研究論文

A TEST FOR TREND CHANGE IN FAILURE RATE[1)]

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Abstract

The problem of trend change in the failure rate is great interest in the reliability and survival analysis. In this paper we develop a test statistic for testing whether or not the failure rate changes its trend based on a complete sample. Monte Carlo simulations are conducted to investigate the speed of convergence of the proposed test statistic.

1. Introduction

Reliabilists find it useful to categorize life distributions(distributions such that $F(x)=0$ for $x<0$) according to different aging properties. These categories are useful for modeling situations where items improve or deteriorate with age. If F has a density f, the failure rate is defined as

$$r(x)=\frac{f(x)}{\overline{F}(x)},$$

where $\overline{F}(x)=1-F(x)$ is the reliability function.

If $r(x)$ increases monotonically over time, the life distribution is said to have increasing failure rate(IFR). If $r(x)$ decreases monotonically, the life distribution has decreasing failure rate (DFR). If $r(x)$ is constant, the life distribution has constant failure rate (CFR). If $r(x)$ is decreasing on the interval $[0,\tau)$ and increasing on the interval $[\tau,\infty)$, the life distribution F has bathtub-shaped failure rate(BTR), and if $r(s)\le r(t)$ for $0\le s\le t<\tau$ and $r(s)\ge r(t)$ for $\tau\le s\le t<\infty$, the life distribution has upside-down bathtub-

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shaped failure rate(UBR). See Guess and Proschan(1988) and the references therein for further applications of the BTR family.

It is well known that $r(x)$ is constant for all $x \geq 0$ if and only if F is an exponential distribution (i.e., for $x \geq 0$, $\mu > 0$, $F(x) = 1 - \exp(-x/\mu)$). Due to this "no-aging" property of the exponential distribution, it is of practical interest to know

whether a given life distribution F is CFR or BTR. Therefore, we consider the problem of testing

H_0 : F is CFR,

against

H_1 : F is BTR (and not CFR),

based on random samples. When the dual model is proposed, we test H_0 against

H_1' : F is UBR (and not CFR).

The following is an example of situations for which such a test is useful. The defective rates of certain products, such as electrical registers, capacitors, etc, are determined through functional test of the products during the burn-in period. In these situations, the manufacturer wishes to determine whether the product exhibits initial failure until the predetermined proportion of defectives of the total products fail. Then the manufacturer can perform the test for CFR against BTR alternative.

Matthews and Farewell (1982) and Matthews, Farewell and Pyke (1985) considered the problem of testing for a CFR against the alternative with two constant failure rates involving a single change-point. Park (1988) proposed a test for CFR versus BTR(UBR), assuming that the proportion of the population that fails at or before the change-point of failure rate is known. Guess, Hollander and Proschan (1986) considered for testing exponentiality against a trend change in mean residual life when the change-point is known or when the proportion of the population that fails at or before the change-point is known.

In this paper we develop a test statistic for testing exponentiality against BTR (UBR) alternative. We assume that the turning point is known. We derive the asymptotic null distribution of our test statistic. To establish the asymptotic distribution of our test statistic, we used the differential statistical function approach. Monte Carlo simulations are conducted to investigate the speed of convergence of our proposed test statistic.

Section 2 is devoted to develop a test statistic for testing exponentiality against BTR(UBR) alternative. Results of simulations are presented in Section 3.

2. Test for Trend Change in Failure Rate

In this section we propose a test

statistic for testing exponentiality against BTR(UBR) alternative. We assume that the turning point τ is known or has been specified by the user. As a measure of the deviation from the null hypothesis H_0 in favor of H_1, we propose the parameter

$$T(F)=\int_0^{\tau}\int_0^t[r(s)-r(t)]\overline{F}(s)\overline{F}(t)\,dsdt+\int_{\tau}^{\infty}\int_{\tau}^t[r(t)-r(s)]\overline{F}(s)\overline{F}(t)\,dsdt.$$

Note that $T(F)$ is zero for the exponential distribution and strictly positive for the BTR. Using integration by parts, we can rewrite $T(F)$ as

$$T\;(F)=(1+\overline{F}(\tau))\int_0^{\tau}\overline{F}(x)dx-2\int_0^{\tau}\overline{F}^2(x)\,dx-\overline{F}(\tau)\int_{\tau}^{\infty}\overline{F}(x)\,dx+2\int_{\tau}^{\infty}\overline{F}^2(x)\,dx.$$

Let $F_n(x)$ be the empirical distribution formed by a random sample $X_1,\cdots,X_n$ from F and let $X_{(1)}<\cdots<X_{(n)}$ be the order statistics of the sample. Then we can estimate $T(F)$ by

$$T(F_n)=\sum_{i=1}^{i^*}B_1\left(\frac{n-i+1}{n}\right)(X_{(i)}-X_{(i-1)})+B_1\left(\frac{n-i^*}{n}\right)(\tau-X_{(i^*)})+B_2\left(\frac{n-i^*}{n}\right)(X_{(i^*+1)}-\tau)+\sum_{i=i^*+2}^{n}B_2\left(\frac{n-i+1}{n}\right)(X_{(i)}-X_{(i-1)}),$$

where $0=X_{(0)}<X_{(1)}<\cdots<X_{(i^*)}\le\tau<X_{(i^*+1)}<\cdots<X_{(n)}$, $B_1(u)=\left(\frac{2n-i^*}{n}\right)u-2u^2$ and

$$B_2(u)=-\left(\frac{n-i^*}{n}\right)u+2u^2.$$

To establish asymptotic distribution of $T(F_n)$, we use the differentiable statistical function approach of von Mises(1947) (cf. Boos and Serfling(1980) and Serfling(1980)). The differential statistical approximation of $T(F_n)$ is defined as

$$T(F_n)=T(F)+d_1T(F)(F_n-F)+R_n(F)$$

where $d_1T(F)(G-F)$ is the first-order Gateaux differential of functional T at the point F in the direction G, and F and G are life distributions in the domain of $T(\cdot)$. For the BTR functional T, the Gateaux differential

$$d_1T(F)(F_n-F)=\frac{1}{n}\sum_{i=1}^{n}d_1T(F)(\delta_{X_i}-F)=4\int_0^{\tau}\overline{F}(x)D_n(x)dx-(1+\overline{F}_n(\tau))\times\int_0^{\tau}D_n(x)\,dx-D_n(\tau)\int_0^{\tau}\overline{F}(x)\,dx+\overline{F}_n(\tau)\int_{\tau}^{\infty}D_n(x)dx+D_n(\tau)\int_{\tau}^{\infty}\overline{F}(x)\,dx-4\int_{\tau}^{\infty}\overline{F}(x)D_n(x)dx$$

where $D_n(x)=\overline{F}(x)-\overline{F}_n(x)$ and $\delta_{X_i}(x)=0$ if $x<X_i$ and $=1$ if $x\ge X_i$. Our proof of asymptotic normality approximates $T(F_n)-T(F)$ by $d_1T(F)(F_n-F)$ and shows that the term $\sqrt{n}R_n(F)$ converges in probability to 0. Let $\mu(T,F)=E_F[d_1T(F)(\delta_{X_1}-F)]$

and $\sigma^2(T,F) = \mathrm{Var}_F[d_1 T(F)(\delta_{X_1} - F)]$. Then we can obtain the following result.

THEOREM 2.1 Let F be the life distribution such that $0 < F(\tau) < 1$ and $\sigma^2(T,F) < \infty$. Then

$$\sqrt{n}(T(F_n) - T(F)) \xrightarrow{d} N(0, \sigma^2(T,F)).$$

PROOF. Applying the classical Lindberg- Levy central limit theorem, we have

$$\sqrt{n} d_1 T(F)(F_n - F) \xrightarrow{d} N(0, \sigma^2(T,F)).$$

Next we show that $\sqrt{n}R_n(F)$ converges in probability to 0. By straightforward calculation, for the life distribution F we have

$$\begin{aligned} R_n(F) &= T(F_n) - T(F) \\ &\quad - d_1 T(F)(F_n - F) \\ &= -2\int_0^\tau (\overline{F}_n(x) - \overline{F}(x))^2 dx \\ &\quad + 2\int_\tau^\infty (\overline{F}_n(x) - \overline{F}(x))^2 dx. \end{aligned}$$

Thus for any $\tau > 0$ and the life distribution F,

$$\begin{aligned} \sqrt{n}|R_n(F)| \le\ & 4\sqrt{n}\int_0^\infty (\overline{F}_n(x) - \overline{F}(x))^2 dx \\ & \le 4\sqrt{n}\sup|\overline{F}_n(\tau) - \overline{F}(\tau)| \\ & \times \int_0^\infty |\overline{F}_n(x) - \overline{F}(x)| dx. \end{aligned}$$

By the classical weak convergence of the empirical process, $\sqrt{n}R_n(F)$ converges in probability to 0. This completes the proof.

Under H_0, (i.e. F is exponential with mean μ), we have

$$\sqrt{n}T(F_n) \xrightarrow{d} N(0, \sigma^2(T,F))$$

where

$\sigma^2(T,F) = \mu^2(\frac{1}{3} - F(\tau) + F^2(\tau))$. Since $\sigma_n^2 \equiv \sigma^2(T, F_n)$ is a consistent estimator of $\sigma^2(T,F)$,

$$T_n^* \equiv \sqrt{n}T(F_n)/\sigma_n \xrightarrow{d} N(0,1).$$

The BTR test procedure rejects H_0 in favor of H_1 at the approximation level α if $T_n^* \ge z_\alpha$, where z_α is the upper α -quantile of standard normal distribution. Analogously, the UBR test rejects H_0 in favor of H_1' at the approximation level α if $T_n^* \le -z_\alpha$.

3. Simulation Study

In this section we perform a Monte Carlo simulation to investigate the speed of convergence of the proposed test statistic, for various τ and n. For Monte Carlo study we use the subroutine IMSL of the package FORTRAN.

To investigate the empirical test size, random numbers are generated from exponential distribution,

$F(x) = 1 - \exp(-x), x \ge 0$, since our test statistics are scale invariant.

Table 3.1 presents the empirical test size of BTR test based on T_n^*. The values in

<Table 3.1> Empirical test size of BTR test based on T_n^*

n	α	$F(\tau)$				
		0.1	0.3	0.5	0.7	0.9
10	.10	.239	.163	.095	.031	.030
	.05	.130	.105	.046	.013	.012
	.01	.028	.026	.006	.001	.001
20	.10	.194	.138	.087	.051	.044
	.05	.101	.086	.042	.023	.013
	.01	.025	.018	.007	.002	.002
30	.10	.164	.140	.104	.063	.053
	.05	.098	.073	.045	.027	.021
	.01	.016	.013	.013	.006	.003
40	.10	.151	.119	.087	.064	.055
	.05	.088	.058	.049	.030	.024
	.01	.014	.013	.013	.005	.004
60	.10	.135	.134	.097	.069	.064
	.05	.078	.068	.039	.031	.032
	.01	.016	.011	.007	.010	.005
80	.10	.120	.118	.086	.072	.056
	.05	.066	.060	.045	.038	.031
	.01	.015	.012	.011	.009	.004
100	.10	.123	.114	.096	.080	.064
	.05	.056	.061	.055	.035	.031
	.01	.011	.013	.011	.009	.005

Table are the fraction of times that H_0 is rejected in favor of H_1 when H_0 is true. The empirical test sizes are calculated based on 1000 replications for; α=0.10, 0.05, 0.01; $n = 10, 20, \cdots, 100$; $F(\tau) = 0.1, 0.3, \cdots, 0.9$.

From Table 3.1, we notice that the convergence to normality is somewhat slow for $F(\tau)=0.1$, is faster for $F(\tau)=$ 0.5, 0.7 and 0.9, and the size of the BTR test is close to the level of significance for $F(\tau)=0.1$ when $n \geq 80$, $F(\tau)=0.3$ when $n \geq 40$ and $F(\tau)=0.5$ when $n \geq 10$.

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