A Study on Singularly Perturbed Open-Loop Systems by Delta Operator Approach

Kyu-Hong Shim and M. Edwin Sawan

Abstract: In this paper, the open-loop state response of the two-time-scale systems by unified approach using the δ-operator is presented with an example of the aircraft longitudinal dynamics. First, the δ-operator system unifies both the continuous system and the discrete system simultaneously, and the δ-operator approach improves the finite word-length characteristics. This saves more computing time than that of the discrete system. Second, the singular perturbation method by block diagonalization reduces the sizes and orders of the systems. This also reduces the floating-point operations (flops). The advantage of those two approaches is shown by comparing our results with the earlier ones in the illustrative example of the longitudinal motion of F-8 aircraft.

Keywords: delta operator, finite word-length characteristics, two-time-scale system, matrix block diagonalization

I. Introduction

The two topics are covered in this paper: One is the unified approach using the δ-operators to improve the finite word-length characteristics. The other is the matrix block diagonalization to reduce the sizes and orders of the two-time-scale systems.

1. The unified approach by using the δ-operators

The discrete models are written in the form of the shift (q) operators. But, equations of the discrete systems by the q-operators are not simple like those of the continuous systems by the operator, d/dt. The δ-operator system unifies both the continuous system and the discrete system. In other words, the equation of the δ-operator system represents both the continuous and the discrete systems simultaneously. As the sampling time Δ in the δ-operator system approaches zero, the unified system becomes the continuous system. Therefore, the δ-operator system includes the whole characteristics of the discrete system and can be handled like the continuous system. The easiness of handling the δ-operator system means to reduce a number of equations in the discrete system.

The normal q-operator systems have the problem of crowding poles within the boundary of the stability circle at small sampling time and the difficulties of the truncation and round-off errors. If the discrete system is converted to the δ-operator system, the problems mentioned above are disappeared since the resolution of the stability circle is increased. Moreover it is the δ-operator system that has the finite word-length characteristics improved compared with the q-operator system [6] [12].

Therefore, the δ-operator approach reduces the computing time of the discrete system; thus, improves the quality of the on-lined operating systems that requires higher accuracy. The analytical work of the unified approach using the δ-operators was fully founded by Middleton and Goodwin [13]. Li and Gevers [7] showed some advantages of the δ-operator state-space realization of the transfer function over that of the q-operator on the minimization of the roundoff noise gain of the realization. They studied that the δ-operator implementation is a special case of residue feedback. Li and Gevers compared the δ-operator with the q-operator state-space realizations in terms of the effects of the finite word-length errors on the actual transfer function[6]. They showed the parameterizations in the δ-operator yielded a superior sensitivity performance over those in the q-operator. Bouslimani *et al.* investigated the monotoninicity properties of the solution of the Riccati equation using the δ-operator formulation [1]. Naidu *et al.* presented an application paper for the H_2 and H_∞ optimal control of a hypersonic vehicle by the δ-operator approach [18]. Shim and Sawan explored the Linear Quadratic Regulator (LQR) design in the singularly perturbed systems by the δ-operator approach [19].

2. The singular perturbation method

Many dynamic systems have the fast and the slow variables in them. This means that there exist the higher and the lower frequencies of the state variables in the systems. When the system eigenvalues gathered by two groups, it is called the two-time-scale system. It is computational burden to solve such fully coupled equations of the two-time-scale systems. Matrix block diagonalization technique is a powerful tool to solve such a problem of the singularly perturbed models because it reduces the sizes and orders of the system [2]-[4]. The method is called the singular perturbation technique and it decouples the two-time-scale systems into the slow and fast subsystems with a quasi-steady-state approximation. Kokotovic *et al.* [5] and Naidu [16] made large contributions in the development of the singular perturbation methods for the continuous systems and the discrete systems, respectively. Naidu and others studied and applied the singular perturbation approach in the flight dynamics [14]-[18].

3. Contribution of this paper

Naidu *et al.* extended the singular perturbation method from the two-time-scale continuous system to the two-time-scale discrete system [14]-[17]. In this paper, we explored to extend the singular perturbation technique to the δ-operator system. The result of the δ-operator solution is compared with the result of the q-operator solution of Naidu *et al.* [14]-[17]. An improved result by the δ-operator approach is obtained and illustrated in the simulation figures.

II. Delta operator

Consider a linear and time-invariant continuous system

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$$\dot{x} = Ax(t) + Bu(t), \tag{1}$$

where x is a n x 1 state vector and u is a r x 1 control vector. A is a n x n matrix and B is n x r matrix. The corresponding sampled-data system with the zero-order hold (ZOH) and sampling interval Δ is given by

$$\begin{aligned} x(k+1) &= A_q x(k) + B_q u(k), \\ y(k) &= C_q x(k), \end{aligned} \tag{2}$$

where $A_q = e^{A\Delta}$, $B_q = \int_0^{\Delta} e^{A(\Delta-\tau)} B d\tau$.

According to Middleton and Goodwin, the delta operator is defined as follow [13]:

$$\delta = \frac{(q-1)}{\Delta}. \tag{3}$$

Assumption 1: The delta operator is identified in terms of the q-operator in (3).

Proof: Given the continuous-time and discrete-time state equations as

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ x_{k+1} &= A_q x_k + B_q u_k, \end{aligned} \tag{4}$$

The shift operator q is defined as

$$\begin{aligned} qx(k) &= qx_k \\ &= x(k+1) \\ &= x_{k+1} \\ &= A_q x_k + B_q u_k. \end{aligned}$$

Now, the delta operator δ, working for both the continuous and the discrete systems, is introduced as follow,

$$\begin{aligned} \delta x_k &= \frac{x_{k+1} - x_k}{\Delta} \\ &= \frac{qx_k - x_k}{\Delta} \\ &= \frac{x(k\Delta + \Delta) - x(k\Delta)}{\Delta} = \frac{(q-I)x_k}{\Delta}. \end{aligned} \tag{5}$$

Therefore, (3) is proved [13].

Assumption 2: Show the parameter identities of the q-operator and the δ-operator in (6) and (7) as in (8).

$$\begin{aligned} qx &= A_q x(k) + B_q u(k), \\ y(k) &= C_q x(k). \end{aligned} \tag{6}$$

$$\begin{aligned} \delta x(\tau) &= A_\delta x(\tau) + B_\delta u(\tau), \\ y(k) &= C_\delta x(k). \end{aligned} \tag{7}$$

$$A_\delta = \frac{(A_q - I)}{\Delta},\ B_\delta = \frac{B_q}{\Delta},\ C_q = C_\delta. \tag{8}$$

Proof: One can write the state equation in the delta form, which represents both the continuous and the discrete systems as (7). Substituting (3) in (7) gives

$$\frac{q-I}{\Delta} x_\delta = A_\delta x_\delta + B_\delta u_\delta. \tag{9}$$

Using (6) for q results in

$$\begin{aligned} &\frac{[A_q + B_q(u_k x_k^{-1})] - I}{\Delta} x_\delta = A_\delta x_\delta + B_\delta u_\delta. \\ &\frac{A_q - I}{\Delta} x_\delta + \frac{B_q}{\Delta}(u_k x_k^{-1}) x_\delta = A_\delta x_\delta + B_\delta u_\delta. \end{aligned} \tag{10}$$

Therefore, one has the identities in (8). Note that all the parameters Cs remain same as

$$y = Cx,\ C = C_q = C_\delta.\ y = C_\delta x_\delta,\ y = C_q x_q, \tag{11}$$

Also note that the δ-operator system contains all the sub-terms of the corresponding discrete expression.

Assumption 3: The parameters between the continuous system and the delta system are identified as

$$A_\delta = \Omega A,\ B_\delta = \Omega B. \tag{12}$$

where

$$\Omega = \frac{1}{\Delta}\int_0^{\Delta} e^{A\tau} d\tau = I + \frac{(A\Delta)}{2} + \frac{(A^2\Delta^2)}{3} + \cdots. \tag{13}$$

Therefore, as Δ goes to zero, Ω becomes the identity matrix:

$$\begin{aligned} A_\delta &= \frac{A_q - I}{\Delta} \\ &= \frac{e^{A\Delta} - I}{\Delta} \\ &= A + \frac{(A^2\Delta)}{2!} + \frac{(A^3\Delta^2)}{3!} + \cdots. \end{aligned} \tag{14}$$

Therefore, as Δ approaches zero, A_δ is identified to A.

Remark 1: When a truncated power series is used to evaluate the matrix exponential as

$$e^{A\Delta} = \sum_{k=1}^{N} \frac{(A\Delta)^k}{k!}, \tag{15}$$

selection of the sampling time Δ as in $\| \Delta A \|_2$ should not be close to 1 because of the numerical difficulty for computing this finite power series. (16) is a comprehensive expression of the state space equation that involves the continuous, the discrete, and the delta equations. This is called a unified approach using the δ-operators in the Middleton and Goodwin [See pp. 44-45 for an example in Ref. 13].

$$\begin{aligned} \rho x(\tau) &= A_\rho x(\tau) + B_\rho u(\tau), \\ y(\tau) &= C_\rho x(\tau). \end{aligned} \tag{16}$$

$$A_\rho = \begin{Bmatrix} A \\ A_q \\ A_\delta \end{Bmatrix},\ B_\rho = \begin{Bmatrix} B \\ B_q \\ B_\delta \end{Bmatrix},\ \rho = \begin{Bmatrix} d/dt \\ q \\ \delta \end{Bmatrix},\ \tau = \begin{Bmatrix} t \\ k \\ \tau_\delta \end{Bmatrix}.$$

It is noted that the first rows, the second rows and the third rows of A_ρ, B_ρ, ρ *and* τ denote the continuous system, the discrete system and the delta system, respectively. When $\Delta \to 0$, then $A\delta \to A$, $B\delta \to B$. This means that, when the sampling time goes to zero, the discrete-like δ system becomes identical with the continuous system. Now the stability regions for various operators are introduced. For continuous systems, the operator is d/dt and the transform variable is s. For discrete systems, the operator is q and the transform variable is z. For unified systems, the operator is δ and the transform variable is γ. The stability regions are as follows:

Case of the continuous system: $\mathrm{Re}(s) < 0$,
Case of the discrete system: $|z| < 1$,
Case of the delta system: $\frac{\Delta}{2}|\gamma|^2 + \mathrm{Re}\{\gamma\} < 0.$ (17)

As Δ approaches zero, the stability inequality of the unified case equals that of the continuous case.

III. Open-Loop response for singularly perturbed systems

1. Unified Systems

Consider the linear system (16) and assume that the system satisfies the condition (17), then we can write *the two-time-scale* system as

$$\begin{bmatrix} \rho x(\tau) \\ \varepsilon \rho z(\tau) \end{bmatrix} = \begin{bmatrix} A_{\delta 11} & A_{\delta 12} \\ A_{\delta 21} & A_{\delta 22} \end{bmatrix} \begin{bmatrix} x(\tau) \\ z(\tau) \end{bmatrix} + \begin{bmatrix} B_{\delta 1} \\ B_{\delta 2} \end{bmatrix} u(\tau), \quad (18)$$

where x and z are n and m dimensional state vectors, u is an r dimensional control vector, and $A_{\delta ij}$ are matrices of appropriate dimensionality. Also, it is required that $A_{\delta 22}$ be non-singular. System (18) has a two-time-scale property, if

$$\begin{aligned} 0 &< |\lambda_{s1}| < |\lambda_{s2}| \cdots \\ &< |\lambda_{sn}| < |\lambda_{f1}| < |\lambda_{f2}| \cdots \\ &< |\lambda_{fm}| < |2/\Delta|, \\ \varepsilon &= |\lambda_{sn}| / |\lambda_{f1}| \ll 1, \end{aligned} \quad (19)$$

where λ denotes eigenvalues of the system. So, we have

$$|\lambda_{\max}(A_{\delta s})| \ll |\lambda_{\min}(A_{\delta f})|.$$

If the norm properties of the invertible matrices are used, this can be equivalent to

$$|A_{\delta f}|^{-1} \ll |A_{\delta s}|^{-1}. \quad (20)$$

Now, we need to de-couple the system (18) into the slow and fast subsystems.

1.1 Block diagonalization [2]-[4]

For decoupling the system (18), rewrite the equations as

$$\begin{aligned} x_s(\tau) &= (I_s - M_\delta L_\delta)x(\tau) - M_\delta z(\tau), \\ z_f(\tau) &= L_\delta x(\tau) + I_{\delta f} z(\tau). \end{aligned} \quad (21)$$

From (21) the slow and fast subsystems are obtained as,

$$\begin{bmatrix} \rho x(\tau) \\ \rho z(\tau) \end{bmatrix} = \begin{bmatrix} A_{\delta s} & 0 \\ 0 & A_{\delta f} \end{bmatrix} \begin{bmatrix} x(\tau) \\ z(\tau) \end{bmatrix} + \begin{bmatrix} B_{\delta s} \\ B_{\delta f} \end{bmatrix} u(\tau), \quad (22)$$

where

$$\begin{aligned} A_{\delta s} &= A_{\delta 11} - A_{\delta 12} L_\delta, \quad A_{\delta f} = A_{\delta 22} + L_\delta A_{\delta 22}, \\ B_{\delta s} &= B_{\delta 1} - M_\delta B_{\delta 2} - M_\delta L_\delta B_{\delta 1}, \quad B_{\delta f} = B_{\delta 2} + L_\delta B_{\delta 1}. \end{aligned} \quad (23)$$

Here, L and M are the solutions of the nonlinear algebraic Riccati-type equations as

$$\begin{aligned} L A_{\delta 11} + A_{\delta 21} - L A_{\delta 12} L - A_{\delta 22} L &= 0, \\ A_{\delta 11} M - A_{\delta 12} L M - M A_{\delta 22} - M L A_{\delta 12} + A_{\delta 12} &= 0, \end{aligned} \quad (24)$$

with initial conditions

$$L_0 = A_{\delta 22}^{-1} A_{\delta 21}, \quad M_0 = A_{\delta 12} A_{\delta 22}^{-1} \quad (25)$$

Therefore

$$A_{\delta 0} = A_{\delta 11} - A_{\delta 12} L_0, \quad B_{\delta 0} = B_{\delta 1} - M_0 B_{\delta 2}. \quad (26)$$

Lemma 1: Assume that A22 is nonsingular and use L0 and A0. If

$$\| A_{\delta 22}^{-1} \| \le \frac{1}{3(\| A_{\delta 0} \| + \| A_{\delta 12} \| . \| L_0 \|)}, \quad (27)$$

then the sequences Lk and Mk are defined by

$$\begin{aligned} L_{k+1} &= A_{\delta 22}^{-1}(A_{\delta 21} + L_k A_{\delta 11} - L_k A_{\delta 12} L_k), \\ M_{k+1} &= \{(A_{\delta 11} + A_{\delta 12} - A_{\delta 12} L_k M_k - M_k L_k A_{\delta 12}\} A_{\delta 22}^{-1}, \end{aligned} \quad (28)$$

and L0 as in (25) converges to a real bounded root of (24.a). Moreover

$$\| L_{k+1} - L \| \le \| L_k - L \|, \quad k = 0,\ 1,\ 2,\ 3\ \cdots. \quad (29)$$

It offers a simple tool for approximating L if $\| A_{\delta 22}^{-1} \|$ is small From (25), (26) and (29) with k = 0, we obtain $L_1 - L_2 = A_{\delta 22}^{-1} L_0 A_0$ and, therefore,

$$\| L_1 - L_0 \| \le \mu \| L_k - L \| \quad if \ \| A_{\delta 22}^{-1} \| \le \mu \| A_{\delta 0} \|^{-1} \quad (30)$$

when (30) is satisfied with a small μ, we can use L0 as an order of μ approximation of L, that is,

$$L = L_0 + O(\mu) = A_{\delta 22}^{-1} A_{\delta 21} + O(\mu). \quad (31)$$

Proof: See [4].

Remark 2: Note that $A_{\delta 22}^{-1} A_{\delta 21}$ needs not be O(μ) because we may allow $A_{\delta 21}$ to be of order 1/μ. It is now easy to interpret (20) as a property of (18).

Lemma 2: The system (18) will have *the two-time-scale* property (20) if *Lemma 1* holds and if (30) is satisfied with $\mu \ll 1$, that is,

$$\| A_{\delta 22}^{-1} \| \ll \| A_{\delta 0} \|^{-1}. \quad (32)$$

Of course, *Remark 2* is valid here. From *Lemma 1* and *Lemma 2*, we conclude that a sufficient condition for a system to possess *the two-time-scale property* is

$$\| A_{\delta 22}^{-1} \| \ll \{\| A_{\delta 0} \| + \| A_{\delta 12} \| \cdot \| L_0 \|\}^{-1}. \quad (33)$$

Now we can produce the approximated expressions for x and z

Proof: See [3].

Lemma 3: When *Lemma 2* holds, then

$$x(\tau)=x_0(\tau)+A_{\delta 12}A_{\delta 22}^{-1}z_0(\tau)+O(\mu),$$
$$z(\tau)=A_{\delta 22}^{-1}A_{\delta 21}x_0(\tau)+z_0(\tau)+O(\mu), \quad (34)$$

where x_{s0} and x_{f0} are obtained from the simplified subsystems as

$$\begin{bmatrix}\dot{x}_0\\ \dot{z}_0\end{bmatrix}=\begin{bmatrix}A_{\delta 0} & 0\\ 0 & A_{\delta 22}\end{bmatrix}\begin{bmatrix}x_0\\ z_0\end{bmatrix}+\begin{bmatrix}B_{\delta 0}\\ B_{\delta 2}\end{bmatrix}u_0. \quad (35)$$

1.2 Quasi-Steady-State approximation [2]-[4]

To give a simple interpretation of (34), we compare (18) with the system

$$\begin{bmatrix}\rho\bar{x}(\tau)\\ 0\end{bmatrix}=\begin{bmatrix}A_{\delta 11} & A_{\delta 12}\\ A_{\delta 21} & A_{\delta 22}\end{bmatrix}\begin{bmatrix}\bar{x}(\tau)\\ \bar{z}(\tau)\end{bmatrix}+\begin{bmatrix}B_{\delta 1}\\ B_{\delta 2}\end{bmatrix}\bar{u}(\tau), \quad (36)$$

where a bar denotes *quasi-steady-state*. The system (38) is reduced to

$$\rho\bar{x}(\tau)=A_{\delta 0}\bar{x}(\tau),\ \ \bar{z}(\tau)=A_{\delta 22}^{-1}(A_{\delta 21}\bar{x}(\tau)+B_{\delta 2}\bar{u}(\tau)) \quad (37)$$

Thus x_0, $A_{\delta 22}^{-1}A_{\delta 21}x_0$ can be interpreted as a quasi-steady-state of x after z has decayed. This state is varying slowly compared with the variations of z. Hence, to get an O(μ) approximation of the slow parts of x(τ) and z(τ), we simply let ρz = 0 in (18).

Remark 3: If the original system possesses a *two-time-scale property*, but the matrix A is not in the form satisfying (33), this property can be exhibited by transformations, such as re-indexing and re-scaling the state variables [3]. Substituting (37) into (36) yields

$$\rho x_s=A_{\delta 0}x_s(\tau)+B_{\delta 0}u_s(\tau),\ \ x_s(0)=x_{s0}, \quad (38)$$

where A0 and B0 are defined in (26). Thus, $\bar{x}=x_s$, $\bar{u}=u_s$, $\bar{y}$ are the slow parts of the corresponding variables in (23). To derive the fast subsystem, we assume that the slow variables are constant during fast modes, that is, ρz = 0 and $\bar{x}$ is constant. From (36) and (37), we have

$$\rho z(\tau)-\rho\bar{z}(\tau)$$
$$=A_{\delta 22}\{z(\tau)-\bar{z}(\tau)\}+B_{\delta 2}\{u(\tau)-u_s(\tau)\}. \quad (39)$$

Letting $z_f=z-\bar{z}$, *and* $u_f=u-u_s$, the fast subsystem of (18) is redefined by

$$\rho z_f(\tau)=A_{\delta 22}z_f(\tau)+B_{\delta 2}u_f(\tau),$$
$$z_f(0)=z_0-\bar{z}(0). \quad (40)$$

2. Discrete-Time systems [16]

The general form for a linear, shift-invariant and singularly perturbed discrete system is given as

$$x(k+1)=A_{d11}x(k)+\varepsilon^{1-j}A_{d12}z(k)+B_{d1}u(k),$$
$$\varepsilon^{2i}z(k+1)=\varepsilon jA_{d21}x(k)+\varepsilon A_{d22}z(k)+\varepsilon^{j}B_{d2}u(k). \quad (41)$$

with 0≤ i ≤1 and 0≤ j ≤1 where x(k) and z(k) are the slow and fast state vectors of *n* and *m* dimensions, respectively. u(k) is an *r* dimensional control vector, ε is the singular perturbation parameter, and Aij and Bi are matrices of appropriate dimensionality. The subscripts δ and d denote the δ-operator system (the continuous-like unified expression) and the discrete system, respectively. Here we choose a special model with i = j = 1 as,

$$x(k+1)=A_{d11}x(k)+A_{d12}z(k)+B_{d1}u(k),$$
$$\varepsilon z(k+1)=A_{d21}x(k)+A_{d22}z(k)+B_{d2}u(k). \quad (42)$$

Consider the system given as (42) with the initial conditions x(0) and z(0). It is assumed that the system is asymptotically stable and that its eigenspectrum consists of a cluster of *n* large eigenvalues and a cluster of *m* small eigenvalues. Then we can arrange the eigenvalues of the system as

$$|1| > |p_{s1}| > |p_{s2}| > \cdots$$
$$> |p_{sn}| > |p_{f1}| > |p_{f2}| \cdots > |p_{fm}|,$$
$$\varepsilon = |p_{f1}|/|p_{sn}| << 1. \quad (43)$$

For block diagonalization of (42), we need the two stages of the linear transformation. First, the A21 block is removed to make the equation (42) an upper triangular matrix by using the transformation:

$$z_f(k)=z(k)+Dx(k), \quad (44)$$

where D(*m* x *n* matrix) is a root of the Riccati-type algebraic equation as

$$A_{d22}D-DA_{d11}+DA_{d12}D-A_{d21}=0. \quad (45)$$

From (42) and (45), we obtain equation (46) as

$$\begin{bmatrix}x(k+1)\\ Z_f(k+1)\end{bmatrix}=\begin{bmatrix}A_{ds} & A_{d12}\\ 0 & A_{df}\end{bmatrix}\begin{bmatrix}x(k)\\ z_f(k)\end{bmatrix}+\begin{bmatrix}B_{d1}\\ B_{df}\end{bmatrix}u(k), \quad (46)$$
$$B_{df}=DB_{d1}+B_{d2}.$$

Second, we have the transformation as

$$x_s(k)=x(k)+Ez_f(k),$$
$$EA_{df}-A_{ds}E+A_{d12}=0, \quad (47)$$

$$E(A_{d22}+DA_{d12})-(A_{d11}-A_{d12}D)E+A_{d12}=0. \quad (48)$$

Then, (46) and (48) results in

$$\begin{bmatrix}x_s(k+1)\\ z_f(k+1)\end{bmatrix}=\begin{bmatrix}A_{ds} & 0\\ 0 & A_{df}\end{bmatrix}\begin{bmatrix}x_s(k)\\ z_f(k)\end{bmatrix}+\begin{bmatrix}B_{ds}\\ B_{df}\end{bmatrix}u(k), \quad (49)$$

where Bs = (Is + ED)B1 + EB2. In addition, from (44) and (47), we have

$$\begin{bmatrix}x(k)\\ z(k)\end{bmatrix}=\begin{bmatrix}I_s & -E\\ -D & (I_f+DE)\end{bmatrix}\begin{bmatrix}x_s(k)\\ z_f(k)\end{bmatrix} \quad (50)$$

The iterative solution of (45) is given as

$$D_{i+1}=(A_{d22}D_i+D_iA_{d12}D_i-A_{d21})A_{d11}^{-1}, \quad (51)$$

with an initial value $D_0=-A_{d21}A_{d11}^{-1}$. Similarly, for the equation (48), we have

$$E_{i+1}=A_{d11}^{-1}(E_iA_{d22}+E_iDA_{d12}+A_{d12}DE_i+A_{d12}), \quad (52)$$

with an initial value of $D_0 = -A_{d21}A_{d11}^{-1}$. Substituting D0 and E0 into (49) yields

$$A_{ds0} = A_{d11} + A_{d12}A_{d21}A_{d11}^{-1},$$
$$A_{df0} = A_{d22} - A_{d21}A_{d11}^{-1}A_{d12},$$
$$B_{ds0} = B_{d1} - A_{d11}^{-1}A_{d12}A_{d22}A_{d11}^{-1}B_{d1} + A_{d11}^{-1}A_{d12}B_{d2},$$
$$B_{df0} = B_{d2} - A_{d21}A_{d11}^{-1}B_{d1}, \quad (53)$$

$$\begin{bmatrix} x_{s0}(k+1) \\ z_{f0}(k+1) \end{bmatrix} = \begin{bmatrix} A_{ds0} & 0 \\ 0 & A_{df0} \end{bmatrix} \begin{bmatrix} x_{s0}(k) \\ z_{f0}(k) \end{bmatrix} + \begin{bmatrix} B_{ds0} \\ B_{df0} \end{bmatrix} u(k). \quad (54)$$

It is shown that a two-time-scale system is de-coupled into the slow and fast subsystems [15]-[17].

IV. Example

An airplane model is given in the discrete system as

$$\begin{bmatrix} x(k+1) \\ z(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x(k) \\ z(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(k),$$

where x1(k) is velocity (ft/sec), x2(k) is pitch angle(deg or rad), x3(k) is altitude(ft), z1(k) is angle of attack (deg or rad), and z2(k) is pitch angular velocity(deg/sec).

$$A = \begin{bmatrix} .923701 & -.308096 & 0 & .053043 & -.090367 \\ .039705 & .995525 & 0 & -.107454 & .588883 \\ .087127 & 1.899490 & 1 & -.635270 & .394015 \\ -.035537 & .010123 & 0 & .007748 & .137407 \\ .069562 & -.012706 & 0 & -.097108 & .287411 \end{bmatrix}$$

1. Open-loop system by the δ-opearator approach

1.1 Zero iterations case

Here we directly use the values Lo and Mo in the continuous-like unified system and have

$$L_0 = \begin{bmatrix} .0219 & -.0076 & 0 \\ -.1006 & .0189 & 0 \end{bmatrix}, \quad Mo = \begin{bmatrix} -.0646 & .1143 \\ .1857 & -.7906 \\ .6815 & -.4215 \end{bmatrix}$$

$$A_{\delta s0} = \begin{bmatrix} -.8655 & -3.0599 & 0 \\ 1.0130 & -.1640 & 0 \\ 1.4067 & 18.8724 & 0 \end{bmatrix},$$

$$A_{\delta f0} = \begin{bmatrix} -9.9028 & 1.3096 \\ -1.0447 & -6.9239 \end{bmatrix}$$

Eigenvalues of Aδ are 0, -0.379 ± j 1.7534, -7.8270 and -9.2712. Eigenvalues of Aδso and Aδfo are 0, -0.5148 ± j 1.7253, -7.4912 and -9.3354.

1.2 Three Iterations Case

We obtained the values after three iterations to solve the Riccati-type nonlinear equation (28) as

$$L_3 = \begin{bmatrix} .0235 & -.0069 & 0 \\ -.1098 & -.0292 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} .0292 & -.2226 \\ .1758 & -.7330 \\ .1341 & -1.5078 \end{bmatrix}$$

$$A_{\delta s3} = \begin{bmatrix} -.8747 & -3.1037 & 0 \\ 1.0688 & .1196 & 0 \\ 1.4529 & 19.0660 & 0 \end{bmatrix},$$

$$A_{\delta f3} = \begin{bmatrix} -9.9027 & 1.3123 \\ -.9980 & -7.1985 \end{bmatrix}.$$

2. Simulation results [15]-[17]

Figures 1-5 show the exact and the zeroth approximated responses of the open-loop system. For *Figures 6-10*, the optimal and the sub-optimal solutions are exactly coincided when the third order approximation is applied.

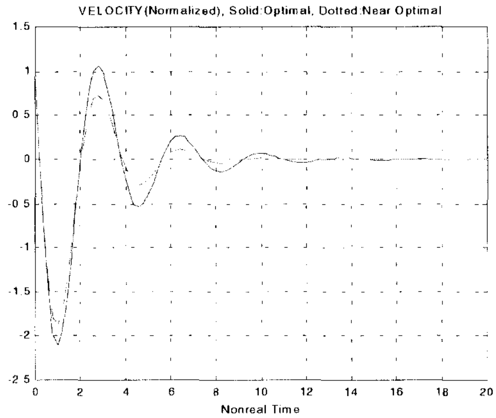

Fig. 1. Velocity for Open-Loop system with the zeroth Approximation.

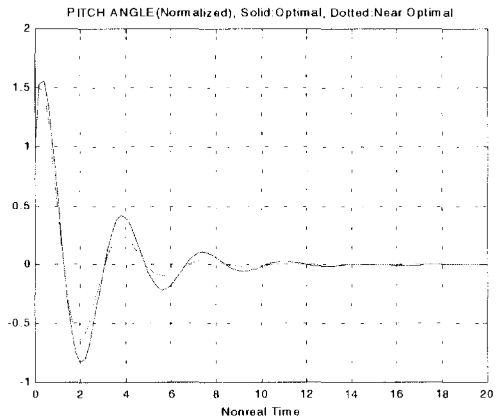

Fig. 2. Pitch angle for Open-Loop system with the zeroth Approximation.

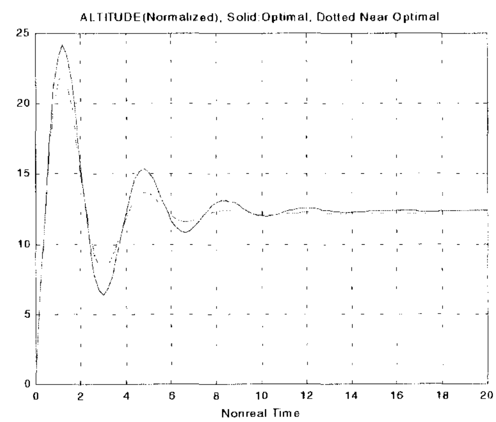

Fig. 3. Altitude for Open-Loop system with the zeroth Approximation.

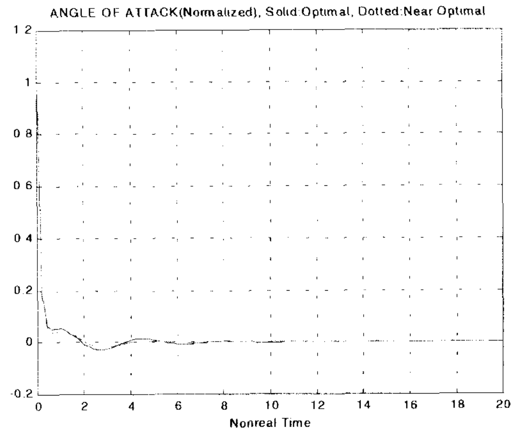

Fig. 4. Angle of Attack for Open-Loop system with the zeroth Approximation.

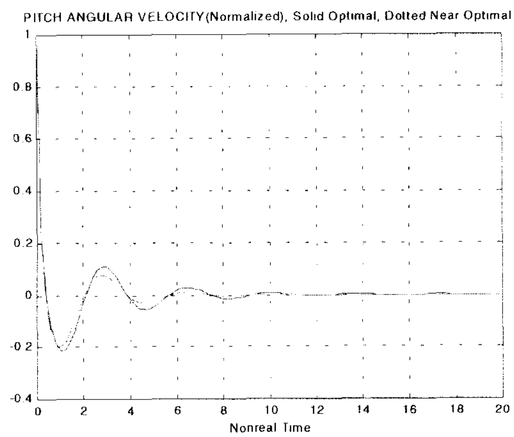

Fig. 5. Pitch angular velocity for Open-Loop system with the zeroth approximation.

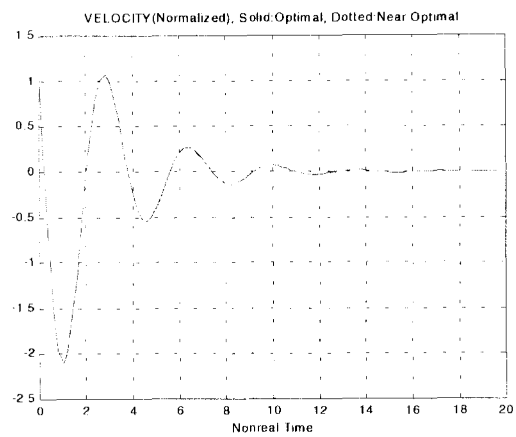

Fig. 6. Velocity for Open-Loop system with the third Approximation.

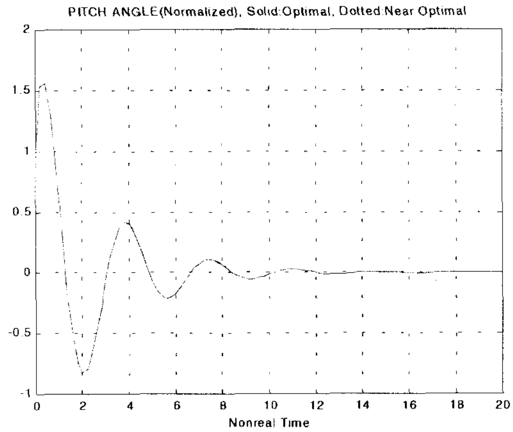

Fig. 7. Pitch angle for Open-Loop system with the third Approximation.

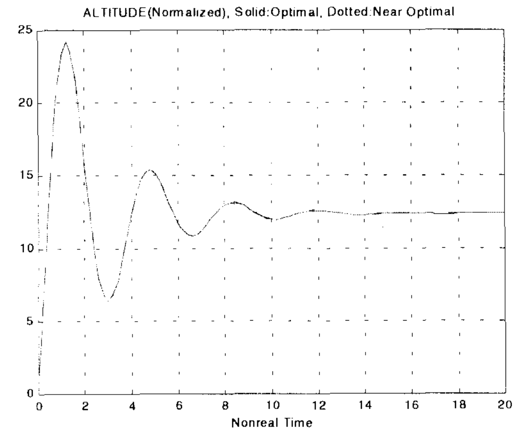

Fig. 8. Altitude for Open-Loop system with the third Approximation.

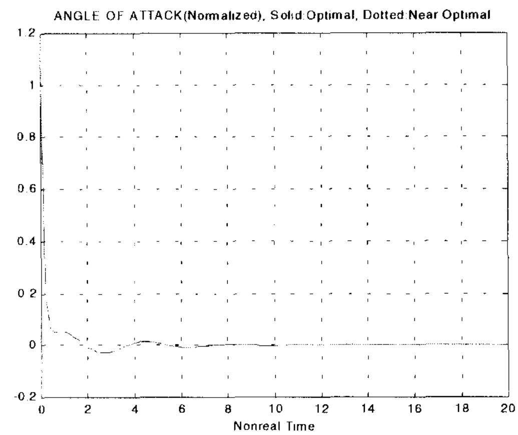

Fig. 9. Angle of attack for Open-Loop system with the third Approximation.

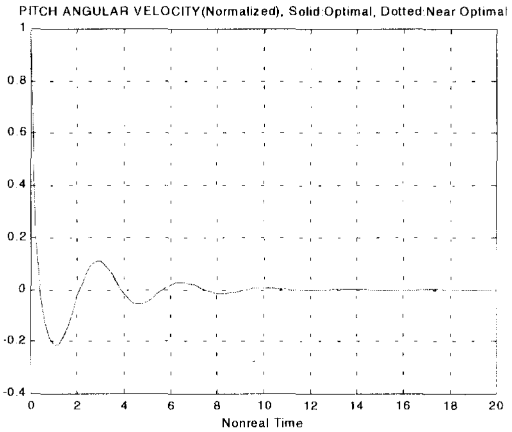

Fig. 10. Pitch Angular Velocity for Open-Loop System with the third Approximation.

V. Conclusion

For the open-loop response of the system, Naidu *et al.* [15]-[17] made the five iterations to obtain the decoupled solution by solving the nonlinear Riccati-type equations (51 and 52) in the discerete system. But we use the three iterations to find the same solution by solving the nonlinear Riccati-type equations (28) in the δ-operator system. In this paper, it is shown that the the δ-operator systems have an improved finite word-length characteristics than the q-operator systems. Also, note that the decoupled solutions of Naidu *et al.* and ours are robust and almost equal to the exact solution.

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