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The Stress Field in a Body Caused by the Tangential Force of a Rectangular Patch on a Semi-Infinite Solid

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Abstract : The stress field in a body caused by the tangential loading of a rectangular patch on a semi-infinite solid has been solved analytically using a potential function. The validity of the results of this study was proved by Saint-Venant's principle in the remote region and by the superposition of point loads in the vicinity of the surface.

Key words : Semi-infinite solid, potential function, subsurface stress and rectangular patch solution

Introduction

Mechanical elements moving relatively to each other accompany wear due to contact force on the contact surfaces. Wear is one of the major causes of mechanical failure. To estimate the life of mechanical elements, it is necessary to evaluate the wear rate quantitatively. The distribution of contact stresses and subsurface stresses due to contact force is very important for a quantitative analysis of wear rates.

Generally, the Hertzian solution [1] has been applied to the evaluation of contact and subsurface stresses in the contacts of negligible friction such as hydrodynamic lubrication or pure rolling. In mixed or boundary lubrication condition in sliding, however, tangential force due to friction can not be neglected any more and makes the subsurface stress distribution different from that of the contact under pure normal force.

The effect of tangential force on the contact stress was first considered by Cattaneo [2] and independently Mindlin [3]. However, their solutions are restricted to elliptic contacts with tangential loading in the direction of the semi-major or minor axis of the ellipse.

Macroscopically, it is reasonable to assume elliptical contact stress distribution because apparent contact shapes of most machine elements in relative moving are spherical or cylindrical. But real contacts of irregular asperities should be considered since normal wears are micro-scale phenomena. The stress distribution for a general contact in an elastic half space was presented by Boussinesq [4] and Cerruti [5] using the potential theory. Stresses and displacements in their solutions are expressed in terms of the space derivative of a potential function. This function is presented as a form of a double integral taken over the pressed area. However, the difficulties in calculating the double integral has been a serious obstacle to the application of their equations.

Most analyses of contact of rough surfaces with irregular asperities have been made by approximating the tip of the asperity into a sphere without mutual effects of asperities. With the recent development of computer, most of the contact analyses have been made numerically by discretizing the contact region of an irregular shape into rectangular patches. In analyzing sub-surface stresses under a distributed normal loading, Love's solution [6] -stresses in a semi-infinite solid due to a uniformly distributed normal loading on a rectangular patch of an elastic half space- has been generally used. However, subsurface stresses due to uniformly distributed tangential loading on a rectangular patch are not yet known in the closed form. As one of trials of this problem, Ahmadi *et al.* [7] expressed the stress distribution in a semi-infinite solid by tangential loading in terms of 52 integrals by integrating Cerruti's solution -the stresses in a semi-infinite solid due to tangential loading taken to be concentrated on a vanishingly small area- over a rectangular patch. However, Ahmadi *et al.*'s solution in the vicinity of the loaded region has the problem of divergence. For the same problem, Kalker [8] numerically evaluated the stress field in a semi-infinite solid by using the technique of asymptotic expansion. However, some value of Kalkers results becomes very large when considered points approach the edge of a rectangle patch. Maria *et al.* [9] presented the corrected solutions to solve the problem in Ahmadi *et al.*'s solution. However, Maria *et al.*'s solution is also not accurate in the remote region from the surface.

This study re-examines the accuracy of the solutions given by Ahmadi *et al.* and Maria *et al.* Also, by using a potential function, the stress field in a body caused by a uniformly distributed tangential loading of a rectangular patch on a semi-infinite solid has been derived in the closed form.

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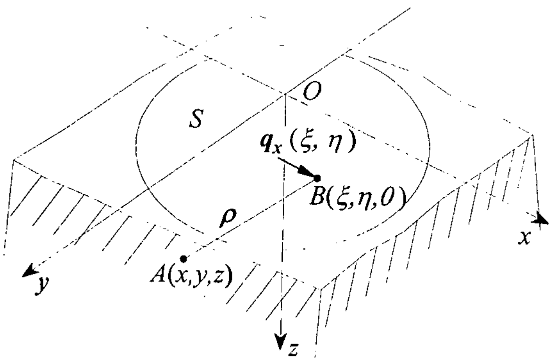


Fig. 1. The elastic half space subject to a distributed tangential loading of an area *S*.

Theory

Tangential loading, as shown in Fig. 1, acts on the arbitrary area S on an elastic half space, whose material is assumed to be isotropic and homogeneous. Stresses in an elastic half space due to $q_x(\xi,\eta)$ can be expressed by introducing a potential function, as follows.

$$F_1 = \iint_S q_x(\xi,\eta)(z\ln(\rho+z)-\rho)d\xi d\eta \quad (1)$$

where ξ and η are the coordinate of any point within the loaded area S in each x, y direction. ρ is the distance between $A(x, y, z)$, positioned at any point in an elastic half space, and $B(\xi, \eta, 0)$, positioned at any point on the loaded area S. The following potential function is introduced by differentiating equation (1) with respect to z.

$$F = \frac{\partial F_1}{\partial z} = \iint_S q_x(\xi,\eta)\ln(\rho+z)d\xi d\eta \quad (2)$$

Love showed that the components of elastic displacement u_x, u_y and u_z at any point $A(x, y, z)$ in the body can be expressed in terms of the above functions, as follows.

$$u_x = \frac{1}{4\pi G}\left\{2\frac{\partial F}{\partial z}+2\nu\frac{\partial^2 F_1}{\partial x^2}-z\frac{\partial^2 F}{\partial x^2}\right\}$$

$$u_y = \frac{1}{4\pi G}\left\{2\nu\frac{\partial^2 F_1}{\partial x\partial y}-z\frac{\partial^2 F}{\partial x\partial y}\right\} \quad (3)$$

$$u_z = \frac{1}{4\pi G}\left\{(1-2\nu)\frac{\partial F}{\partial x}-z\frac{\partial^2 F}{\partial x\partial z}\right\}$$

where G is the shear modulus. The elastic displacements in equation (3) satisfy the condition of compatibility because the potential functions, F_1 and F_2, are continuous and differentiable for all orders. The displacement having been found, the stresses at point $A(x, y, z)$ are calculated by Hooke's law, as follows.

$$\sigma_x = \frac{\nu+1}{\pi}\frac{\partial^2 F}{\partial x\partial z}+\frac{1}{2\pi}\left\{2\nu\frac{\partial^3 F_1}{\partial x^3}-z\frac{\partial^3 F}{\partial x^3}\right\}$$

$$\sigma_y = \frac{\nu}{\pi}\frac{\partial^2 F}{\partial x\partial z}+\frac{1}{2\pi}\left\{2\nu\frac{\partial^3 F_1}{\partial x\partial y^2}-z\frac{\partial^3 F}{\partial x\partial y^2}\right\}$$

$$\sigma_z = \frac{-z}{2\pi}\frac{\partial^3 F}{\partial x\partial z^2} \quad (4)$$

$$\tau_{xy} = \frac{1}{2\pi}\left\{\frac{\partial^2 F}{\partial z\partial y}+2\nu\frac{\partial^3 F_1}{\partial x^2\partial y}-z\frac{\partial^3 F}{\partial x^2\partial y}\right\}$$

$$\tau_{yz} = \frac{-z}{2\pi}\frac{\partial^3 F}{\partial x\partial y\partial z}$$

$$\tau_{zx} = \frac{1}{2\pi}\left\{\frac{\partial^2 F}{\partial z^2}-z\frac{\partial^3 F}{\partial x^2\partial z}\right\}$$

where n is Poisson's ratio. Stresses must satisfy the equation of equilibrium, as follows.

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z} = 0$$

$$\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{yz}}{\partial z} = 0 \quad (5)$$

$$\frac{\partial\sigma_z}{\partial z}+\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y} = 0$$

Substitution of the stresses by equation (4) into equation (5) gives expressions including $\nabla^2 F$ and $\nabla^2 F_1$, where ∇^2 denotes the operator $\vartheta^2/\vartheta x^2+\vartheta^2/\vartheta y^2+\vartheta^2/\vartheta z^2$, and F_1 and F_2 are the harmonic functions of x, y and z. I.e, they both satisfy the Laplacian equation. Therefore, the amount of the stresses given by equation (4) automatically satisfies the equation of equilibrium. In addition, boundary conditions must be satisfied. For the half space shown in Fig. 1, they are as follows. On the boundary z = 0, outside the area S loaded by a tangential loading, the surface is free from stresses. Within the loaded region, the surface is subjected to tangential loading. Finally, all the components of displacements and stresses given by equations (3) and (4) must be zero at an infinite distance ($\rho\rightarrow\infty$) from the loaded region. It follows from the characteristics of F_1 and F_2 that the amount of the stresses given by equation (5) satisfies the boundary conditions.

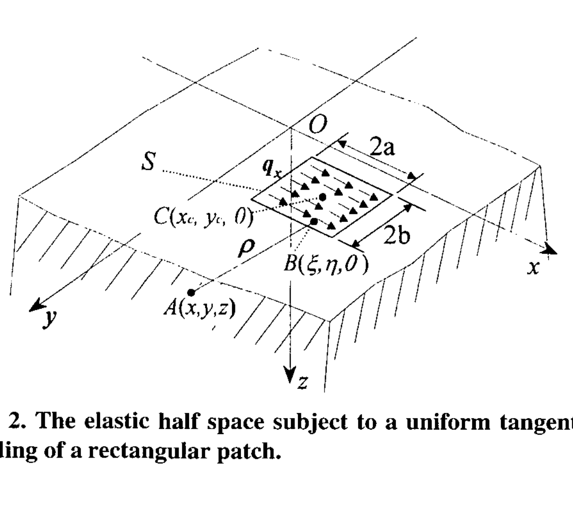
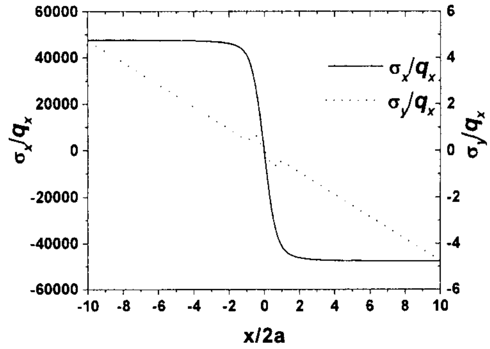
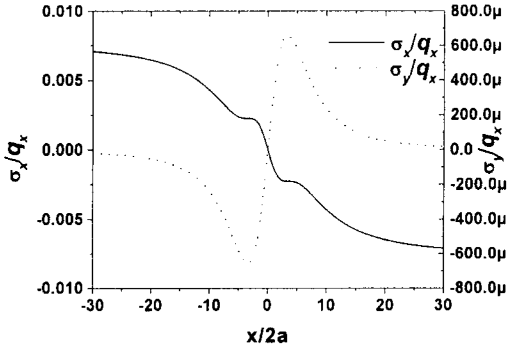


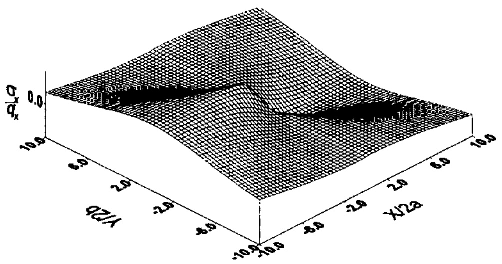
Fig. 2. The elastic half space subject to a uniform tangential loading of a rectangular patch.



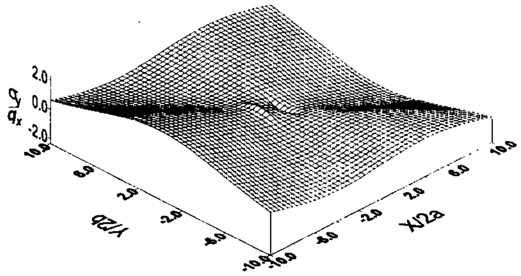
(a) σ_x, σ_y at y=0, z/2a=0.002



(b) σ_x, σ_y at y=0, z/2a=5($\mu = 10^{-6}$)



(c) 3–D view of σ_x at z/2a=0.002



(d) 3–D view of σ_y at z/2a=0.002

Fig. 3. Stress distribution in Ahmadi *et al.*'s Result.

Results and Discussion

Generally, contact problems due to contact regions with arbitrary shapes, as shown in Fig. 1, can be solved numerically by discretizing the area S into a rectangular patch, as shown in Fig. 2. This study evaluated the stresses at any point $A(x, y, z)$ in a body caused by uniformly distributed tangential loading acting parallel to the x-axis on a rectangular patch of dimension $2a \times 2b$ centered at $C(x_c, y_c, 0)$ on the surface of an elastic half space. Also, we examined the accuracy of the results derived in this study and that of the existing results by Ahmadi *et al.* and Maria *et al.* The results derived in this study are presented in the appendix, and the figures presented in this paper are expressed by equating $C(x_c, y_c, 0)$ to the origin. The distribution of σ_x, σ_y in Ahmadi *et al.*'s solution is illustrated along the x-axis, as shown in Fig. 3. As $|x/2a| \to \infty$, all the stresses in a solid should converge to zero. However, σ_x and σ_y diverge in the vicinity of the loaded region (Fig. 3(a)), and σ_x at some distance from the loaded region (Fig. 3(b)) has finite values, and do not converge to zero. Figs. 3 (c), (d) illustrates 3-D view of and in the vicinity of the loaded region. 3-D distribution of and also do not converge to zero.

To solve these problems, Maria *et al.* presented the corrected solution obtained by correcting Ahmadi *et al.*'s solution. However, Maria *et al.*'s solution also has a problem, as shown in Fig. 4. Fig. 4 shows the difference between patch solutions(J_p) – which refer to the Von Mises equivalent stress for Maria *et al.*'s result or the present result – and Cerruti's solution(J_c) – which refers to the Von Mises equivalent stress for the stresses in the body by a point load, which acts at the center point of a rectangular patch and has the same magnitude

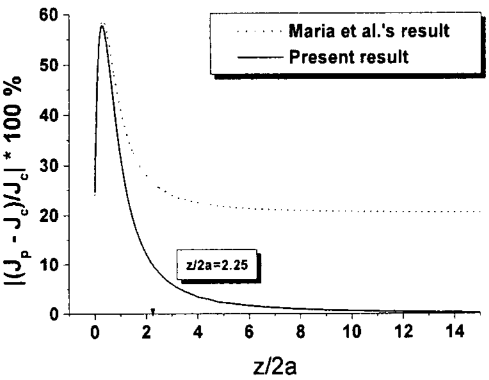
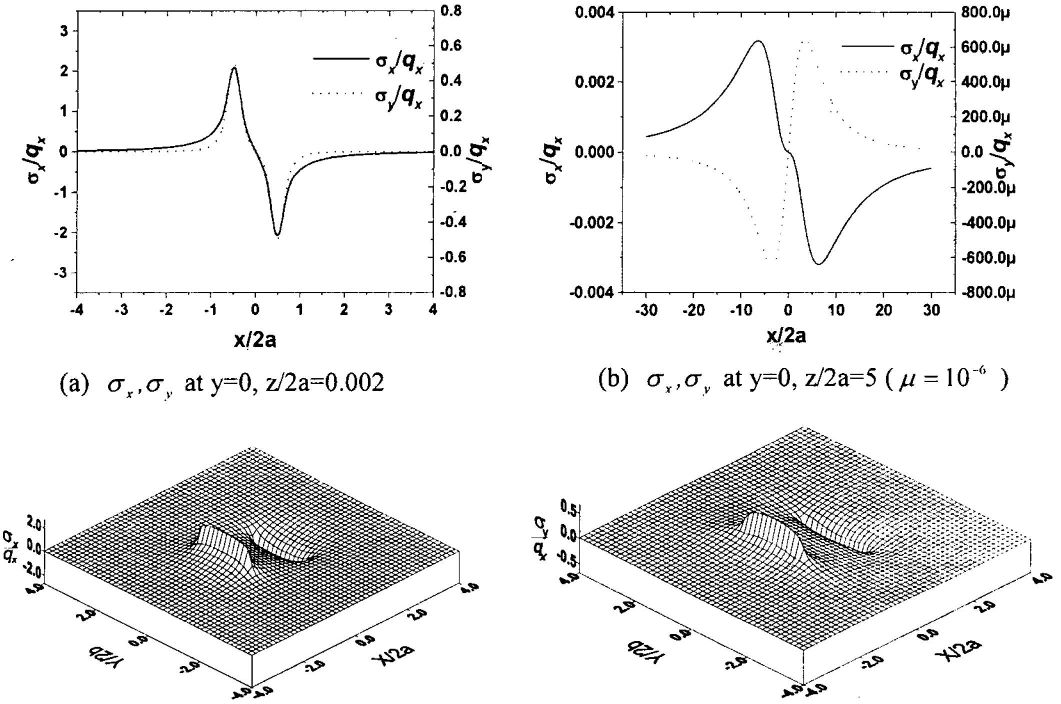


Fig. 4. Difference of Mises stresses between patch solution (J_p) and Cerrutis solution (J_c) along vertical line x/2a = 0.25, y/2b = 0.25.

as the resultant stress of a uniformly distributed tangential loading over a rectangular patch – along the vertical line x/2a = 0.25, y/2b = 0.25. According to the Saint-venant's principle, the difference between a patch solution (J_p) and Cerruti's solution (J_c) should converge to zero as the considered point $A(x, y, z)$ recedes from the loaded region. In Fig. 4, Maria *et al.*'s results in the vicinity of the loaded region is similar to the present studys results. However, Maria *et al.*'s results in the distance from the loaded region still show about a 20% error, and do not converge to zero. However, the difference between the present studys solution and Cerruti's solution becomes less than 10% for about z/2a>2.25 and converges to zero as z/2a. Therefore, this present studys results correspond well with Saint-venant's principle. This fact



(a) σ_x, σ_y at y=0, z/2a=0.002

(b) σ_x, σ_y at y=0, z/2a=5 ($\mu = 10^{-6}$)

(c) 3–D view of σ_x at z/2a=0.002

(d) 3–D view of σ_y at z/2a=0.002

Fig. 5. Stress distribution in the present Result (ν = 0.3, q_x = 1.0, a = half length of a rectangular patch, a = b).

demonstrates the good accuracy of this studys result in the distance from the loaded region. The distribution of σ_x, σ_y in this study is illustrated in Fig. 5. The values of σ_x, σ_y in Fig. 5 converge to zero regardless of the magnitude of z/2a, as the magnitude of $|x/2a| \rightarrow \infty$ is increased.

The accuracy of this studys solution near the loaded region can be proved by comparing the solution with the solution by the superposition of point loads, where the latter refers to the summation of the effects of point loads on a point in a semi-infinite solid. Johnson [10] showed that the stresses in a body resulting from any distributed loading on a surface could be approximated by the superposition of point loads. Johnson's method is as follows. First, divide a loaded rectangular patch by N × N, the number of infinitesimal rectangular patches. Then, equate each uniformly distributed tangential load with an infinitesimal concentrated load acting at the center point of infinitesimal patches. Finally, the stresses in a body can be approximated by superposing the influences on any point in the body by infinitesimal concentrated loads. Fig. 6 shows the difference between this studys solution (J_p) and the solution (J_{nc}) obtained by the superposition of point loads. The sizes of 2a and 2b, the length of a side on a single rectangular patch in each x, y direction, are 2.0. The length of a side on the infinitesimal rectangular patch is $2a_{nc}$ = 0.02 because the length

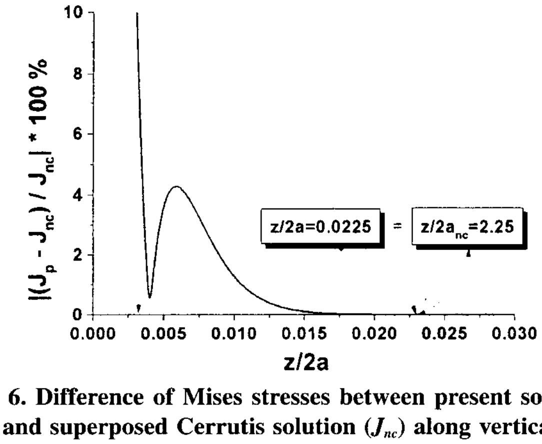


Fig. 6. Difference of Mises stresses between present solution (J_p) and superposed Cerrutis solution (J_{nc}) along vertical line x/2a = 0.25, y/2b = 0.25. (a = half length of the rectangular patch, a_{nc} = half length of a finite union of a rectangular patch, a = b).

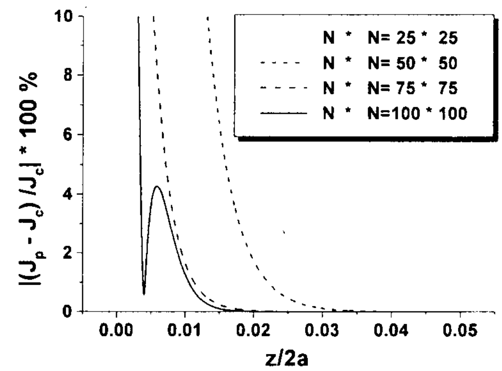


Fig. 7. Difference between J_p and J_{nc} for the various number of infinitesimal rectangular patches at x/2a = 0.25, y/2b = 0.25. (a = b).

of a side on a single rectangular patch is divided by 100. As illustrated in Fig. 6, the difference between this studys solution (J_p) and that (J_{nc}) obtained by the superposition of point loads almost converges to zero for about z/2a > 0.0225. It was shown in Fig. 4 that the difference between J_p in this studys results and J_{nc} becomes less than 10% for about z/2a > 2.25. In Fig. 6, z/2a = 0.0225 is equal to z/2a$_{nc}$ = 2.25. This indicates that the results of this study are accurate within the error of 10% for z/2a > 0.0225. The differences among the various numbers of infinitesimal rectangular patches are illustrated by Fig. 7. The greater the number of infinitesimal elements, the smaller the difference between the two solutions. Therefore, the accuracy of this studys solution in the vicinity of the loaded region has also been well proved.

Judging from the previously illustrated results, the solution derived from this study is believed to be the exact solution, being very accurate for all the regions in a semi-infinite solid.

Conclusion

The stress field in a solid caused by the tangential loading of a rectangular patch on a semi-infinite solid was studied analytically. We derived a closed form solution that is very accurate for all the regions in a semi-infinite solid by using Cerruti's potential function. In the remote region from the loaded area, the difference between this studys solution and Cerruti's solution converges to zero. Thus, the results of this study follow well Saint-venant's principle. In the vicinity of the loaded region, the difference between the superposed Cerruti's solution and this studys solution is infinitesimal. Therefore, the accuracy of the studys solution in the vicinity of the loaded region has been well proved.

Appendix

Partial differential terms in equation (4) are as follows.

$$\frac{\partial^2 F}{\partial x \partial z} = q_x log\frac{(\rho_3+y_3)(\rho_2+y_2)}{(\rho_4+y_2)(\rho_1+y_1)}$$

$$\frac{\partial^3 F_1}{\partial x^3} = q_x\left\{z\left(\frac{y_2}{x_2^2+y_2^2}-\frac{y_1}{x_2^2+y_1^2}-\frac{zy_2(\rho_4^2+x_2^2)}{\rho_4(y_2^2z^2+x_2^2\rho_4^2)}+\frac{zy_1(\rho_3^2+x_2^2)}{\rho_3(y_1^2z^2+x_2^2\rho_3^2)}\right.\right.$$
$$\left.-\frac{y_2}{x_1^2+y_2^2}+\frac{y_1}{x_1^2+y_1^2}+\frac{zy_2(\rho_2^2+x_1^2)}{\rho_2(y_2^2z^2+x_1^2\rho_2^2)}-\frac{zy_1(\rho_1^2+x_1^2)}{\rho_1(y_1^2z^2+x_1^2\rho_1^2)}\right)$$
$$log\frac{(\rho_4+y_2)(\rho_1+y_1)}{(\rho_2+y_2)(\rho_3+y_1)}+x_2^2\left(\frac{1}{\rho_4(\rho_4+y_2)}-\frac{1}{\rho_2(\rho_2+y_2)}\right)-x_1^2\left(\frac{1}{\rho_2(\rho_2+y_2)}-\frac{1}{\rho_1(\rho_1+y_1)}\right)$$

$$\frac{\partial^3 F}{\partial x^3} = q_x\left\{\frac{y_2}{x_2^2+y_2^2}-\frac{y_1}{x_2^2+y_1^2}-\frac{zy_2(\rho_4^2+x_2^2)}{\rho_4(y_2^2z^2+x_2^2\rho_4^2)}+\frac{zy_1(\rho_3^2+x_2^2)}{\rho_3(y_1^2z^2+x_2^2\rho_3^2)}\right.$$
$$\left.-\frac{y_2}{x_1^2+y_2^2}+\frac{y_1}{x_1^2+y_1^2}+\frac{zy_2(\rho_2^2+x_1^2)}{\rho_2(y_2^2z^2+x_1^2\rho_2^2)}-\frac{zy_1(\rho_1^2+x_1^2)}{\rho_1(y_1^2z^2+x_1^2\rho_1^2)}\right)$$

$$\frac{\partial^3 F_1}{\partial x \partial y^2} = q_x\left\{\frac{y_2}{\rho_4+z}-\frac{y_1}{\rho_3+z}-\frac{y_2}{\rho_2+z}+\frac{y_1}{\rho_1+z}\right\}$$

$$\frac{\partial^3 F}{\partial x \partial y^2} = q_x\left\{-\frac{y_2}{\rho_4(\rho_4+z)}+\frac{y_1}{\rho_3(\rho_3+z)}+\frac{y_2}{\rho_2(\rho_2+z)}-\frac{y_1}{\rho_1(\rho_1+z)}\right\}$$

$$\frac{\partial^3 F}{\partial x \partial z^2} = q_x\left\{-\frac{z}{\rho_4(\rho_4+y_2)}+\frac{z}{\rho_3(\rho_3+y_1)}+\frac{z}{\rho_2(\rho_2+y_2)}-\frac{z}{\rho_1(\rho_1+y_1)}\right\}$$

$$\frac{\partial^2 F}{\partial z \partial y} = q_x log\frac{(\rho_3+x_3)(\rho_2+x_2)}{(\rho_4+x_2)(\rho_1+x_1)}$$

$$\frac{\partial^3 F_1}{\partial x^2 \partial y} = q_x\left\{\frac{x_2}{\rho_4+z}-\frac{x_2}{\rho_3+z}-\frac{x_1}{\rho_2+z}+\frac{x_1}{\rho_1+z}\right\}$$

$$\frac{\partial^3 F}{\partial x^2 \partial y} = q_x\left\{-\frac{x_2}{\rho_4(\rho_4+z)}+\frac{x_2}{\rho_3(\rho_3+z)}+\frac{x_1}{\rho_2(\rho_2+z)}-\frac{x_1}{\rho_1(\rho_1+z)}\right\}$$

$$\frac{\partial^3 F}{\partial x \partial y \partial z} = q_x\left\{\frac{1}{\rho_4} - \frac{1}{\rho_3} - \frac{1}{\rho_2} + \frac{1}{\rho_1}\right\}$$

$$\frac{\partial^3 F}{\partial x^2 \partial z} = q_x\left\{\frac{x_2}{\rho_4(\rho_4 + y_2)} + \frac{x_2}{\rho_3(\rho_3 + y_1)} + \frac{x_1}{\rho_2(\rho_2 + y_2)} - \frac{x_1}{\rho_1(\rho_1 + y_1)}\right\}$$

$$\frac{\partial^2 F}{\partial z^2} = q_x\left\{-tan^{-1}\frac{x_2 y_3}{z\rho_4} + tan^{-1}\frac{x_2 y_1}{z\rho_3} + tan^{-1}\frac{x_1 y_2}{z\rho_2} - tan^{-1}\frac{x_1 y_1}{z\rho_1}\right\}$$

where $x_1 = x_c - a - x, x_2 = x_c + a - x, y_1 = y_c - b - y, y_2 = y_c + b - y$

$$\rho_1 = \sqrt{x_1^2 + y_1^2 + z^2},\ \rho_2 = \sqrt{x_1^2 + y_2^2 + z^2},\ \rho_3 = \sqrt{x_2^2 + y_1^2 + z^2},\ \rho_4 = \sqrt{x_2^2 + y_2^2 + z^2}$$

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