AN APPROXIMATE ANALYTICAL SOLUTION OF A NONLINEAR HYDRO-THERMO COUPLED DIFFUSION EQUATION

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Abstract: An approximate analytical solution of a nonlinear hydro-thermo coupled diffusion equation is derived using the dimensionless form of the equation and transformation method. To derive an analytical solution, it is drastically assumed that the product of first order derivatives in the non-dimensionalized governing equation has little influence on the solution of heat and moisture behavior problem. The validity of this drastic assumption is demonstrated. Some numerical simulation is performed to investigate the applicability of a derived approximate analytical solution. The results show a good agreement between analytical and numerical solutions. The proposed solution may provide a useful tool in the verification process of the numerical models. Also, the solution can be used for the analysis of one-dimensional coupled heat and moisture movements in unsaturated porous media.

Key Words: approximate analytical solution, hydro-thermo coupled diffusion equation

1. INTRODUCTION

The process of simultaneous heat and moisture transfer in porous materials is of interest in a variety of engineering applications. Typical applications include the disposals of high level nuclear waste and heat emitting industrial buried materials, underground energy storage system, buried electric cables, thermal soil remediation, etc. In all these applications, thermal and hydraulic gradients are thought to be important factors that should be evaluated because both are the driving forces that influence the process of heat and moisture movements in unsaturated medium. Heat sources can induce a high temperature gradients and lead to significant moisture movement. For example, excessive moisture movement can cause localized moisture content depletion, which will give rise to shrinkage cracks of the geologic material. Thus, better understanding of the coupled heat and moisture process is required. For this reason, there are a lot of efforts to examine the coupled heat and mass flows in unsaturated porous medium (Philip and de Vries, 1957; Cassel et al., 1969; Yong et al., 1990, etc.). In particular, Philip and de Vries (1957) formulated a nonlinear hydro-thermo coupled diffusion equation to govern the moisture movement and heat transfer. Based on this diffusion equation, much research has been extensively done. Dempsey (1978) has applied the Philip and de Vries (1957) model in conjunction with a one-dimensional implicit finite difference method to study the coupled heat transfer and moisture movement in an unsaturated soil. A two-dimensional integrated finite-difference method which incorporates the Philip and de Vries (1957) model is presented by Radhakrishna et al. (1984). A number of investigators have applied finite element techniques to two-dimensional problem (Abdel-Hadi and and Mitchell, 1981; Geraminegad and Saxena, 1986; Thomas, 1987). However, most studies are concentrated on numerical techniques and experiments for the analysis of heat and moisture movement problem. Little previous efforts have been made to develop an analytical approach for solving a nonlinear hydro-thermo coupled diffusion equation. Despite the availability of various numerical methods for solving such a diffusion equation, analytical methods should be pursued for computational efficiency and verification of the numerical solutions. For example, numerical methods have the disadvantage of consuming very long simulation time when the variations of temperature and moisture profiles have to be investigated for over a few hundreds years. It doesn't matter to analytical methods.

Recently, Basha and Selvadurai (1998) developed an analytical solution to the problem of heat-induced moisture movement in the vicinity of a spherical heat source, but diffusion parameters are assumed to be constant without considering the variability of those parameters with the changes of temperature and moisture. This is the primary motivation for the present study. So, an attempt is made to derive an approximate analytical solution of a nonlinear hydro-thermo coupled diffusion equation with variable diffusion parameters. To this end, we make the dimensionless form of diffusion equation proposed by Philip and de Vries (1957). It is drastically assumed that the product of first order derivatives in a dimensionless governing equation has little influence on the solution of heat and moisture behavior problem. The validity of this drastic assumption is demonstrated. Some numerical simulation is carried out to investigate the applicability of a derived approximate analytical solution. In this study, we only focus on one-dimensional horizontal problem. Even though one-dimensional analysis is not sufficient to represent regional problem related to heat-moisture movement, it may be important in case of local problem. For example, in a laboratory test, one-dimensional analysis is needed to estimate diffusion parameters governing heat and moisture movement.

The proposed analytical solution may provide a useful tool in the verification process of the numerical models. Also, the solution can be used for the analysis of one-dimensional coupled heat and moisture movements in local unsaturated porous media.

2. DERIVATION OF ANALYTICAL SOLUTION

2.1 Governing equation

Philip and de Vries (1957) presented a theory to describe the process of heat transfer and moisture transport in a non-deforming porous medium. Our study is based on this theory. In this theory, the coupled heat and moisture flows which are governed by Fourier's law and Darcy's law, respectively are described by

$$\frac{\partial \theta}{\partial t} = \nabla \cdot (D_T \nabla T) + \nabla \cdot (D_\theta \nabla \theta) + \frac{\partial K_\theta}{\partial z} \tag{1a}$$

$$\rho c \frac{\partial T}{\partial t} = \nabla \cdot (\lambda \nabla T) + \rho L_v \nabla \cdot (D_{\theta v} \nabla \theta) \tag{1b}$$

where θ is the volumetric water content, T is the temperature, t is the time, D_T is the thermal moisture diffusivity[L^2/T], D_θ is the moisture diffusivity[L^2/T], K_θ is the unsaturated hydraulic conductivity[L/T], c is the specific heat[L^2/T^2], λ is the thermal conductivity[ML/T^3], ρ is the density[M/T^3], L_v is the latent heat vaporization[L^2/T^2], and $D_{\theta v}$ is the isothermal vapor diffusivity[L^2/T]. Eqs. (1a) and (1b) take the forms of nonlinear partial differential equations because diffusion parameters such as D_T, D_θ, λ are function of θ and T .To obtain the solution of a coupled nonlinear equation, a considerable mathematical effort would be required. Therefore it is desirable to examine a simplified model. It has been shown that the gravity effects of third term of RHS in Eq. (1a) and the vaporization effects of second term of RHS in Eq. (1b) have little influence on the process of heat and moisture, especially for the cases involving localized high temperature (Basha and Selvadurai, 1998). We focus on a one-dimensional horizontal problem. The coupled system of equations then is reduced to

$$\frac{\partial \theta}{\partial t} = \nabla \cdot (D_T \nabla T) + \nabla \cdot (D_\theta \nabla \theta) \tag{2a}$$

$$\rho c \frac{\partial T}{\partial t} = \nabla \cdot (\lambda \nabla T) \tag{2b}$$

By using the Eqs. (2a) and (2b), an approximate analytical solution is derived for one-dimensional movements of moisture and temperature.

2.2 Approximate analytical solution

Consider an unsaturated porous bar with length L, confined at two ends A and B. This is shown in Fig. 1. End A has θ_1, T_1(℃) and end B has θ_2, T_2(℃). Under this system, simplified dimensionless forms of one-dimensional equations of (2a) and (2b) can be written as

$$\frac{\partial \theta^*}{\partial \tau} = D_\theta^* \frac{\partial^2 \theta^*}{\partial \zeta^2} + D_T^* \frac{\partial^2 T^*}{\partial \zeta^2} \tag{3a}$$

$$\frac{\partial T^*}{\partial \tau} = \frac{\partial^2 T^*}{\partial \zeta^2} \tag{3b}$$

where $\theta^* = \dfrac{\theta - \theta_2}{\theta_1 - \theta_2}$, $T^* = \dfrac{T - T_2}{T_1 - T_2}$, $\tau = \dfrac{\lambda}{\rho c}\dfrac{t}{L^2}$, $\zeta = \dfrac{x}{L}$, $\lambda^* = \dfrac{\lambda}{\rho c}$, $D_\theta^* = \dfrac{D_\theta}{\lambda^*}$, $D_T^* = \dfrac{T_1 - T_2}{\theta_1 - \theta_2}\dfrac{D_T}{\lambda^*}$. When we derive the Eqs. (3a) and (3b), a drastic assumption is made. The validity of the assumption will be discussed in the following chapter.

The initial and boundary conditions are

$$T(0, t) = T_1 \quad \text{at } x = 0$$
$$T(L, t) = T_2 \quad \text{at } x = L \tag{4a}$$
$$\theta(0, t) = \theta_1 \quad \text{at } x = 0$$
$$\theta(L, t) = \theta_2 \quad \text{at } x = L \tag{4b}$$
$$\theta(x, 0) = \theta_0 \quad \text{at } t = 0$$
$$T(x, 0) = T_0 \quad \text{at } t = 0 \tag{4c}$$

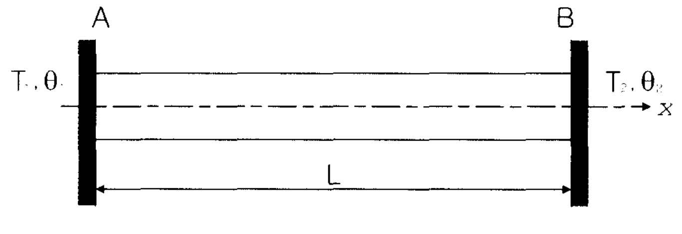


Fig. 1. Schematic diagram of an unsaturated porous bar system

where subscript 0 means the initial value. For the analytical solutions of Eqs. (3a) and (3b), integral transforms are used. The space derivatives may be replaced by finite Fourier sine transform and time derivatives by Laplace transform. After applying these transformations and the required procedures, the following solutions can be obtained.

$$T^*(\zeta,\tau) = (1-\zeta) + \sum_{n=0}^{\infty} \frac{2}{n\pi}\left[T_0^*(1-(-1)^n) - 1\right] e^{-n^2\pi^2\tau} \sin(n\pi\zeta) \tag{5a}$$

$$\theta^*(\zeta,\tau) = (1-\zeta) + \sum_{n=0}^{\infty} \frac{2}{n\pi}\left[\frac{D_T^* - T_0^* D_T^*\left(1-(-1)^n\right)}{1-D_\theta^*}\left(e^{-n^2\pi^2 D_\theta^*\tau} - e^{-n^2\pi^2\tau}\right) - e^{-n^2\pi^2 D_\theta^*\tau} + \theta_0^*\left(1-(-1)^n\right) e^{-n^2\pi^2 D_\theta^*\tau}\right]\sin(n\pi\zeta) \tag{5b}$$

where n is the Fourier transform variable, θ_0^* is the non-dimensional initial volumetric water content, and T_0^* is the non-dimensional initial temperature. The detailed solution procedure is given in the Appendix. Equations of (5a) and (5b) have implicit forms because both sides are function of θ and T. Thus the solutions of these equations can be obtained by successive iterations with the help of Newton-Raphson technique.

3. VERIFICATION

3.1 Numerical model

A numerical model is established in order to verify an approximate analytical solution. Finite difference method is used to solve Eqs. (2a) and (2b) numerically. Temporal discretization of Eqs. (2a) and (2b) using a backward Euler method may be written as

$$\frac{\theta^{n+1} - \theta^n}{\Delta t} - \nabla\cdot(D_T^{n+1}\nabla T^{n+1}) - \nabla\cdot(D_\theta^{n+1}\nabla\theta^{n+1}) = 0 \tag{6a}$$

$$\rho c\frac{T^{n+1} - T^n}{\Delta t} - \nabla\cdot(\lambda^{n+1}\nabla T^{n+1}) = 0 \tag{6b}$$

where θ^n and T^n denote the approximate values at the nth discrete time level ($t = t^n$), $\Delta t = t^{n+1} - t^n$ is the time step, D_T^{n+1}, D_θ^{n+1}, and λ^{n+1} identify thermal moisture diffusivity, moisture diffusivity, and thermal conductivity evaluated using θ^{n+1} and T^{n+1}, respectively. D_T and D_θ are functions of θ and T, therefore Eqs. (6a) and (6b) are linearized as Eqs. (7a) and (7b) using the Picard iteration scheme. The latest estimates of D_T^{n+1}, D_θ^{n+1}, and λ^{n+1} are used to estimate the unknown values of θ^{n+1} and T^{n+1}.

$$\frac{\theta^{n+1,m+1} - \theta^n}{\Delta t} - \nabla\cdot(D_T^{n+1,m}\nabla T^{n+1,m+1}) - \nabla\cdot(D_\theta^{n+1,m}\nabla\theta^{n+1,m+1}) = 0 \tag{7a}$$

$$\rho c\frac{T^{n+1,m+1} - T^n}{\Delta t} - \nabla\cdot(\lambda^{n+1,m}\nabla T^{n+1,m+1}) = 0 \tag{7b}$$

where m is the iteration level. These equations may be rewritten in the following equivalent forms,

$$\frac{\theta^{n+1,m+1} - \theta^{n+1,m}}{\Delta t} - \nabla\cdot D_T^{n+1,m}(\nabla T^{n+1,m+1} - \nabla T^{n+1,m}) - \nabla\cdot D_\theta^{n+1,m}(\nabla\theta^{n+1,m+1} - \nabla\theta^{n+1,m}) = -\frac{\theta^{n+1,m} - \theta^n}{\Delta t} + \nabla\cdot(D_T^{n+1,m}\nabla T^{n+1,m}) + \nabla\cdot(D_\theta^{n+1,m}\nabla\theta^{n+1,m}) \equiv R_\theta^{n+1,m} \tag{8a}$$

$$\rho c \frac{T^{n+1,\,m+1} - T^{n+1,\,m}}{\Delta t} - \nabla \cdot \lambda^{n+1,\,m} (\nabla T^{n+1,\,m+1} - \nabla T^{n+1,\,m})$$
$$= -\rho c \frac{T^{n+1,\,m} - T^{n}}{\Delta t} - \nabla \cdot (\lambda^{n+1,\,m} \nabla T^{n+1,\,m}) \equiv R_T^{n+1,\,m} \tag{8b}$$

where $R_\theta^{n+1,\,m}$ and $R_T^{n+1,\,m}$ are the residuals of the discretized equations. The solutions of the Eqs. (8a) and (8b) can be obtained by iteration until both residuals and the differences ($\theta^{n+1,\,m+1} - \theta^{n+1,\,m}$, $T^{n+1,\,m+1} - T^{n+1,\,m}$) go to zero.

3.2 Comparison of analytical and numerical solutions

Some simulation of one-dimensional moisture and temperature movements is carried out analytically and numerically. Then, analytical and numerical computations are compared. The applicability of an approximated analytical solution is also discussed.

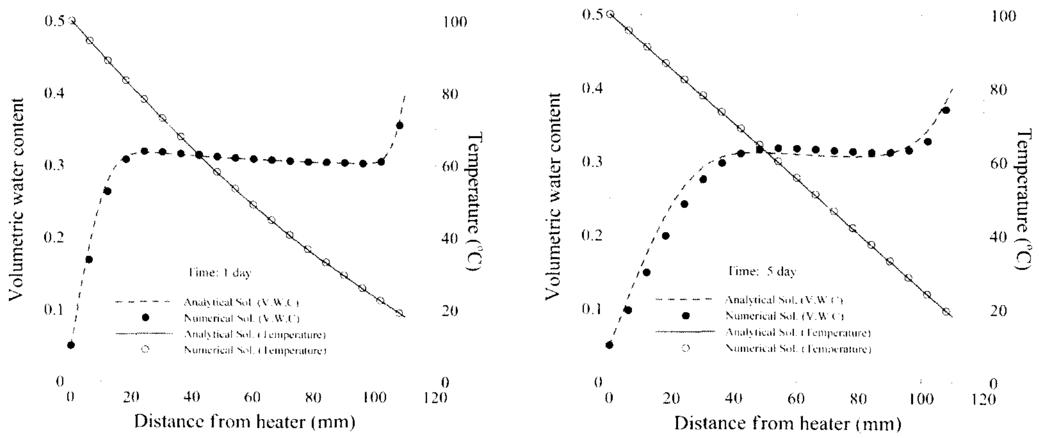
In this paper, the following linear functions have been used in order to express the diffusivity parameters as a function of volumetric water content and temperature. These relationships are determined by Yong et al. (1997), which could be obtained from the experimentally measured moisture and temperature profiles in a clay material with length of $L = 110$ mm.

$$D_\theta = (0.1 + 0.398T + 0.12\theta) \ (\text{mm}^2/\text{day}) \tag{9a}$$

$$D_T = (0.00646 + 0.003267T + 1.54\theta) \ (\text{mm}^2 \ ℃/\text{day}) \tag{9b}$$

$$\lambda / \rho c = (2504.95 + 1.137T) \ (\text{mm}^2/\text{day}) \tag{9c}$$

The boundary and initial conditions are given by $\theta_1 = 0.05$, $\theta_2 = 0.4$, $T_1 = 100$ ℃, $T_2 = 18$ ℃, $\theta_0 = 0.3$, $T_0 = 18$ ℃. These values are corresponding to the conditions of experiment conducted by Yong et al. (1997). Computed results of volumetric water content and temperature along the distance are presented in Fig. 2, which in general show a good agreement between the analytical and numerical solutions. Fig. 2 (a) shows that an approximate analytical solution after 1 day gives an excellent match to the numerical result by FDM. A discrepancy between analytical and numerical solutions increases as



(a) Variation of volumetric water content and temperature along the distance (time = 1 day)

(b) Variation of volumetric water content and temperature along the distance (time = 5 days)

Fig. 2. Comparison of analytical and numerical solutions

time elapses (time = 5 days). This discrepancy can reach maximum where a maximum moisture gradient is found. However, the error induced by the simplification of governing non-dimensionalized equation is very small in spite of the extreme case in which temperature gradient is severe. These illustrate the capability of an approximate analytical solution.

4. VALIDITY OF DRASTIC ASSUMPTION

Extended dimensionless form of Eq. (2a) can be expressed as Eq. (10).

$$\frac{\partial \theta^*}{\partial \tau} = \underbrace{D_\theta^* \frac{\partial^2 \theta^*}{\partial \zeta^2} + D_T^* \frac{\partial^2 T^*}{\partial \zeta^2}}_{\textit{left two terms}} + \underbrace{\frac{\partial D_\theta^*}{\partial \zeta}\frac{\partial \theta^*}{\partial \zeta} + \frac{\partial D_T^*}{\partial \zeta}\frac{\partial T^*}{\partial \zeta}}_{\textit{right two terms}} \tag{10}$$

However, the last two terms of RHS in Eq. (10) are not shown in Eq. (3a). This is due to the drastic assumption that we made when deriving an analytical solution of a one-dimensional coupled heat and mass problem. It is assumed that the last two terms of RHS in Eq. (10) are negligible compared to the first two ones. Under this assumption we omit the last two terms. To demonstrate the validity of this drastic assumption, the magnitude of the first two terms and that of the last two terms of RHS in Eq. (10) are compared. If the former is much larger than the latter, we may say that the omitting terms, which are composed of products of first order derivatives, have little influence on the solution. Based on the numerical results, the dimensionless variables of moisture, temperature, and diffusion parameters along the dimensionless distance ($\theta^*(\zeta), T^*(\zeta), D_\theta^*(\zeta), D_T^*(\zeta)$) are calculated and each of those variables is fitted to a polynomial form expressed as Eq. (11).

$$f(\zeta) = a + b\zeta + c\zeta^2 + d\zeta^3 + e\zeta^4 + \cdots \tag{11}$$

where f is any of dimensionless variables mentioned above, and a, b, c, d, e are the fitted coefficients. The fitted polynomial equations and derivatives of them are inserted into the first two terms and the last two terms. Then magnitude of the first two terms and that of the last two terms of Eq. (10) are plotted in Fig. 3. As shown in Fig. 3, the non-dimensional magnitude of the first two terms is relatively much larger than that of the last two terms. This result means that the product of first order derivatives neglected in the non-dimensionalized governing equation has little effect on the solution of heat and moisture behavior problem.

In similar way, the above procedure is carried out for Eq. (12) with dimensional variables of moisture, temperature, diffusion parameters.

$$\frac{\partial \theta}{\partial \tau} = D_\theta \frac{\partial^2 \theta}{\partial x^2} + D_T \frac{\partial^2 T}{\partial x^2} + \frac{\partial D_\theta}{\partial x}\frac{\partial \theta}{\partial x} + \frac{\partial D_T}{\partial x}\frac{\partial T}{\partial x} \tag{12}$$

The result is shown in Fig. 4. It is found out that the difference of magnitude between the first and the last two terms is insignificant. This suggests that the first two terms and the last two terms are of the same order. That is, both play almost equivalent roles in solution of equation (12). Consequently, non-dimensionalization of governing equation has significantly reduced the effect of non-linearity on the solution.

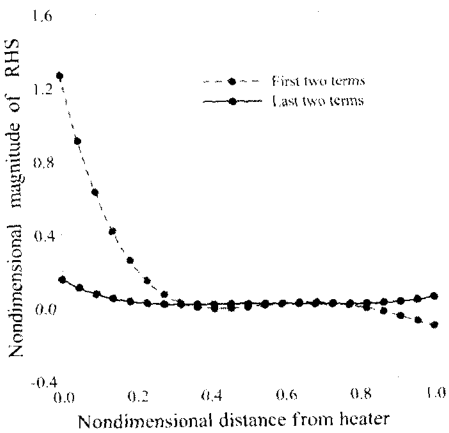


Fig. 3. Non-dimensional magnitude of RHS terms

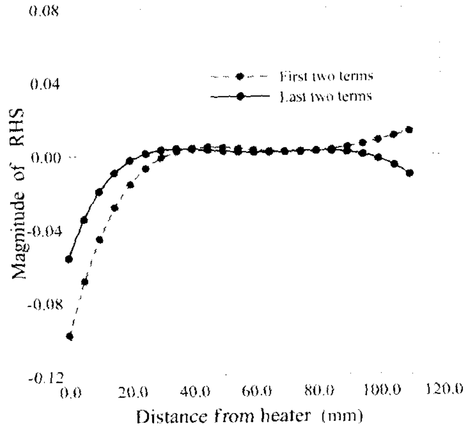


Fig. 4. Dimensional magnitude of RHS terms

5. CONCLUSIONS

Both thermal and hydraulic gradients are thought to be important factors that should be evaluated in geo-environmental problems. Thus a lot of studies have been done on coupled heat and mass flows in porous materials by using numerical methods. However, little previous efforts have been made to develop the analytical solution of a nonlinear hydro-thermo coupled diffusion equation. Therefore, an attempt is made to derive an analytical solution in this paper.

An approximate analytical solution of a nonlinear hydro-thermo coupled diffusion equation is derived using non-dimensionalized form of equation and transformation method. Then some numerical simulation is carried out to investigate the applicability of a derived approximate analytical solution. The results show a good agreement between analytical and numerical solutions. It is also found out that the product of first order derivatives neglected in the non-dimensionalized governing equation has little influence on solution of heat and moisture behavior problem. The proposed approximate analytical solution may provide a useful tool in the verification process of the numerical models. Also, the solution may be used for the analysis of the coupled heat and moisture movements in unsaturated porous media.

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APPENDIX

This appendix gives a detailed derivation for approximate analytical solutions of Eqs. (2a) and (2b). The procedure is as follows:

$$f_s\{\theta^*\} = \theta^{**} \quad \text{(A1)}$$

$$f_s\{T^*\} = T^{**} \quad \text{(A2)}$$

$$f_s\{\frac{\partial \theta^*}{\partial \tau}\} = \frac{\partial \theta^{**}}{\partial \tau} \quad \text{(A3)}$$

$$f_s\{\frac{\partial T^*}{\partial \tau}\} = \frac{\partial T^{**}}{\partial \tau} \quad \text{(A4)}$$

$$f_s\{\frac{\partial^2 \theta^*}{\partial \zeta^2}\} = n\pi[\theta^*(0,\tau) - (-1)^n \theta^*(1,\tau)] - n^2\pi^2\theta^{**} \quad \text{(A5)}$$

$$f_s\{\frac{\partial^2 T^*}{\partial \zeta^2}\} = n\pi[T^*(0,\tau) - (-1)^n T^*(1,\tau)] - n^2\pi^2 T^* \quad \text{(A6)}$$

where $f_s\{\ \}$ is operation of taking the finite Fourier sine transformation (Churchill, R.V., 1972) and n is the transform variable. When boundary conditions (Eqs. (4a), (4b)) are inserted into equations (A5) and (A6):

B.C ; $\theta^*(0,\tau) = 1$, $\theta^*(1,\tau) = 0$, $T^*(0,\tau) = 1$, $T^*(1,\tau) = 0$

$$f_s\left\{\frac{\partial^2 \theta^*}{\partial \zeta^2}\right\} = n\pi - n^2\pi^2\theta^{**} \quad \text{(A7)}$$

$$f_s\left\{\frac{\partial^2 T^*}{\partial \zeta^2}\right\} = n\pi - n^2\pi^2 T^{**} \quad \text{(A8)}$$

are obtained. If the finite Fourier sine transform of equations (2a) and (2b) are taken and the expressions given by equations (A3), (A4), (A7) and (A8) are considered, then the following equations can be obtained:

$$\frac{\partial \theta^{**}}{\partial \tau} = D_\theta^*(n\pi - n^2\pi^2\theta^{**}) + D_T^*(n\pi - n^2\pi^2 T^{**}) \quad \text{(A9)}$$

$$\frac{\partial T^{**}}{\partial \tau} = n\pi - n^2\pi^2 T^{**} \quad \text{(A10)}$$

When the finite Fourier sine transforms of the initial conditions are taken, equation (4c) become

$$\theta^{**}(n,0) = \frac{\theta_0^*}{n\pi}[1-(-1)^n] \tag{A11}$$

$$T^{**}(n,0) = \frac{T_0^*}{n\pi}[1-(-1)^n] \tag{A12}$$

Using the Laplace transform, one may write:

$$L\{T^{**}\} = \overline{T} \tag{A13}$$

$$L\{\theta^{**}\} = \overline{\theta} \tag{A14}$$

$$L\left\{\frac{\partial T^{**}}{\partial \tau}\right\} = p\overline{T} - T^{**}(n,0) = p\overline{T} - \frac{T_0^*}{n\pi}[1-(-1)^n] \tag{A15}$$

$$L\left\{\frac{\partial \theta^{**}}{\partial \tau}\right\} = p\overline{\theta} - \theta^{**}(n,0) = p\overline{\theta} - \frac{\theta_0^*}{n\pi}[1-(-1)^n] \tag{A16}$$

where $L\{\ \}$ is operation of taking the Laplace transform, p is the Laplace transform variable.

When the Laplace transforms of equations (A9) and (A10) are taken, and equations (A13) ~ (A16) are considered;

$$\frac{n\pi}{p} - n^2\pi^2\overline{T} = p\overline{T} - \frac{T_0^*}{n\pi}[1-(-1)^n] \tag{A17}$$

$$D_\theta^*\left(\frac{n\pi}{p} - n^2\pi^2\overline{\theta}\right) + D_T^*\left(\frac{n\pi}{p} - n^2\pi^2\overline{T}\right) = p\overline{\theta} - \frac{\theta_0^*}{n\pi}(1-(-1)^n) \tag{A18}$$

$\overline{T}$ and $\overline{\theta}$ can be solved from equations (A17) and (A18) as:

$$\overline{T} = \frac{1}{n\pi}\left(\frac{1}{p} - \frac{1}{p+n^2\pi^2}\right) + \frac{T_0^*\left(1-(-1)^n\right)}{n\pi(p+n^2\pi^2)} \tag{A19}$$

$$\overline{\theta} = \frac{1}{p+n^2\pi^2 D_\theta^*}\left[\frac{n\pi D_\theta^*}{p} + \frac{n\pi D_T^*}{p+n^2\pi^2} - \frac{n\pi D_T^* T_0^*\left(1-(-1)^n\right)}{p+n^2\pi^2} + \frac{\theta_0^*}{n\pi}\left(1-(-1)^n\right)\right]$$
$$= \frac{1}{n\pi}\left(\frac{1}{p} - \frac{1}{p+n^2\pi^2 D_\theta^*}\right) + \frac{D_T^*}{n\pi(1-D_\theta^*)}\left(\frac{1}{p+n^2\pi^2 D_\theta^*} - \frac{1}{p+n^2\pi^2}\right)$$
$$- \frac{D_T^* T_0^*\left(1-(-1)^n\right)}{n\pi(1-D_\theta^*)}\left(\frac{1}{p+n^2\pi^2 D_\theta^*} - \frac{1}{p+n^2\pi^2}\right) + \frac{\theta_0^*}{n\pi}\left(\frac{1-(-1)^n}{p+n^2\pi^2 D_\theta^*}\right)$$

From Laplace transform tables (Churchill, 1972) the inversion of equations (A19) and (A20) will yield:

$$T^{**} = \frac{1}{n\pi}\left[1 - e^{-n^2\pi^2\tau} + T_0^*\left(1-(-1)^n\right)e^{-n^2\pi^2\tau}\right] \tag{A21}$$

$$\theta^{**} = \frac{1}{n\pi}\left[1 - e^{n^2\pi^2 D_\theta^*\tau} + \frac{D_T^*}{1-D_\theta^*}\left(e^{-n^2\pi^2 D_\theta^*\tau} - e^{n^2\pi^2\tau}\right)\right.$$
$$\left. - \left(\frac{D_T^* T_0^*\left(1-(-1)^n\right)}{1-D_\theta^*}\right)\left(e^{n^2\pi^2 D_\theta^*\tau} - e^{-n^2\pi^2\tau}\right) + \theta_0^*\left(1-(-1)^n\right)e^{n^2\pi^2 D_\theta^*\tau}\right] \tag{A22}$$

When the inverse finite Fourier sine transform is applied to equations (A21) and (A22), the resulting equations are

$$T^*(\zeta,\tau) = (1-\zeta) + \sum_{n=0}^{\infty}\frac{2}{n\pi}\left[T_0^*(1-(-1)^n)-1\right] e^{-n^2\pi^2\tau}\sin(n\pi\zeta) \tag{A23}$$

$$\theta^*(\zeta,\tau) = (1-\zeta) + \sum_{n=0}^{\infty}\frac{2}{n\pi}\left[\frac{D_T^* - T_0^* D_T^*\left(1-(-1)^n\right)}{1-D_\theta^*}\left(e^{n^2\pi^2 D_\theta^*\tau} - e^{n^2\pi^2\tau}\right) - e^{-n^2\pi^2 D_\theta^*\tau} + \theta_0^*\left(1-(-1)^n\right)e^{n^2\pi^2 D_\theta^*\tau}\right]\sin(n\pi\zeta) \tag{A24}$$

If $T_0 = T_2$ and $\theta_0 = \theta_2$ (that is, $T_0^* = 0$, $\theta_0^* = 0$) then equations (A23) and (A24) become

$$T^*(\zeta,\tau) = (1-\zeta) - \sum_{n=0}^{\infty} \frac{2}{n\pi} e^{-n^2\pi^2\tau} \sin(n\pi\zeta) \quad \text{(A25)}$$

$$\theta^*(\zeta,\tau) = (1-\zeta) + \sum_{n=0}^{\infty} \frac{2}{n\pi} \left[\frac{D_T^*}{1-D_\theta^*} \left(e^{-n^2\pi^2 D_\theta^* \tau} - e^{-n^2\pi^2\tau} \right) - e^{-n^2\pi^2 D_\theta^* \tau} \right] \sin(n\pi\zeta) \quad \text{(A26)}$$

Finally, θ and T can be obtained from

$$\theta^* = \frac{\theta - \theta_2}{\theta_1 - \theta_2}, \quad T^* = \frac{T - T_2}{T_1 - T_2}.$$

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