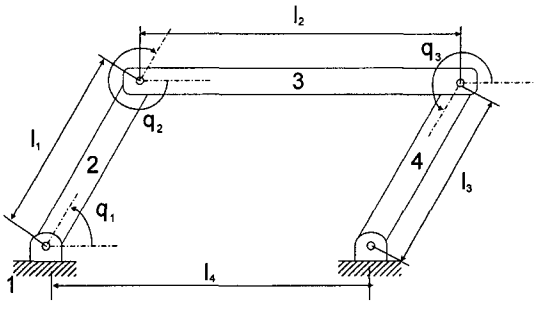

Figure 1. A parallel 4-bar linkage.

$$\Phi=\begin{bmatrix}\Phi^1\\ \Phi^2\end{bmatrix}=$$

$$\begin{bmatrix}l_1\cos q_1+l_2\cos(q_1+q_2)+l_3\cos(q_1+q_2+q_3)-l_4\\ l_1\sin q_1+l_2\sin(q_1+q_2)+l_3\sin(q_1+q_2+q_3)\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \quad (21)$$

The Jacobian matrix is:

$$\Phi_q=$$

$$\begin{bmatrix}-l_1\sin q_1-l_2\sin q_{12}-l_3\sin q_{123} & -l_2\sin q_{12}-l_3\sin q_{123} & -l_3\sin q_{123}\\ l_1\cos q_1+l_2\cos q_{12}+l_3\cos q_{123} & l_2\cos q_{12}+l_3\cos q_{123} & l_3\cos q_{123}\end{bmatrix} \quad (22)$$

where $q_{12}=q_1+q_2$ and $q_{123}=q_1+q_2+q_3$. To simplify the expression, $l_1=l_3=1$m, $l_2=l_4=2$m. Then,

$$\sin(q_1+q_2)=0,\ \cos(q_1+q_2)=1$$
$$\cos(q_1+q_2+q_3)=-\cos q_1,\ \sin(q_1+q_2+q_3)=-\sin q_1 \quad (23)$$

Substituting the above equation in the Jacobian matrix of equation (22), the final result is:

$$\Phi_q=\begin{bmatrix}0 & \sin q_1 & \sin q_1\\ 2 & 2-\cos q_1 & -\cos q_1\end{bmatrix} \quad (24)$$

Using this Jacobian matrix, the inverse matrix and projection matrix appear as:

$$\Phi_q\Phi_q^T=\begin{bmatrix}2\sin^2 q_1 & 2\sin q_1(1-\cos q_1)\\ 2\sin q_1(1-\cos q_1) & 2\cos^2 q_1-4\cos q_1+8\end{bmatrix} \quad (25)$$

$$(\Phi_q\Phi_q^T)^{-1}=$$

$$\frac{1}{12\sin^2 q}\begin{bmatrix}2\cos^2 q_1-4\cos q_1+8 & -2\sin q_1(1-\cos q_1)\\ 2\sin q_1(1-\cos q_1) & 2\cos^2 q_1\end{bmatrix} \quad (26)$$

From the equation $I_R=\Phi_q^T(\Phi_q\Phi_q^T)^{-1}$,

$$I_R=\frac{1}{12\sin^2 q_1}\begin{bmatrix}-4(\sin q_1-\sin q_1\cos q_1) & 4\sin^2 q_1\\ 2\sin q_1\cos q_1+4\sin q_1 & 2\sin^2 q_1\\ -2\sin q_1\cos q_1+8\sin q_1 & -2\sin^2 q_1\end{bmatrix} \quad (27)$$

Thus, the projection matrix P_N becomes,

$$P_N=I-I_R\Phi_q=\frac{1}{3}\begin{bmatrix}1 & -1 & 1\\ -1 & 1 & -1\\ 1 & -1 & 1\end{bmatrix} \quad (29)$$

With the derived matrices, numerical simulations are carried out to show the effectiveness of the proposed hybrid algorithm. Figure 2 illustrates vertical acceleration of the coupler. As shown in Figure 2, the LU method shows incorrect result after it reaches to the second singular configuration at 1.0 second. However, both the projection method and the hybrid method give correct answers. As shown in Figure 3, vertical displacements of the two methods are in a good agreement. Thus, it is evident that the proposed hybrid coordinate partitioning method in this paper can easily overcome the singular configuration without divergence of solution.

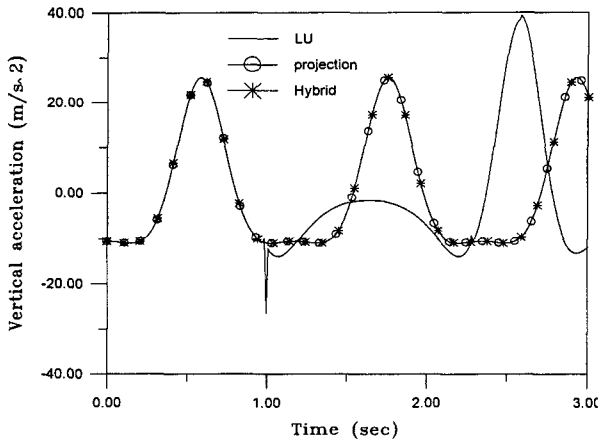

Figure 2. Vertical acceleration of the coupler.

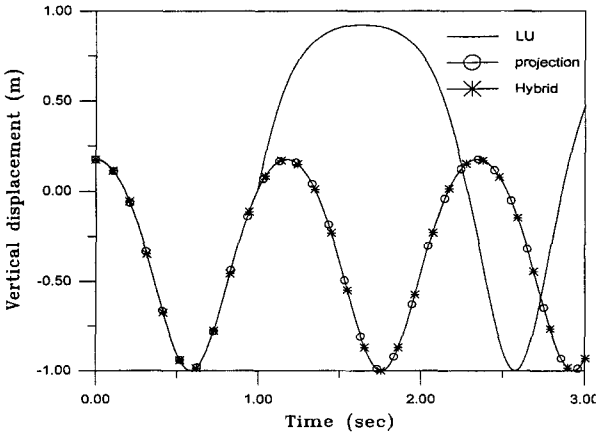

Figure 3. Vertical position of the coupler.

Figure 4 illustrates the total energy of the system during the simulation. Since the system is a conservative one, the total energy should have equal values. While the LU method can not preserve the energy of the system, the other two methods maintain the same level of energy. It can be said that the choice of independent coordinates in the LU method is not adequate near the second singular configuration and the calculated dependent velocities have a large error.

Table 1 compares the CPU times used for three

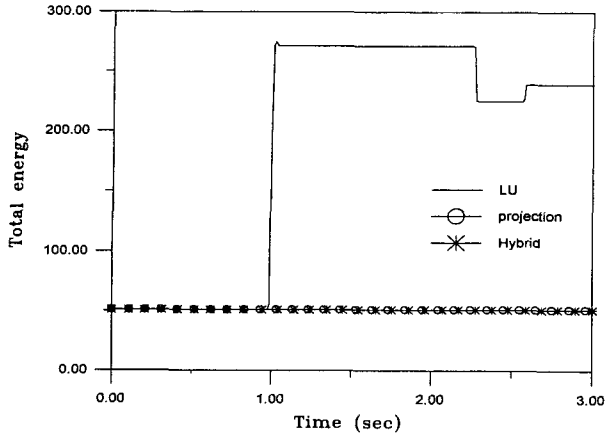

Figure 4. Total energy of the system.

Table 1. Comparison of CPU times.

	LU Method	Projection method	Hybrid Method
CPU time (seconds)	2.30	15.63	9.65

methods. The simulation is carried out on SGI (Silicon Graphics Inc.) INDIGO 2 workstation equipped with a R10000 chip. As shown in table 1, it is considered that the LU method is desirable if accuracy of solution is guaranteed. But for this kind of example with singular points, the hybrid method has advantage of projection matrix in terms of accuracy around singular points.

7. CONCLUSION

A null space modification method for constrained multibody system is suggested and a hybrid coordinate partitioning scheme is proposed. From the derivation and numerical results, the following conclusion is obtained.

The LU decomposition method is efficient and desirable if accuracy of the solution is guaranteed. But for a simulation containing singular configurations, solutions near the singular points may diverge with the LU method. The projection method, on the other hand, can preserve the accuracy around the singular points, but it takes longer simulation times. The hybrid method proposed in this paper combines advantages of two methods, which preserves accuracy and efficiency. Thus, the proposed method combining the LU method and the projection method of null space is useful for the simulation of systems including singular configurations.

ACKNOWLEDGEMENT–This research was supported by the KOSEF (Project No. 97-0200-1001-5).

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