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Analysis of the Dispersion Relation of Elastic Waves Propagating on Vibrating Cylindrical Shells

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Abstract

This paper examines the dispersion relation governing the wave propagation on cylindrical shells. The assumption of thin shells allows the dispersion relation to be separated into three relations related to the propagation of flexural waves and two types of membrane waves. Those relations are used to identify the characteristics of the wave number curves. The dispersion relation provides two and three closed wave number curves below and above the ring frequency. Above the ring frequency three wave number curves are clearly identified to be those of flexural, shear and longitudinal waves, respectively. Below the ring frequency, the characteristics of two wave number curves are identified with dependence of the direction of wave propagation.

Keywords: *Dispersion relation, Wave number curve, Elastic wave, Cylindrical shell*

I. Introduction

Vibration of cylindrical shells can be regarded as a superposition of disturbances due to elastic waves propagating on the shells. The analysis from a wave viewpoint can give physical insight about how disturbance propagates on vibrating shells. Pierce[1] introduced the basic idea that waves propagating on a point-excited cylindrical shell behave like waves propagating in a two-dimensional unbounded homogeneous anisotropic medium with excitation forces that are periodic in the transverse direction. Pierce and Kil[2] used the idea to analyze the wave propagation on the cylindrical shell. Their numerical results yielded the implication that major features of the thin plate model's prediction could hold relatively well, down to frequencies as low as twice the ring frequency of the cylindrical shell. Means, Kouzoupis and Pierce[3] examined the wave propagation on a cylindrical shell well below the ring frequency including a fluid-loading. But their analysis was performed mainly in a low frequency limit. The present paper examined the dispersion relation which governs wave propagation on the cylindrical shells. The assumption of thin shells allows the dispersion relation to be separated into three relations related to the propagation of flexural waves and two types of membrane waves, Those relations are used to identify the characteristics of the wave number curves. Above the ring frequency three wave number curves are clearly identified to be those of flexural, shear and longitudinal waves, respectively. Below the ring frequency, the characteristics of two wave number curves are

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identified with dependence of the direction of wave propagation.

II. Formulation

2.1. Shell Dynamic Equation

Consider an infinite thin-walled cylindrical shell with nominal radius a and thickness h in Figure 1 (The dispersion of waves is independent of presence of the ends of the shell). The displacement vector **v** is described in terms of variables u, v and w in the axial (z), circumferential (ϕ) and radial (r) directions. In order to describe the displacement field on the shell, the Donnell's thin shell dynamic equations[4] are used. However, the methodology introduced in this paper, for the most part, is independent of the adopted simple shell model.

With the allowance for a radial point-force excitation, the shell dynamic equations take the forms

$$\{l\}\ \mathbf{v} = \{F\} \tag{1}$$

where $\{l\}$ is a linear operator that can be found in Ref.[5]. $\{F\}$ denotes the external force vector applied at $z=0$ on the shell.

The response of the point excited cylindrical shell can be regarded as a superposition of propagating disturbances due to elastic waves which circum-navigate the cylinder. This physical idea allows the response to be taken as a spatial Fourier integral[6]. The axial component of the displacement vector, for example, with the suppressed time dependence $e^{-i\omega t}$,

$$u(\phi, z, w) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} U_n(k_z, w)\, e^{in\phi}\, e^{ik_z z} dk_z \tag{2}$$

where k_z is the axial wavenumber. n corresponds to the circumferential wave number $k_\phi = n/a$. The Fourier transform U_n can be regarded as the complex amplitude of the axial displacement component, which is associated with waves propagating in the direction of the related wave number vector **k** as

$$\mathbf{k} = k_\phi \widetilde{e_\phi} + k_z \widetilde{e_z} = k \sin\theta_k \widetilde{e_\phi} + k\cos\theta_k \widetilde{e_z} \tag{3}$$

where $\theta_k = \tan^{-1}(n/k_z a)$. $\widetilde{e_z}$ and $\widetilde{e_\phi}$ correspond to the unit vector in the corresponding directions.

The algebraic equations governing the Fourier transforms U_n, V_n and W_n are obtained from Eq.(1). Those can be expressed in matrix form as

$$[L]\begin{pmatrix} U_n \\ V_n \\ W_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ Q_n \end{pmatrix} \tag{4}$$

where elements of the matrix $[L]$ take the forms

$$L_{11} = \Omega^2 - k_z a^2 - \frac{1-\nu}{2} n^2,\ \ L_{12} = -\frac{1+\nu}{2} n k_z a\ (= L_{21})$$
$$L_{13} = i\nu k_z a\ \ (= L_{31}),\ \ L_{22} = \Omega^2 - \frac{1-\nu}{2} k_z^2 a^2 - n^2,$$
$$L_{23} = in\ \ (= L_{32}),\ \ L_{33} = 1 + \varepsilon(k_z^2 a^2 + n^2)^2 - \Omega^2 \tag{5}$$

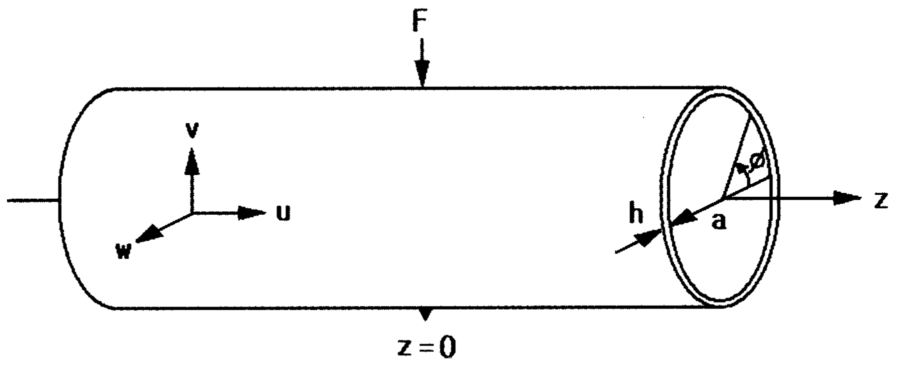

Figure 1. Coordinate system of a cylindrical shell.

Here $\varepsilon = h^2/(12a^2)$ and Q_n corresponds to the Fourier transform of the radial point force $F = F_0\delta(z)\delta(\phi)e^{-i\omega t}$. In Eq.(5) the dimensionless frequency is used as $\Omega = \omega/\omega_r$ where $\omega_r = (1/a)[E/\rho(1-\nu^2)]^{1/2}$ denotes the ring frequency of the cylinder. Here E, ρ and ν correspond to Young's modulus, density and Poisson's ratio, respectively.

For a nontrivial solution of Eq.(4), the determinant of the coefficient matrix must be zero as

$$det([L]) = D(n, k_z, w) = 0 \tag{6}$$

that takes the form of a fourth-order equation in k_z^2. Eq.(6) is called the dispersion relation governing the wave propagation on the shells.

2.2. Dispersion Relation

Naturally occurring plane wave solutions of shell dynamic equations are governed by the dispersion relation $D(n, k_z, w) = 0$ in Eq.(6). Depending on the nature of the roots of the dispersion relation, the solution describes propagating or exponentially decaying waves. The propagating waves include flexural, longitudinal and shear waves.

The thickness of the shell is regarded small compared with the radius of the shell like $h/a \leq 1/20$ (or $\varepsilon = h^2/(12a^2) \leq 2.1\times10^{-4}$) which is generally accepted for thin shell dynamics[7]. Such a small ε allows the function D to be factored to

$$D \approx D_{flex} D_{mem} = D_{flex} D_{mem1} D_{mem2} \tag{7}$$

where

$$D_{flex} = \varepsilon k_0^4 - \Omega^2 + (1-v^2)\cos^4\theta_k \tag{8}$$

$$D_{mem} = \frac{(1-\nu)}{2} D_{mem1} D_{mem2} \tag{9}$$

Here,

$$D_{mem1} = (H_1 - k_o^2)\ ;\ D_{mem2} = (H_2 - k_o^2) \tag{10}$$

where

$$H_1 = \frac{I_4 + (I_8)^{1/2}}{I_2}\ ;\ H_2 = \frac{I_4 - (I_8)^{1/2}}{I_2} \tag{11}$$

$$I_2 = (1-\nu)[(1-\nu^2)\cos^4\theta_k - \Omega^2]\ ,$$

$$I_4 = -\frac{3-v}{2}\Omega^4 + \frac{1-\nu}{2}[1+2(1+\nu)\cos^2\theta_k]\Omega^2$$

$$I_8 = \frac{(1+\nu)^2}{4}\Omega^8 + \frac{(1-\nu^2)}{2}[1-2(3-\nu)\cos\theta_k + 4(1-\nu)\cos^4\theta_k]\Omega^6 + \frac{(1-\nu^2)}{4}[1+4(1+\nu)\cos^2\theta_k - 4(1-\nu^{2)}\cos^4\theta_k]\Omega^4$$

Here $k_o = ka$ represents the dimensionless wavenumber. The factor D_{flex} gives the dispersion relation of flexural waves. The factor D_{mem} is independent of the parameter ε (i.e. thickness h), and $D_{mem} = 0$ gives the dispersion relation of membrane waves. The factor D_{mem} is exactly factored to $D_{mem1} D_{mem2}$ as shown in Eqs.(9)-(11). This factorization is effective in all frequency range. Each of $D_{mem1} = 0$ and $D_{mem2} = 0$ correspond to the dispersion relation of the different types of membrane waves.

In the limit of high frequency, $\Omega \gg 1$, those three factors can be approximated by

$$D_{flex} \approx \varepsilon k_o^4 - \Omega^2 \tag{12}$$

$$D_{mem1} \approx \Omega^2 - k_o^2,\ = D_{long} \tag{13}$$

$$D_{mem2} \approx \frac{2}{1-\nu}(\Omega^2 - \frac{1-\nu}{2}k_o^2),\ = D_{shear} \tag{14}$$

The three factors, when separately equated to zero, correspond to dispersion relations for flexural, longitudinal, and shear waves propagating on thin plates, respectively. Improved approximations that hold down to moderate values of Ω

$$D_{long} = \Omega^2 - k_o^2 - [\sin^2\theta_k + \nu\cos^2\theta_k]^2 \tag{15}$$

$$D_{shear} = \Omega^2 - \frac{1-\nu}{2}k_o^2 - 2(1-v)\sin^2\theta_k\cos^2\theta_k \tag{16}$$

When $\theta_k = 0$, such that the propagation is in the axial direction, D_{mem1} and D_{mem2} Eq.(10) are

$$D_{mem1} = \frac{2}{(1-\nu)}(\Omega^2 - \frac{1-\nu}{2}k_0^2) \tag{17}$$

$$D_{mem2}=\frac{\Omega^2(\Omega^2-1)}{\Omega^2-(1-\nu^2)}-k_0^2 \quad (18)$$

at Ω lower than Ω_m where

$$\Omega_m=\left[\frac{(1-\nu)(1+2\nu)}{(1+\nu)}\right]^{1/2} \quad (19)$$

which corresponds to $\Omega=0.94$ with $\nu=0.283$. But, at Ω higher than Ω_m, those take the opposite forms

$$D_{mem1}=\frac{\Omega^2(\Omega^2-1)}{\Omega^2-(1-\nu^2)}-k_0^2 \quad (20)$$

$$D_{mem2}=\frac{2}{(1-\nu)}(\Omega^2-\frac{1-\nu}{2}k_0^2) \quad (21)$$

Here, Eqs.(17) and (21), when equated to zero, correspond to the dispersion relation of shear waves on plates. Therefore, for Ω lower than Ω_m those relations indicate that D_{mem1} and D_{mem2}, when $\theta_k=0$, are associated with shear and longitudinal waves, respectively, even though they are related to longitudinal and shear waves, respectively, at Ω higher than Ω_m. In low frequency limit ($\Omega \ll 1$), the factor D_{mem2} in Eq.(18) is best approximated by

$$D_{mem2}=\frac{\Omega^2}{(1-\nu^2)}-k_0^2 \quad (22)$$

For the case when the propagation is in the axial direction, it indicates that the phase velocity of a longitudinal wave in the low frequency limit is $(E/\rho)^{1/2}$, which corresponds to the so-called "bar-velocity".

When $\theta_k=\pi/2$, such that the propagation in the circumferential direction, D_{mem1} and D_{mem2} in Eq.(10) take the forms

$$D_{mem1}=(\Omega^2-1)-k_o^2 \quad (23)$$

$$D_{mem2}=\frac{2}{(1-\nu)}\ (\Omega^2-\frac{1-\nu}{2}k_0^2) \quad (24)$$

It indicates that D_{mem1} and D_{mem2}, when $\theta_k=\pi/2$, are associated to longitudinal and shear waves, respectively.

III. Results

The characteristics of wave propagation on the shell are examined by plotting the wave number curve of constant frequency, along which the dispersion relation is satisfied in wave number space. Figures 2 and 3 show the wavenumber curves below and above the ring frequency ($\Omega=1$), respectively. Those are predicted from the dispersion relations such as $D_{mem1}=0$, $D_{mem2}=0$, $D_{flex}=0$ in Eqs.(8)-(11). Those are also compared to the wavenumber curves evaluated from the exact dispersion relation in Eq.(6). In these computations, $\varepsilon=3.3\times10^{-5}$ ($h/a=1/50$) was used. The results show that the approximate factorization gives very good approximations at the dimensionless frequency below and above the ring frequency.

Figures 2(a)-(c) show that the wavenumber curve for flexural waves does not form a closed curve at dimensionless frequencies below a certain frequency, which is determined as $\Omega_f=(1-\nu^2)^{1/2}$ (corresponding to 0.96 for $\nu=0.283$) from Eq.(8). Those figures also show that the flexural waves propagate in the particular region, which is defined as

$$|\theta_f|<|\theta_k|<|\pi-\theta_f| \quad (25)$$

Here, the angle θ_f corresponds to $\theta_f=\cos^{-1}[\Omega^2/(1-\nu^2)]$ determined from Eq.(8).

Figures 2(a)-(c) show that the wave number curve represented by $D_{mem1}=0$ forms two open curves up to Ω_f, which approximately resemble parabolas near k_z axis. When the finite value of ε is taken account, these curves continue to the curve represented by $D_{flex}=0$ at dimensionless frequency below Ω_f. The wave number curve represented by $D_{mem1}=0$ do not exist between Ω_f and $\Omega=1$ as illustrated in Figure 2(d). Above the ring frequency, it forms a closed curve which corresponds to longitudinal waves as shown in Figures 3(a) and (b). However, the wave number curve represented by $D_{mem2}=0$ forms a closed curve in all frequency as shown Figures 2 and 3. The waves governed by $D_{mem2}=0$ have

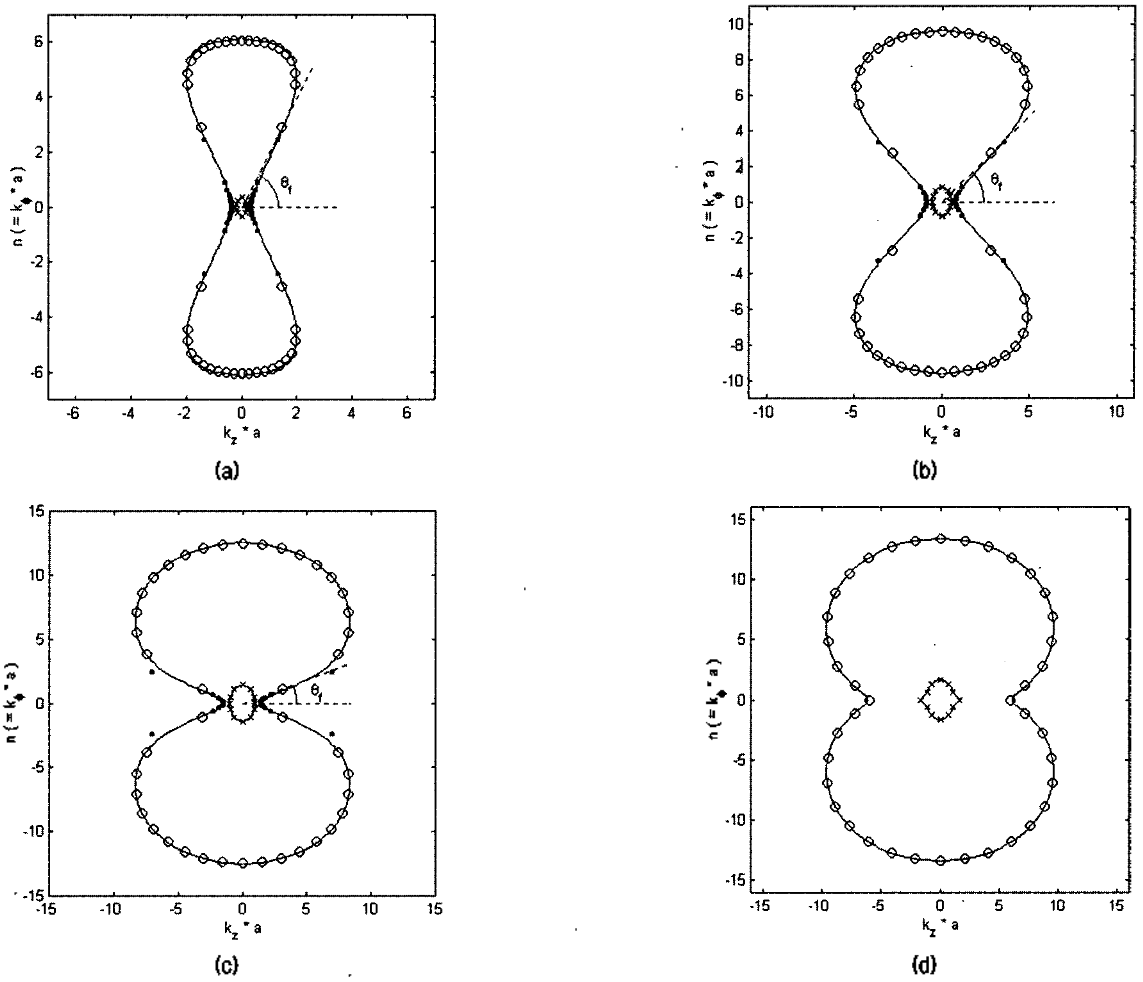

Figure 2. Wave number curves for waves propagating on the cylindrical shell excited at the dimensionless frequencies below the ring frequency ($\Omega=1$) : (a) Ω=0.2, (b) Ω=0.5, (c) Ω=0.85 and (d) Ω=0.98. The curves represented by (• • •), (xxx) and (○ ○ ○) correspond to waves governed by the equations such as $D_{mem1}=0$, $D_{mem2}=0$ and $D_{flex}=0$, respectively. The solid lines represent the wave number curves evaluated from the exact dispersion relation $D=0$.

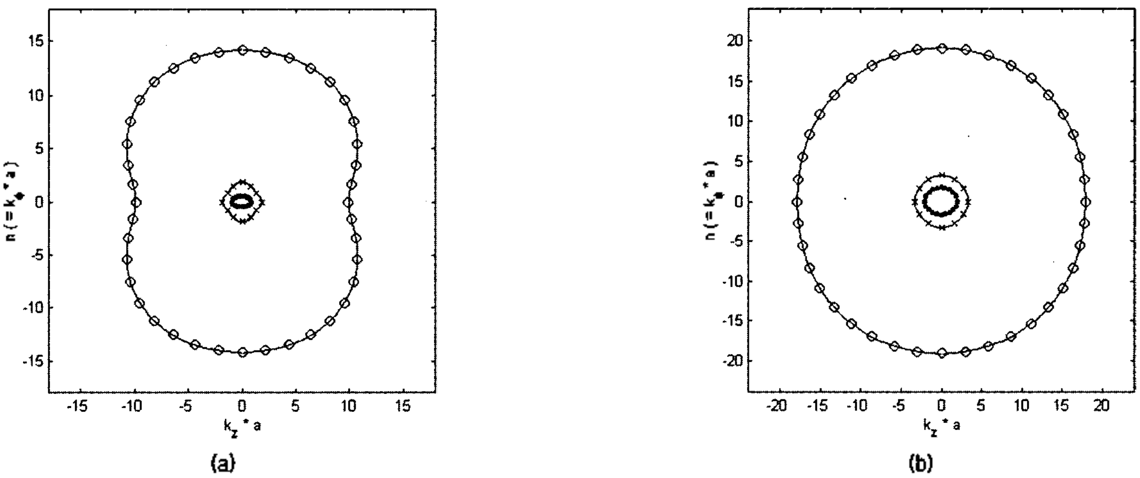

Figure 3. Wave number curves for waves propagating on the cylindrical shell excited at the dimensionless frequencies above the ring frequency ($\Omega=1$) : (a) Ω=1.1 and (b) Ω=2.0. The curves represented by (• • •), (xxx) and (○ ○ ○) correspond to waves governed by the equations such as $D_{mem1}=0$, $D_{mem2}=0$ and $D_{flex}=0$, respectively. The solid lines represent the wave number curves evaluated from the exact dispersion relation $D=0$.

characteristics for propagation near the axial direction: longitudinal waves below the dimensionless frequency Ω_m, and shear waves between Ω_m and Ω_f as shown in Eqs. (18) and (21). Above the ring frequency the waves governed by $D_{mem1}=0$ and $D_{mem2}=0$ correspond to longitudinal and shear waves.

One of characteristics below the ring frequency is a figure-8 shaped wavenumber curve. As shown in Figs. 2(a)-(c), the top and bottom of the figure-8 in the region defined by Eq.(25) is formed by $D_{flex}=0$ which represents flexural waves. The dispersion relation $D_{mem1}=0$ in Eq.(10) for membrane waves forms the center of the figure-8. Those waves have characteristics for propagation near the axial direction: shear waves below the dimensionless frequency Ω_m, and longitudinal waves between Ω_m and Ω_f as shown in Eqs. (17) and (23). In other directions up to the direction defined by θ_f, those are associated with membrane waves whose phase velocities are much smaller than the speeds of either longitudinal or shear waves propagating in the axial direction.

In the point of view for the homogeneous isotropic elastic material, it is expected that waves spreads with energy out from the source point over the shell so that the wavefronts form circles. However, the waves on the shell have the anisotropic nature of propagation up to frequencies somewhat higher than the ring frequency. The figure-8 of the wavenumber curve is one of typical examples which shows the anisotropic nature of propagation. The outward normal direction to the wavenumber curve at any given point gives the group velocity direction, or the direction at which energy flows, for a wave with the corresponding wavenumber and phase velocity direction. Thus, the less circular the wavenumber curve becomes, the more anisotropic the wave propagation on the shell are. The further research about the anisotropic characteristics of waves on the cylindrical shells need to be performed.

IV. Conclusions

The dispersion relation has been analyzed to identify the characteristics of waves propagating on the cylindrical shells. The assumption of thin shells allowed the dispersion relation to be separated into three relations related to the propagation of flexural waves and two types of membrane waves. Those relations have been used to identify the characteristics of the wave number curves. The dispersion relation provided two and three closed wave number curves below and above the ring frequency. Above the ring frequency three wave number curves have been clearly identified to be those of flexural, shear and longitudinal waves, respectively. Below the ring frequency, the characteristics of two wave number curves have been identified with dependence of the direction of wave propagation.

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[Profile]

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