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EQUIVALENT CONDITIONS FOR A DIRECT INJECTIVE MODULE

SU-JEONG CHOI AND CHANG-WOO HAN

ABSTRACT. The purpose of this paper is to find the necessary and sufficient conditions for a module to be a direct injective module. Moreover, we focus on the possibility that a direct injective module can be related with arbitrary module and Hom functor like an injective module.

1. INTRODUCTION

Throughout, R is a ring with unity, all modules are unitary R-modules and all maps are R-maps. A module M is said to be direct injective if given a direct summand N of M with inclusion $i : N \longrightarrow M$ and any monomorphism $f : N \longrightarrow M$, there exists an endomorphism g of an R-module M such that the diagram

$$\begin{array}{ccccc} & & M & & \\ & & i\uparrow & \nwarrow g & \\ O & \longrightarrow & N & \xrightarrow{f} & M \end{array}$$

commutes, i.e., $g \circ f = i$. The concept of a direct injective module as a generalization of a quasi-injective module was introduced by Nicholson [2] in 1976. Xue [3] showed the characterizations of hereditary rings and semisimple rings by using direct projective modules and direct injective modules.

In this paper, we obtain the necessary and sufficient conditions for a module to be a direct injective module. As the results of it, we obtain equivalent conditions for a module to be a direct injective module.

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2. MAIN RESULTS

Theorm 2.1. *A module M is direct injective if and only if for any direct summand N of M and a monomorphism $f : N \longrightarrow A$, there exists a map $f' : A \longrightarrow N$ such that $f' \circ f = I_N$, where A is a submodule of M.*

Proof. Assume that M is a direct injective module. Let N be a direct summand of M, A be a submodule of M, and $g : A \longrightarrow M$ be a monomorphism. Then for each monomorphism $f : N \longrightarrow A$, there exists a map $h \in \mathrm{End}(M)$ which completes the following

$$\begin{array}{ccccc} & & M & & \\ & {}^{i}\nearrow & & \nwarrow^{h} & \\ N & \xrightarrow{\ f\ } & A & \xrightarrow{\ g\ } & M \end{array}$$

as a commutative diagram, i.e., $h \circ g \circ f = i$.

Let $p : M \longrightarrow N$ be the projection map and define a map $k : A \longrightarrow M$ by $k = h \circ g$.

Let $f' = p \circ k$. Then we have

$$\begin{aligned} f' \circ f &= (p \circ k) \circ f \\ &= p \circ (h \circ g) \circ f \\ &= p \circ (h \circ g \circ f) \\ &= p \circ i = I_N, \end{aligned}$$

i.e.,

$$\begin{array}{ccccc} & & M & & \\ & {}^{i}\nearrow\!\swarrow_{p} & {}^{g}\uparrow & \nwarrow^{h} & \\ N & \underset{f'}{\overset{f}{\rightleftarrows}} & A & \xrightarrow{\ g\ } & M. \end{array}$$

Hence, $f' \circ f = I_N$.

Conversely, assume that N is a direct summand of a module M and $i : N \longrightarrow M$ be the inclusion map and let $f' : M \longrightarrow N$ be a map such that $f' \circ f = I_N$, i.e.,

$$\begin{array}{ccccc} & & M & & \\ & & {\scriptstyle i}\uparrow & & \\ O & \longrightarrow & N & \underset{f'}{\overset{f}{\rightleftarrows}} & M. \end{array}$$

Define a map $h : M \longrightarrow M$ by $h = i \circ f'$, then

$$h \circ f = i \circ f' \circ f = i \circ I_N = i.$$

So we have the diagram

$$\begin{array}{ccccc} & & M & & \\ & & {\scriptstyle i}\uparrow & \nwarrow^{h} & \\ O & \longrightarrow & N & \xrightarrow{f} & M \end{array}$$

as a commutative diagram, i.e., $h \circ f = i$. Hence, M is a direct injective module. This completes the proof. □

Theorm 2.2. *A module M is direct injective if and only if, given any direct summand N of M and any map $g : N \longrightarrow M$, for each monomorphism $f : N \longrightarrow M$, there exists a map $h \in \mathrm{End}(M)$ such that the following diagram*

$$\begin{array}{ccccc} & & M & & \\ & & {\scriptstyle g}\uparrow & \nwarrow^{h} & \\ O & \longrightarrow & N & \xrightarrow{f} & M \end{array}$$

commutes, i.e., $h \circ f = g$.

Proof. Assume that a module M is direct injective. Then by Theorem 2.1, there exists a map $f' : M \longrightarrow N$ such that $f' \circ f = I_N$. Therefore define a map $h : M \longrightarrow M$ by $h = g \circ f'$. Then

$$h \circ f = g \circ f' \circ f = g \circ I_N = g.$$

Hence there exists a map $h \in \mathrm{Hom}(M)$ such that $h \circ f = g$.

Conversely, suppose that given any direct summand N of M and any map $g : N \longrightarrow M$, for each monomorphism $f : N \longrightarrow M$, there exists a map $h \in \mathrm{End}(M)$

such that $h \circ f = g$. i.e., the diagram

$$\begin{array}{ccccc} & & M & & \\ & & g\uparrow & \nwarrow h & \\ O & \longrightarrow & N & \xrightarrow{f} & M \end{array}$$

is commutative. If we take an inclusion map $i : N \longrightarrow M$ instead of arbitrary map $g : N \longrightarrow M$, then we have an immediate consequence from the above assumption and the definition of direct injective module. □

Theorm 2.3. *A module M is direct injective if and only if, given an exact sequence*

$$O \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow O$$

for a direct summand A of M and submodules B, C of M, then

$$O \longrightarrow \mathrm{Hom}(C, M) \xrightarrow{F\beta} \mathrm{Hom}(B, M) \xrightarrow{F\alpha} \mathrm{Hom}(A, M) \longrightarrow O$$

is exact sequence.

Proof. Assume that a module M is direct injective and let A be a direct summand of M and B, C be submodules of M. Since $\mathrm{Hom}(, M)$ is a left exact contravariant functor, for an exact sequence

$$O \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow O,$$

to prove that

$$O \longrightarrow \mathrm{Hom}(C, M) \xrightarrow{F\beta} \mathrm{Hom}(B, M) \xrightarrow{F\alpha} \mathrm{Hom}(A, M) \longrightarrow O$$

is exact sequence, it is enough to show that $F\alpha$ is an epimorphism. For arbitrary map $g \in \mathrm{Hom}(A, M)$, since a module M is direct injective, by Theorem 2.1, we have $\alpha' : B \longrightarrow A$ with $\alpha' \circ \alpha = I_A$. Define a map $f : B \longrightarrow M$ by $f = g \circ \alpha'$. Then there is a map $f \in \mathrm{Hom}(B, M)$ such that

$$F\alpha(f) = f \circ \alpha = g \circ \alpha' \circ \alpha = g \in \mathrm{Hom}(A, M).$$

The diagram

$$\begin{array}{ccccc} & & M & & \\ & & g\uparrow & \nwarrow f & \\ O & \longrightarrow & A & \underset{\alpha'}{\overset{\alpha}{\rightleftarrows}} & B \end{array}$$

commutes. Hence $F\alpha$ is an epimorphism.

Conversely, let A be a direct summand of M and B, C be submodules of M. Assume that for an exact sequence

$$O \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow O,$$

the sequence

$$O \longrightarrow \mathrm{Hom}(C, M) \xrightarrow{F\beta} \mathrm{Hom}(B, M) \xrightarrow{F\alpha} \mathrm{Hom}(A, M) \longrightarrow O$$

is exact. If we take M instead of B from the above assumption, then for an arbitrary map $g : A \longrightarrow M$ and each monomorphism $\alpha : A \longrightarrow M$, there exists a map $h \in \mathrm{End}(M)$ such that $h \circ \alpha = g$, i.e.,

$$\begin{array}{ccccc} & & M & & \\ & & g\uparrow & \nwarrow h & \\ O & \longrightarrow & A & \xrightarrow{\alpha} & M. \end{array}$$

Hence by Theorem 2.2, the module M is direct injective. □

From Theorem 2.3, we know that direct injective modules can be defined by Hom functor with some conditions.

Theorm 2.4. *A module M is direct injective if and only if for each monomorphism $f : N \longrightarrow M$, a direct summand N and a module K, any map $g : N \longrightarrow K$ can be extended to a map $h : M \longrightarrow K$ which completes the diagram*

$$\begin{array}{ccccc} & & K & & \\ & & g\uparrow & \nwarrow h & \\ O & \longrightarrow & N & \xrightarrow{f} & M \end{array}$$

commutes, i.e., $h \circ f = g$.

Proof. ($\Rightarrow$) Suppose that M is a direct injective module. Let N be a direct summand of M. Then by Theorem 2.1, for each monomorphism $f : N \longrightarrow M$, there is a map $f' : M \longrightarrow N$ such that

$$f' \circ f = I_N.$$

For an arbitrary module K and any map $g : N \longrightarrow K$, define a map $h : M \longrightarrow K$ by

$$h = g \circ f'.$$

Then

$$h \circ f = g \circ f' \circ f = g \circ I_N = g.$$

Therefore, there is a map $h : M \longrightarrow K$ such that

$$h \circ f = g.$$

Hence, we have the diagram

$$\begin{array}{ccccc} & & K & & \\ & & g\big\uparrow & \nwarrow h & \\ O & \longrightarrow & N & \underset{f'}{\overset{f}{\rightleftarrows}} & M \end{array}$$

as a commutative diagram.

($\Leftarrow$) The converse proof is trivial by taking inclusion map i instead of arbitrary map g. □

Theorm 2.5. *For a module M, let A be a direct summand of M and B, C be submodules of M. M is a direct injective module if and only if, given an exact sequence*

$$O \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow O$$

we have

$$O \longrightarrow \mathrm{Hom}(C,K) \xrightarrow{F\beta} \mathrm{Hom}(B,K) \xrightarrow{F\alpha} \mathrm{Hom}(A,K) \longrightarrow O$$

as an exact sequence, for any module K.

Proof. Assume that M is a direct injective module. Let A be a direct summand of M and B, C be submodules of M. Since Hom($,K$) is a left exact contravariant functor, in order to prove that

$$O \longrightarrow \text{Hom}(C,K) \xrightarrow{F\beta} \text{Hom}(B,K) \xrightarrow{F\alpha} \text{Hom}(A,K) \longrightarrow O$$

is an exact sequence for an exact sequence

$$O \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow O$$

and an arbitrary module K, we will show that $F\alpha$ is an epimorphism, i.e., we want to complete the diagram

$$\begin{array}{ccccc} & & K & & \\ & & g\uparrow & & \\ O & \longrightarrow & A & \xrightarrow{\alpha} & B \end{array}$$

for an arbitrary map $g \in \text{Hom}(A,K)$. Since M is a direct injective module, we have $\alpha' : B \longrightarrow A$ such that $\alpha' \circ \alpha = I_A$. Define a map $f : B \longrightarrow K$ by

$$f = g \circ \alpha'.$$

Then

$$F\alpha(f) = f \circ \alpha = g \circ \alpha' \circ \alpha = g \in \text{Hom}(A,K).$$

Hence, we have the diagram

$$\begin{array}{ccccc} & & K & & \\ & & g\uparrow & \nwarrow f & \\ O & \longrightarrow & A & \underset{\alpha'}{\overset{\alpha}{\rightleftarrows}} & B \end{array}$$

as a commutative diagram. Therefore, $F\alpha$ is an epimorphism.

Conversely, let A be a direct summand of M and C be a submodule of M. Assume that for an exact sequence

$$O \longrightarrow A \xrightarrow{\alpha} M \xrightarrow{\beta} C \longrightarrow O$$

and an arbitrary module K, the sequence

$$O \longrightarrow \text{Hom}(C,K) \xrightarrow{F\beta} \text{Hom}(M,K) \xrightarrow{F\alpha} \text{Hom}(A,K) \longrightarrow O$$

is an exact sequence, i.e., we have the diagram

$$\begin{array}{ccccc} & & K & & \\ & & {\scriptstyle g}\uparrow & \nwarrow & \\ O & \longrightarrow & A & \xrightarrow{\alpha} & M \end{array}$$

commutative. Then for an arbitrary map $g : A \longrightarrow K$ and each monomorphism $\alpha : A \longrightarrow M$, there exists a map $h : M \longrightarrow K$ which completes the following diagram

$$\begin{array}{ccccc} & & K & & \\ & & {\scriptstyle g}\uparrow & \nwarrow^{h} & \\ O & \longrightarrow & A & \underset{\alpha'}{\overset{\alpha}{\rightleftarrows}} & M \end{array}$$

as a commutative diagram, i.e., $h \circ \alpha = g$. This implies that M is a direct injective module. This completes the proof. □

Theorm 2.6. *M is a direct injective module if and only if for every pair of direct summands A, B of M, the injection $i : A \longrightarrow M$, and every monomorphism $f : A \longrightarrow B$, there exists a map $g : B \longrightarrow M$ which completes the following diagram*

$$\begin{array}{ccccc} & & M & & \\ & & {\scriptstyle i}\uparrow & \nwarrow^{g} & \\ O & \longrightarrow & A & \xrightarrow{f} & B \end{array}$$

as a commutative diagram, i.e., $g \circ f = i$.

Proof. ($\Rightarrow$) Suppose that M is a direct injective module. Then for the injection maps $i : A \longrightarrow M$, $i' : B \longrightarrow M$ and each monomorphism $f : A \longrightarrow B$, there exists a map $h \in \mathrm{End}(M)$ which completes the diagram

$$\begin{array}{ccccc} & & M & & \\ & {}^{i}\nearrow & \uparrow{\scriptstyle g} & \nwarrow^{h} & \\ A & \xrightarrow{f} & B & \xrightarrow{i'} & M \end{array}$$

commutes, i.e., $h \circ i' \circ f = i$. Let

$$g = h \circ i'$$

then we have $g \circ f = i$. Therefore, there exists a map $g : B \longrightarrow M$ such that

$$g \circ f = i.$$

This completes the proof of "$\Rightarrow$" part.

($\Leftarrow$) The converse case is omitted since it is the same as the "$\Rightarrow$" part by replacing B with M. □

Through the above long proofs, we know that from Theorem 2.1 to Theorem 2.6, they are equivalent. We want to focus on the possibility that a direct injective module can be related with arbitrary module and Hom functor like an injective module.

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(S.-J. CHOI) DEPARTMENT OF MATHEMATICS, DONG-A UNIVERSITY, HADAN-DONG, SAHA-GU, BUSAN 604-714, KOREA
E-mail address: purity100@hotmail.com

(C.-W. HAN) DEPARTMENT OF MATHEMATICS, DONG-A UNIVERSITY, HADAN-DONG, SAHA-GU, BUSAN 604-714, KOREA
E-mail address: cwhan@mail.donga.ac.kr