〈논 문〉

Vibration Analysis for Beams on Variable Two-Parameter Elastic Foundations Using Differetial Transformation

Differential Transformation에 의한 가변 2 파라미터 탄성기초에 설치된 보의 진동해석

Young-Jae Shin†, Chae-Ho Kim* and Jeong-Ki Hwang*
신 영 재 · 김 재 호 · 황 정 기

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ABSTRACT

This paper presents the application of the technique of differential transformation to the vibration analysis of beams resting on variable two-parameter elastic foundations. The closed form series solutions for beams are obtained for various boundary conditions. Numerical calculations are carried out and compared with previously published results. The results obtained by the present method agree very well with those reported in the previous works. The present analysis shows the usefulness and validity of differential transformation in solving nonlinear problem of the free vibration.

1. INTRODUCTION

Beams resting on an elastic foundation are common in engineering. The Winkler elastic foundation model is one of the simplest one-parameter models, which consist of an infinite number of closely spaced springs uniformly distributed along the beam. To overcome the limitation of this model which disregards the interaction between springs, several two-parameter models have been suggested.[1~4] All these are equivalent mathematical models. The only difference is the definition of the foundation parameters.

Since the coefficients of the corresponding differential equation are constants when these parameters are constant along the length of the beam, the solution of the differential equation can be obtained easily. If the foundation parameters vary along the beam, the differential equation in most of the cases cannot be solved exactly, and numerical techniques should be applied.

Dynamics and stability behavior of the Winkler foundation model have been thoroughly investigated by both approximate methods[3] and exact approaches.[6, 7] The same beam on constant two-parameter foundation has been analyzed in an exact way by Valsangkar et al.[8] and the corresponding Timoshenko beam has been studied

† Correesponding Auther Member, School of Mechanical Engineering, Andong National University
E-mail : yjshin@andong.ac.kr
Tel : (054) 820-5435, Fax : (054) 823-5495
* Member, School of Mechanical Engi- neering, Andong National University

by Rosa.[9] Eisenberger[10] presented the closed form series solution for the vibration of the beam on variable two-parameter elastic foundations by using the infinite power series.

In this study, a transformation technique called differential transformation method is applied to analyze the free vibration of beams on variable elastic foundations. The concept of this transformation was first proposed by Zhou[11] in 1986 and was applied to solve linear and nonlinear initial value problems in electric circuit analysis. Recently this method seems to attract researcher's interest in solving eigenvalue problems.[12~15] In this paper, the free vibration analysis for beams resting on variable two-parameter elastic foundation is considered. Using differential transformation method, the closed form series solutions for the beam are obtained for various boundary conditions. Numerical comparisons are carried out and compared with previously published results and the accuracy and efficiency of the present method are discussed.

2. Governing Equation of Motion

The partial differential equation of motion governing the bending vibration of beams on variable two-parameter elastic foundation is given as follows:

$$EI\frac{\partial^4 z}{\partial x^4}-\frac{\partial}{\partial x}\left[\overline{K_1}(x)\frac{\partial z}{\partial x}\right]+\overline{K}(x)z=-\rho A\frac{\partial^2 z}{\partial x^2} \qquad (1)$$

Where EI is the flexural rigidity of the beam, $z(x, t)$ is the lateral displacement, $\overline{K}(x)$ is the Winkler foundation parameter along the beam, $\overline{K_1}(x)$ is the second foundation parameter along the beam, ρ is the material density of the beam, and A is the cross-sectional area. The right term of equation (1) represents the inertia forces during vibrations.

Assuming a steady-state solution

$$z(x,t)=w(x)e^{i\omega t} \qquad (2)$$

Substituting equation (2) into equation (1), we can obtain the ordinary differential equation of mode shape as follows

$$EI\frac{d^4 w}{dx^4}-\frac{d}{dx}\left[\overline{K_1}(x)\frac{dw}{dx}\right]+\overline{K}(x)w-\rho A\omega^2 w=0 \qquad (3)$$

The coefficients in equation (3) are treated to have the following polynomial variation along the beam:

$$\overline{K}(x)=\sum_{j=0}^{p}k_j x^j \qquad (4)$$

$$\overline{K_1}(x)=\sum_{j=0}^{q}k_{1,j} x^j \qquad (5)$$

Here p and q are integers representing the number of terms in each series. Using this representation, the coefficient functions can be represented, exactly or up to any desired accuracy.

In order to solve the equation (3), two boundary conditions are needed for both ends of $x=0$ and $x=L$. The boundary conditions considered here are as follows:

Simply supported end (S):

$$w=0 \quad \text{and} \quad \frac{d^2 w}{dx^2}=0 \qquad (6)$$

Clamped end (C):

$$w=0 \quad \text{and} \quad \frac{dw}{dx}=0 \qquad (7)$$

Free end (F):

$$\frac{d^2 w}{dx^2}=0 \quad \text{and} \quad \frac{d^3 w}{dx^3}=0 \qquad (8)$$

3. Differential Transformation

Let $y(x)$ be an analytic in domain D and $x=0$ be a point in D. Then there exists precisely one power series with center at $x=0$ which represents

$y(x)$: this series, the Maclaurin series of the function $y(x)$, is of the form

$$y(x)=\sum_{k=0}^{\infty}\frac{x^k}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=0} \quad \text{for} \quad \forall x \subset D \tag{9}$$

If we define differential transformation of function $y(x)$ as follows

$$Y(k)=\frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=0} \tag{10}$$

and substitute equation (10) into equation (9), equation (9) becomes

$$y(x)=\sum_{k=0}^{\infty}x^k Y(k) \tag{11}$$

$Y(k)$ is the differential transformation(T-function) for the original function $y(x)$, and equation (11) is the differential inverse transformation of $Y(x)$.[12, 13]

From the above definition of the differential transformation of a function, we can derive the rules of transformational operations: some of these, which are useful in the following analysis, are as follows:

Original function	T-function
$w(x)=y(x)\pm z(x)$	$W(k)=Y(k)\pm Z(k)$
$z(x)=\lambda y(x)$	$Z(k)=\lambda Y(k)$
$z(x)=\frac{dy(x)}{dx}$	$Z(k)=(k+1)Y(k+1)$
$w(x)=\frac{d^2y(x)}{dx^2}$	$W(k)=(k+1)(k+2)Y(k+2)$
$w(x)=y(x)z(x)$	$W(k)=\sum_{l=0}^{k}Y(l)Z(k-l)$
$w(x)=x^m y(x)$	$W(k)=\begin{matrix} Y(k-m) & k-m\ge 0 \\ 0 & k-m<0\end{matrix}$

In actual applications, the function $y(x)$ may be expressed by a finite series and equation (11) can be written as

$$y(x)=\sum_{k=0}^{n}x^k Y(k) \tag{12}$$

Equation (12) implies that $\sum_{k=n+1}^{\infty}x^k Y(k)$ is neglected. Generally, n is decided by the desired convergence of the natural frequency.

4. Application of Differential Transformation to Beams on Variable Two-Parameter Foundations

Taking differential transformation of equation (3) and using the transformational operations mentioned above, we obtain

$$\begin{aligned}&EI(k+1)(k+2)(k+3)(k+4)W(k+4)\\&-\sum_{l=0}^{k}(l+1)(k-l+1)K_1(l+1)W(k-l+1)\\&-\sum_{l=0}^{k}(k-l+1)(k-l+2)K_1(l)W(k-l+2)\\&+\sum_{l=0}^{k}K(l)W(k-l)-\omega^2\rho AW(k)=0\end{aligned}$$

$$\text{for } k=0, 1, \cdots\cdots, n\text{-}4 \tag{13}$$

Where $W(k)$, $K_1(k)$, and $K(k)$ are T-functions of $w(x)$, $\overline{K}_1(x)$, and $\overline{K}(x)$, respectively.

The transformed boundary condition equations at each end should be obtained by differential transformation method as follows

At the end $x=0$

Simply supported end (S):

$$W(0)=0 \quad \text{and} \quad W(2)=0 \tag{14}$$

Clamped end (C):

$$W(0)=0 \quad \text{and} \quad W(1)=0 \tag{15}$$

Free end (F):

$$W(2)=0 \quad \text{and} \quad W(3)=0 \tag{16}$$

At the end $x=L$

Simply supported end (S):

$$\sum_{k=0}^{n}W(k)=0 \quad \text{and} \quad \sum_{k=0}^{n}k(k-1)W(k)=0 \tag{17}$$

Clamped end (C):

$$\sum_{k=0}^{n} W(k)=0 \quad \text{and} \quad \sum_{k=0}^{n} kW(k)=0 \tag{18}$$

Free end (F):

$$\sum_{k=0}^{n} k(k-1)W(k)=0 \quad \text{and}$$

$$\sum_{k=0}^{n} k(k-1)(k-2)W(k)=0 \tag{19}$$

5. Numerical Analysis and Discussions

In order to obtain the natural frequencies of beams resting on two-parameter elastic foundations, we should take advantage of the transformed equation (13) and the four corresponding boundary condition equations (14~19). These equations can be arranged in matrix form as

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & a_{1,n+1} \\ a_{2,1} & a_{2,2} & & a_{2,n} & a_{2,n+1} \\ \cdot & \cdot & & \cdot & \cdot \\ a_{n+1,1} & a_{n+1,2} & & a_{n+1,n} & a_{n+1,n+1} \end{bmatrix} \begin{bmatrix} W(0) \\ W(1) \\ \cdot \\ W(n) \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \tag{20}$$

A non-trivial solution exists when the determinant of the coefficient matrix vanishes. This condition leads to the following frequency equation:

$$\begin{vmatrix} a_{1,1} & a_{1,2} & \cdots\cdots & \cdots & a_{1,n} & a_{1,n+1} \\ a_{2,1} & a_{2,2} & \cdots\cdots & \cdots & a_{2,n} & a_{2,n+1} \\ \cdot & \cdot & \cdots\cdots & \cdots & \cdot & \cdot \\ a_{n+1,1} & a_{n+1,2} & \cdots\cdots & \cdots & a_{n+1,n} & a_{n+1,n+1} \end{vmatrix} = 0 \tag{21}$$

The solution of equation (21) yields the desired frequency parameter $\omega_i^{(n)}$. The number of the available frequency parameter obtained is not more than $n+1$ since the equation (21) could have the imaginary roots. $\omega_i^{(n)}$ is the i-th estimated natural frequency corresponding to n, with n being decided by the following desired convergence (accuracy) of the frequency parameter.

$$\left|\omega_i^{(n)} - \omega_i^{(n-1)}\right| \le \varepsilon \tag{22}$$

where $\omega_i^{(n-1)}$ is the i-th estimated natural frequency corresponding to $n-1$ and ε is a desired tolerance (accuracy).

Substituting $\omega=\omega_i^{(n)}$ into equation (20), solving $W(0), W(1), W(2), \ldots W(n)$ and substituting these into equation (11), we obtain the i-th mode shape.

For the first example, the free vibration of a cantilevered beam on variable one-parameter (Winkler) elastic foundation that was analyzed in reference (10) is considered.

Since this beam has the boundary conditions of free-clamped ends, the transformed boundary condition equations become equations (16) and (18).

The beam properties are as follows

$$EI = 1.5\times10^5\, Nm^2,$$
$$\overline{K}(x) = (4x - 3x^2 + x^3)\times10^5\, N/m,$$
$$\rho A = 1.5\times10^3 \text{ kg/m} \text{ and } L = 3 \text{ m}$$

In Table 1, the first three natural frequencies of the beam obtained by the present method are given and the convergence of differential transformation solution is presented. A glance at Table 1 reveals that the calculated natural frequencies are very much the same as those obtained by Eisenberger.[10] The convergence of the first three natural frequencies are shown in Fig. 1. Here, the first three natural frequencies converge to 10.00796, 27.63314 and 70.06638 respectively.

As the second example, the free vibration of the cantilevered beam on variable two-parameter elastic foundations that was analyzed in reference (10) is considered.

The beam properties considered here are , $EI=500$ kNm^2, $\overline{K}_1(x) = (6x - x^2)\times10^2$ kN, $\overline{K}(x) = (3x - x^2/2)\times$ N/m, $\rho A = 1.0\times10^3$ kg/m and $L=3$ m.

Table 1 The first three natural frequencies of a beam on variable one-parameter elastic foundations.

Natural frequency	Present method						Eisenberger's result
1st	Series size (n)						
	14	18	22	26	30	34	
	9.39758	9.68149	9.99334	10.00694	10.00791	10.00796	10.00796
2nd	Series size (n)						
	17	21	25	29	33	37	
	22.48810	27.12761	27.60154	27.63148	27.63310	27.63314	27.63314
3rd	Series size (n)						
	20	24	28	32	36	40	
	57.50623	68.05435	69.92909	70.06102	70.06624	70.06638	70.06638

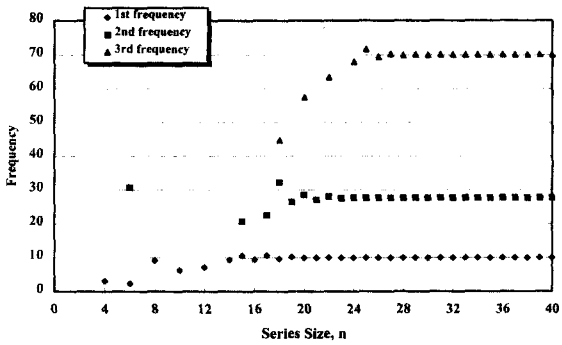

Fig. 1 The convergence of differential transformation solution for a cantilevered beam on variable one-parameter foundation in reference (10)

In Table 2, the first three natural frequencies of the beam obtained by the present method are given and the convergence of differential transformation solution is presented. Table 2 also shows that the agreement between the calculated natural frequencies and the results obtained by Eisenberger (10) is very good. The convergence of the first three natural frequencies are shown in Fig. 2. Here, the first three natural frequencies converge to 40.38242, 83.76883 and 173.07397 respectively.

Table 2 The first three natural frequencies of a beam on variable two-parameter elastic foundations.

Natural frequency	Present method						Eisenberger's result
1st	Series size (n)						
	13	17	21	25	29	33	
	35.93595	41.21221	40.40688	40.38321	40.38265	40.38242	40.38242
2nd	Series size (n)						
	16	20	24	28	32	36	
	99.45347	84.53887	83.59031	83.77716	83.76869	83.76883	83.76883
3rd	Series size (n)						
	18	22	26	30	34	38	
	151.62021	180.79170	173.05139	173.06339	173.07431	173.07397	173.07397

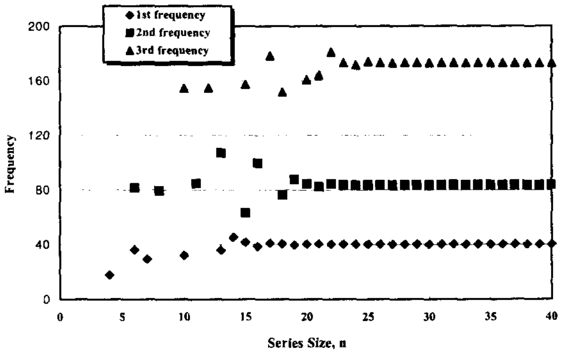

Fig. 2 The convergence of differential transformation solution for a cantilevered beam on variable two-parameter foundation in reference (10)

To extend the application of the present method to general vibration analysis, the free vibrations of beams on variable two-parameter elastic foundations were analyzed for the various boundary conditions of simply supported(S), clamped(C) and free(F) ends of beams.

The free vibration analysis of beams on variable two-parameter elastic foundations ($\bar{K}_1(x) = (1+6x-x^2)\times 10^2\, kN$, $\bar{K}(x) = (1+3x-x^2/2)\times 10^3\, N/m$) were performed. The results obtained are presented in Table 3. The beam properties are the same as the above second vibration example.

The free vibration analysis of beams on the elastic foundations of $\bar{K}_1(x) = (2+6x-x^2)\times 10^2\, kN$ and $\bar{K}(x) = (2+3x-x^2/2)\times 10^3\ N/m$ were examined. The calculated results are shown in Table 4.

5. Conclusions

In this paper, the free vibration analysis of beams resting on variable two-parameter elastic foundations is performed by using differential transformation. The steps processed in this study are, deriving the governing partial differential equations, getting an ordinary differential equations with variable coefficients, transforming the ordinary differential equations into algebraic equations, solving the algebraic equations, and finally inverting the solution of the equations to obtain natural frequency.

The results obtained by the present method are presented for various boundary conditions and

Table 3 The first three natural frequencies of beams on variable two-parameter elastic foundations of $\bar{k}_1(x) = (1+6x-x^2)\times 10^2\ kN$ and $\bar{k}(x) = (1+3x-x^2/2)\times 10^3\ kN/m$

Natural frequency	Boundary conditions of the beam					
	S-S	C-S	C-C	F-S	F-C	F-F
1st	73.48655	83.07517	90.94202	50.95693	51.00052	50.83447
2nd	129.35662	152.52305	176.88876	82.50252	90.41046	72.05633
3rd	244.11446	281.10782	319.69212	152.94236	177.38891	91.41607

Table 4 The first three natural frequencies of beams on variable two-parameter elastic foundations of $\bar{k}_1(x) = (1+6x-x^2)\times 10^2\ kN$ and $\bar{k}(x) = (1+3x-x^2/2)\times 10^3\ kN/m$

Natural frequency	Boundary conditions of the beam					
	S-S	C-S	C-C	F-S	F-C	F-F
1st	80.70996	89.57061	96.97978	59.73714	59.77627	59.64606
2nd	134.79251	157.28323	181.10050	89.14084	96.58646	78.79165
3rd	248.14948	284.72076	322.95552	157.70963	181.60017	97.44573

agree very well with those reported in the previous works. The present analysis shows the usefulness and validity of differential transformation in solving nonlinear problem arising in the vibration problem.

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