J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 8, Number 2 (November 2001), Pages 153–162

ON ε-BIRKHOFF ORTHOGONALITY AND ε-NEAR BEST APPROXIMATION

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ABSTRACT. In this paper, the notion of ε-Birkhoff orthogonality introduced by Dragomir [*An. Univ. Timişoara Ser. Ştiinţ. Mat.* **29** (1991), no. 1, 51–58] in normed linear spaces has been extended to metric linear spaces and a decomposition theorem has been proved. Some results of Kainen, Kurkova and Vogt [*J. Approx. Theory* **105** (2000), no. 2, 252–262] proved on e-near best approximation in normed linear spaces have also been extended to metric linear spaces. It is shown that if (X, d) is a convex metric linear space which is pseudo strictly convex and M a boundedly compact closed subset of X such that for each $\varepsilon > 0$ there exists a continuous ε-near best approximation $\phi : X \to M$ of X by M then M is a chebyshev set.

1. INTRODUCTION

The notion of Birkhoff orthogonality (cf. [2]) in normed linear spaces was used to prove some results on best approximation (see [11]). This notion of orthogonality was extended to metric linear spaces and some results on best approximation were proved in Narang [8]. A generalization of Birkhoff orthogonality, called ε-Birkhoff orthogonality was introduced by Dragomir [4] in normed linear spaces and this notion was used to prove a decomposition theorem (cf. [4, Theorem 3]). We extend this notion of ε-Birkhoff orthogonality and prove the decomposition theorem in metric linear spaces (see Theorem 1).

It was shown by Kainen-Kurkova-Vogt [6] that the existence of a continuous ε-near best approximation in a strictly convex normed linear spaces X and taking values in a suitable subset M implies that M has the unique best approximation property. We extend this result to convex metric linear spaces (see Theorem 2). We

Received by the editors June 26, 2001.
2000 *Mathematics Subject Classification.* 46B20, 41A65.
Key words and phrases. Proximinal set, Chebyshev set, approximatively compact set, pseudo/strict convexity, ε-Birkhoff orthogonality, ε-near best approximation.
*The first author is thankful to U. G. C., India for awarding her Teacher-Fellowship.

also extend some other results on ε-near best approximation proved in [6] to metric linear spaces (see Theorem 3 and its corollaries).

2. PRELIMINARIES

To start with, we recall a few definitions. Let A and B be non empty sets. A mapping $f : A \to B$ is called a *retraction* of A onto B if

(i) B is a subset of A.
(ii) $f(x) = x \quad \forall\, x \in B$.

A non-empty set K of a linear space (X, d) is said to be *convex* if $\alpha x + (1-\alpha)y \in K$ for all $x, y \in K$ and $\alpha \in [0,1]$.

Let (X, d) be a metric space, M a subset of and $P_M(x) = \{m \in M : d(x,m) = d(x,M)\}$. An element of $P_M(x)$ is called *a best approximation to x in M*. If $P_M(x)$ is non empty for each $x \in X$ then M is called a *proximinal* set. If $P_M(x)$ is a singleton for each x in X then M is called a *Chebyshev* set.

A set G in a metric space (X, d) is said to be *boundedly compact* (Klee [7]) if every bounded sequence in G has a subsequence converging to a point of the space X. Equivalently, if the closure of $G \cap B$ is compact for each closed ball B in X.

A set G in a metric space (X, d) is said to be *approximately compact* (Efimov-Steckin [5]) if for every $x \in X$ and every sequence $< g_n >$ in G with

$$\lim_{n \to \infty} d(x, g_n) = d(x, G)$$

there exists a subsequence $< g_n >$ converging to an element of G.

An approximatively compact set in a metric space is proximinal (Efimov-Steckin [5]) but a proximinal set need not be approximatively compact (Singer [11, p. 389]).

Given a non-empty subset A of a metric space (X, d) and a positive number ε, *ε-near best approximation of A by M* is a map $\phi : A \to M$ such that

$$d(x, \phi(x)) \leq d(x, M) + \varepsilon \quad \text{for all } x \text{ in } A.$$

A metric linear space (X, d) over a field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) is said to be *pseudo strictly convex* (P.S.C.) if given $x \neq 0, y \neq 0, d(x+y, 0) = d(x, 0) + d(y, 0)$ implies $y = tx$ for some $t > 0$.

The notion of pseudo strict convexity in a metric linear space is a variant of strict convexity (see e.g. [1]) and was introduced and discussed by Sastry-Naidu-Kishore

[9] and [10]. For normed linear spaces, strict convexity and pseudo strict convexity are equivalent (see e.g. [3, p. 122]).

A metric linear space (X, d) is said to be *convex* if for all $x, y \in X, \lambda \in [0,1]$

$$d(u, \lambda x + (1-\lambda)y) \leq \lambda d(u,x) + (1-\lambda)d(u,y) \text{ for all } u \in X.$$

Clearly every normed linear space is a convex metric linear space.

For a metric linear space (X, d) over a field $\mathbb{K}$ and $\varepsilon \in [0,1]$, an element $x \in X$ is said to be *ε-Birkhoff orthogonal* over $y \in X$ [4] if $d(x + \alpha y, 0) \geq (1-\varepsilon)d(x,0)$ for all $\alpha \in \mathbb{K}$ and we denote it by $x \perp y(\varepsilon - B)$.

If A is a non-empty subset of X then by *ε-Birkhoff orthogonal complement* $A^{\perp}(\varepsilon - B)$, we denote the set of all elements which are ε-Birkhoff orthogonal to A, i.e.,

$$A^{\perp}(\varepsilon - B) = \{y \in X : y \perp x(\varepsilon - B) \text{ for all } x \in A\}.$$

Since $A^{\perp}(\varepsilon - B) = \{y \in X : y \perp x(\varepsilon - B) \text{ for all } x \in A\}$, $O \in A^{\perp}(\varepsilon - B)$ as $O \perp x(\varepsilon - B)$ for all $x \in A(d(O + \alpha x, O) \geq (1-\varepsilon)d(0,0)$ for all $x \in A)$.

We claim that $A \cap A^{\perp}(\varepsilon - B) \subseteq \{0\}$ for every $\varepsilon \in [0,1[$.

Let $y \in A \cap A^{\perp}(\varepsilon - B)$. Then $y \in A$ and $y \in A^{\perp}(\varepsilon - B)$. Now

$$\begin{aligned} y \in A^{\perp}(\varepsilon - B) &\Rightarrow y \perp x(\varepsilon - B) \text{ for all } x \in A. \\ &\Rightarrow y \perp x(\varepsilon - B) \\ &\Rightarrow d(y + \alpha y) \geq (1-\varepsilon)\, d(y,0) \quad \text{for all} \quad \alpha \in \mathbb{K} \\ &\Rightarrow 0 \geq (1-\varepsilon)\, d(y,0) \text{ by taking } \alpha = -1 \\ &\Rightarrow \varepsilon d(y,0) \geq 0 \\ &\Rightarrow y = 0 \end{aligned}$$

and so $A \cap A^{\perp}(\varepsilon - B) \subseteq \{0\}$.

Now we prove a lemma to be used in the proof of next decomposition theorem.

Lemma 1. *Let G be a closed linear subspace of a metric linear space $(X, d), G \neq X$. Then for any $\varepsilon \in]0,1[$, the ε-Birkhoff orthogonal complement of G is non-zero.*

Proof. Let $Y \in X \setminus G$. Since G is closed, $d(Y, G) = r > 0$. Thus there exists $Y_\varepsilon \in G$ such that

$$r \leq d(y, y_\varepsilon) \leq r/(1-\varepsilon),$$

i.e.,

$$r \geq d(y, y_\varepsilon, 0) \leq r/(1-\varepsilon).$$

Put $x_\varepsilon = y - y_\varepsilon$, we have $x_\varepsilon \neq 0$ and for all $y \in G$ and $\lambda \in \mathbb{K}$, we obtain

$$\begin{aligned} d(x_\varepsilon + \lambda y_1, 0) &= d(y - y_\varepsilon + \lambda y_1, 0) \\ &= d(y, y_\varepsilon - \lambda y_1) \\ &\geq r \\ &\geq (1-\varepsilon) d(x_\varepsilon, 0), \end{aligned}$$

i.e., $x_\varepsilon \perp y_1(\varepsilon - B)$ and so $x_\varepsilon \in G^\perp(\varepsilon - B)$.

The following decomposition theorem was proved in normed linear spaces in [6]. We extended this to metric linear spaces. □

Theorm 1. *Let G be a closed linear subspace of a metric linear space (X, d). Then for any $\varepsilon \in]0,1[$ We have $X = G \bigoplus G^\perp(\varepsilon - B)$.*

Proof. Suppose $G \neq X$ and $x \in X$. If $x \in G$, then $x = x + 0 \in G + G^\perp(\varepsilon - B)$. If $x \notin G$, then there exists an element $y_\varepsilon \in G$ such that

$$0 < r = d(x, G) \leq d(x, y_\varepsilon) \leq r/(1-\varepsilon)$$

Since $x_\varepsilon = x - y_\varepsilon \in G^\perp(\varepsilon - B)$ (by the above lemma), we have

$$x = y_\varepsilon + x_\varepsilon \in G + G^\perp(\varepsilon - B).$$

Since $\{0\} \subseteq G \cap G^\perp(\varepsilon - B) \subseteq \{0\}$; we get, $X = G \bigoplus G^\perp(\varepsilon - B)$. □

The following theorem shows that the continuity of ε-near best approximation is enough to guarantee the uniqueness of best approximation in convex metric linear spaces which are pseudo strictly convex.

Theorm 2. *Let (X, d) be a convex metric linear space which is pseudo strictly convex and M a boundedly compact closed subset of X. Suppose that for each $\varepsilon > 0$, there exists a continuous ε-near best approximation $\phi : X \dashrightarrow M$ of X by M then M is a Chebyshev set.*

Proof. Since a boundedly compact closed set in a metric space is proximinal (see [11, p. 383]), $P_M(x)$ is non-empty for each $x \in X$. Let $m \in P_M(x)$. We choose a point $x_0 \in X$ with $r = d(x_0, M) > 0$. Given an integer $n \geq 1$, let $\phi_n : X \dashrightarrow M$ be continuous with

$$d(x, \phi_n(x)) \leq d(x, M) + 1/n \quad \text{for all } x \text{ in } X.$$

Then $\phi_n : B(x_0, r) \dashrightarrow M$ and $d(\phi_n(x), x_0) \geq r$ for all x in the closed ball $B(X_0, r)$.

Let π be a mapping defined by

$$\pi(x) = x_0 + r(x - x_0)/d(x, x_0), \quad x \in X.$$

We claim that

$$\pi = \{x : d(x, x_0) \geq r\} \dashrightarrow \{x : d(x, x_0) = r\} \equiv \partial B(x_0, r)$$

is a radial retraction, i.e.,

(i) $d(\pi(x), x_0) = r$,

(ii) for $x \in \partial B(x_0, r)$, $\pi(x) = x$.

Consider

$$\begin{aligned} d(\pi(x), x_0) &= d(x_0 + r(x - x_0)/d(x - x_0), x_0) \\ &= d(r(x - x_0)/d(x, x_0), 0), \\ &\leq \frac{r}{d(x, x_0)} d(x - x_0, 0), \text{ by the convexity of } (x, d) \\ &= \frac{r}{d(x, x_0)} d(x, x_0) \\ &= r \end{aligned}$$

Thus,

$$d(\pi(x), x_0) \leq r \tag{$*$}$$

As $\pi(x) = x_0 + [r(x - x_0)]/d(x, x_0) = rx/d(x, x_0) + [(1 - r)/d(x, x_0)]x_0$, i.e., $\pi(x) \in [x, x_0]$ and so

$$d(x, \pi(x)) + d(\pi(x), x_0) = d(x, x_0) \tag{$**$}$$

Now

$$\begin{aligned} d(\pi(x), x) &= d(x_0 + [r(x - x_0)]/d(x, x_0), x) \\ &= d(r(x - x_0)/d(x, x_0), x - x_0) \\ &\leq [1 - r/d(x, x_0)]d(0, x - x_0), \text{ by convexity of } X \\ &= [1 - r/d(x, x_0)]d(x, x_0) \\ &= d(x, x_0) - r \end{aligned}$$

Hence, $-d(\pi(x), x) \geq r - d(x, x_0)$.

So $(**)$ implies $d(\pi(x), x_0) \geq d(x, x_0) + [r - d(x, x_0)] = r$, i.e.,

$$d(\pi(x), x_0) \geq r \tag{$***$}$$

Combining $(*)$ and $(***)$, we get $d(\pi(x), x_0) = r$.

For $x \in \partial B(x_0, r)$, i.e., $d(x, x_0) = r$. We get

$$\pi(x) = x_0 + r(x - x_0)/d(x, x_0) = x, i.e., \pi(x) = x \,\forall\, x \in \partial B(x_0, r).$$

Thus $\pi : \{x : d(x, x_0) \geq r\} \dashrightarrow \{x : d(x, x_0) = r\}$ is a radical retraction and $\pi_0\phi_n : B(x_0, r) \dashrightarrow \partial B(x_0, r)$.

Now $\phi_n(x)$ for x in $B(x_0, r)$, satisfies

$$\begin{aligned} d(\phi_n(x), x_0) &\leq (x, M) + 1/n + d(x, x_0) \\ &\leq d(x, x_0) + d(x_0, M) + 1/n + d(x, x_0) \\ &= d(x_0, M) + 1/n + 2d(x, x_0) \\ &\leq 3r + 1 \end{aligned} \tag{1}$$

Hence $\phi_n(B(x_0, r)) \subseteq M \cap B(x_0, 3r + 1)$ and $\phi_n(B(x_0, r))$ is a bounded subset of M. So $\mathrm{cl}(\phi_n(B(x_0, r)))$ is compact since M is given to be boundedly compact.

Let $P : X \dashrightarrow X$ be the reflection through x_0, i.e.,

$$P(y) = x_0 + (x_0 - y). \tag{2}$$

Then $\mathrm{cl}(P_0\pi_0\phi_n(B(x_0, r))) = P_0\pi(\mathrm{cl}\ \phi_n(B(x_0, r)))$ is a compact subset of $\partial B(x_0, r)$ and $P_0\pi_0\phi_n$ is a continuous function from $B(x_0, r)$ into $\partial B(x_0, r)$.

Since in a convex metric linear space $B(x_0, r)$ is convex, by Rothe's theorem, a version of Schauder's theorem (see [12], p. 27) for each n, $P_0\pi_0\phi_n$ has a fixed point x_n(say) in $B(x_0, r)$ Thus

$$x_n = P_0\pi_0\phi_n(x_n) = P_0(\pi_0\phi_n(x_n)) = 2x_0 - (\pi_0\phi_n(x_n)) \qquad \text{(using (2))}$$

and so $(\pi_0\phi_n)(x_n) = 2x_0 - x_n$.

We claim that $x_n, x_0, 2x_0 - x_n = \pi_0\phi_n(x_n)$ and $\phi_n(xn)$ are consecutive collinear points.

Since $2x_0 - x_n = \pi_0\phi_n(x_n)$ implies $2x_0 - x_n - \pi_0\phi_n(x_n) = 0$, i.e., $\alpha x_0 + \beta x_n + \gamma\pi_0\phi_n(x_n) = 0$ with $\alpha + \beta + \gamma = 0$, i.e., $x_0 = (\beta x_n + \gamma \cdot \pi_0\phi_n(xn))/\beta + \gamma$.

Also, by definition of $\pi(x)$, we have

$$\begin{aligned} \pi(\phi_n(x_n)) &= x_0 + (r(\phi_n(x_n) - x_0))/d(\phi_n(x_n), x_0) \\ &= r\phi_n(x_n)/d(\phi_n(x_n), x_0) + (1 - r/[d(\phi_n(x_4), x_0])x_0 \end{aligned}$$

$$\begin{aligned}
&\Rightarrow \quad 1 \cdot \pi_0\phi_n(x_n) - r\phi_n(x_n)/d(\phi_n(x_n), x_0) - (1 - r/d(\phi_n(x_n, x_0))x_0 = 0 \\
&\Rightarrow \quad \alpha \cdot \pi_0\phi_n(x_n) + \beta\phi_n(x_n) + \gamma \cdot x_0 = 0 \\
&\qquad\qquad \text{with } \alpha + \beta + \gamma = 1 - r/d(\phi_n(x_n), x_0) - 1 + r/d(\phi_n(x_n), x_0) = 0 \\
&\Rightarrow \quad \pi_0\phi_n(x_n) = (\beta\phi_n(x_n) + \gamma \cdot x_0)/(\beta + \gamma)
\end{aligned}$$

and so

$$\begin{aligned}
d(\phi_n(x_n), x_n) &\geq d(\pi_0\phi_n(x_n), x_n) \\
&= d(2x_0 - x_n, x_n) \\
&= d(x_n, x_0) + d(x_0, 2x_0 - x_n) \\
&\qquad\qquad (\text{as points } x_n, x_0 \text{ and } 2x_0 - x_n \text{ are collinear}) \\
&= d(x_n, x_0) + d(x_n, x_0) \\
&= 2d(x_n, x_0)
\end{aligned}$$

Now we prove that $d(x_n, x_0) = r$.

Since $\pi_0\phi_n : B(x_0, r) \dashrightarrow \partial B(x_0, r)$ and $x_n \in B(x_0, r)$ implies $(\pi_0\phi_n)(x_n) \in \partial B(x_0, r)$ and so $d(\pi_0\phi_n(x_n), x_0) = r$, i.e., $d(2x_0 - x_n, x_n) = r$, i.e., $d(x_n, x_0) = r$. Hence $d(\phi_n(x_n), x_n) \geq 2r$. In addition for each m in M,

$$\begin{aligned}
d(x_n, m) &\geq d(x_n, \phi_n(x_n)) - 1/n && \text{(using (1))} \\
&\geq 2r - 1/n && (3)
\end{aligned}$$

Again M is boundedly compact, the sequence $\{\phi_n(x_n)\}$ in $M \cap B(x_0, 3r + 1)$ has a convergent subsequence with limit u in X. Then the sequence $\{P_0\pi_0\phi_n(x_n)\}$ has a convergent subsequence with limit $P_0\pi(u) = x_\infty \in \partial B(x_0, r)$. Moreover, for each m in M,

$$\begin{aligned}
d((x_\infty - x_0) + (x_0 - m), 0) &= d(x_\infty - m, 0) \\
&= d(x_\infty, m) \\
&\geq 2r \qquad \text{(using (3))} && (4)
\end{aligned}$$

If m is in $P_M(x_0)$, then $d(x_0, m) = d(x_0, M) = r$. Also $d(x_\infty, x_0) = r$ as $x_\infty \in \partial B(x_0, r)$. So

$$\begin{aligned} d((x_\infty - x_0) + (x_0 - m), 0) &= d(x_\infty - x_0, m - x_0) \\ &\leq d(x_\infty - x_0, 0) + d(m - x_0, 0) \\ &= r + r \\ &= 2r \end{aligned}$$

implies

$$d((x_\infty - x_0) + (x_0 - m), 0) \leq 2r \tag{5}$$

Combining (4) and (5) we have

$$\begin{aligned} d((x_\infty - x_0) + (x_0 - m), 0) &= 2r \\ &= r + r \\ &= d(x_\infty - x_0, 0) + d(x_0 - m, 0) \end{aligned} \tag{6}$$

Since (X, d) is pseudo strictly Convex, (6) implies $x_\infty - x_0 = t(x_0 - m)$ for same $t > 0$, i.e., $m = [(1+t)x_0 - x_\infty]/t$ implying $P_M(x_0) = [(1+t)x_0 - x_\infty]/t$ for some $t > 0$. Hence M is Chebyshev. □

In strictly convex normed linear spaces this theorem was proved by Kainen-Kurkova-Vogt [6] and the above proof is an extension of the one given in [6].

Corollary 1. *Let (X, d) be a convex metric linear space, M a boundedly compact subset of X and x an element of X with $r = d(x_0, M) > 0$. Suppose that for some ε, with $0 < \varepsilon < 2r$ there exists a continuous ε-near best approximation $\phi : B(x, r) \dashrightarrow M$ of $B(x, r)$ by M. Then there exists a point x_1 in $\partial B(x, r)$ such that $d(x_1, m) \geq 2r - \varepsilon$.*

Proof. The proof is contained in the first part of the proof of Theorem 3 (upto equation (3)).

If M is an approximatively compact set in a metric space, then $P_M(x)$ is compact for each x in X. Indeed, any sequence $\{m_n\}$ in $P_M(x)$ is a sequence in M with $d(x, m_n) = d(x, M)$ and by the definition of approximative compactness, has a convergent subsequence with limit in M and hence in $P_M(x)$. □

Using this, we have:

Theorm 3. *Let M be an approximatively compact set in a metric linear space (X,d) and x an element of X. Suppose that for each $\varepsilon > 0$, there is a continuous ε-near best approximation $\phi_\varepsilon : \{x\} * P_M(x) \dashrightarrow M$ of $\{x\} * P_M(x)$ by M. Then $P_M(x)$ is connected.*

For normed linear spaces the proof of Theorem 3 is given in [6] and that proof can easily be extended to metric linear spaces.

Corollary 2. *Let (X,d) be a metric linear space and M an approximately compact subset of X which is countably proximinal (i.e., $P_M(x)$ is non-empty and countable for each x in X). Suppose that for each $\varepsilon > 0$ there exists a continuous ε-near best approximation $\phi : X \to M$ of X by M. Then M is a Chebyshev set.*

Proof. By Theorem 3, for each $x, P_M(x)$ is connected and since the only countable connected set is a singleton, M is Chebyshev. □

Corollary 3. *Let (X,d) be a metric linear space, M a closed, boundedly compact subset of X, and x an element of X with $r = d(x,M) > 0$. If for each $\varepsilon > 0$, there exists a continuous ε-near best approximation $\phi : B(x,r) \dashrightarrow M$ of $B(x,r)$ by M then $P_M(x)$ is connected.*

Proof. Since a closed, boundedly compact subset is approximatively compact (Singer [11, p. 383]), the proof follows from Theorem 3. □

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