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CERTAIN IDENTITIES ASSOCIATED WITH GENERALIZED HYPERGEOMETRIC SERIES AND BINOMIAL COEFFICIENTS

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ABSTRACT. The main object of this paper is to present a transformation formula for a finite series involving ${}_3F_2$ and some identities associated with the binomial coefficients by making use of the theory of Legendre polynomials $P_n(x)$ and some summation theorems for hypergeometric functions ${}_pF_q$. Some integral formulas are also considered.

1. INTRODUCTION AND PRELIMINARIES

There have been many transformation formulas for the generalized hypergeometric function ${}_pF_q$ (cf. Bailey [1], Whipple [6]) with p numerator and q denominator parameters defined by

$$
(1.1) \qquad {}_pF_q\left[\begin{matrix}\alpha_1, \cdots, \alpha_p\,; \\ \beta_1, \cdots, \beta_q\,;\end{matrix}\; z\right] = {}_pF_q(\alpha_1, \cdots, \alpha_p\,; \beta_1, \cdots, \beta_q\,; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!},
$$

where $(\alpha)_n$ denotes the Pochhammer symbol (or the shifted factorial) defined by

$$
(1.2) \qquad \alpha_n := \begin{cases} 1 & (n = 0) \\ \alpha(\alpha+1)\cdots(\alpha+n-1) & (n \in \mathbb{N} := \{1, 2, 3, \cdots\}), \end{cases}
$$

which can also be rewritten in the form:

$$
(1.3) \qquad (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}),
$$

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Γ being the well-known Gamma function whose Euler's integral is, among several equivalent forms, given by

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt \quad (\Re(z) > 0), \tag{1.4}$$

which is often called the *second Eulerian integral*, whereas the familiar Beta function $B(\alpha,\, \beta)$ defined by

$$B(\alpha,\, \beta) = \int_0^1 t^{\alpha-1}\,(1-t)^{\beta-1}\, dt \quad (\Re(\alpha) > 0;\, \Re(\beta) > 0) \tag{1.5}$$

is often referred to as the *first Eulerian integral.*

There is a well-known relationship between the first and second Eulerian integrals:

$$B(\alpha,\, \beta) = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{1.6}$$

The binomial coefficient, in view of (1.2), may be expressed as

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} = \frac{(-1)^n\,(-\alpha)_n}{n!} \tag{1.7}$$

or, equivalently, as

$$\binom{\alpha}{n} = \frac{\Gamma(\alpha+1)}{n!\,\Gamma(\alpha-n+1)}. \tag{1.8}$$

It follows from (1.7) and (1.8) that

$$\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)} = (-1)^n\,(-\alpha)_n, \tag{1.9}$$

which, for $\alpha = \beta - 1$, yields

$$\frac{\Gamma(\beta-n)}{\Gamma(\beta)} = \frac{(-1)^n}{(1-\beta)_n} \quad (\beta \neq 0,\, \pm 1,\, \pm 2, \cdots). \tag{1.10}$$

Equations (1.3) and (1.10) suggest the definition (cf. Srivastava and Manocha [5, p. 22]):

$$(\beta)_{-n} = \frac{(-1)^n}{(1-\beta)_n} \quad (n \in \mathbb{N};\, \beta \neq 0,\, \pm 1,\, \pm 2, \cdots). \tag{1.11}$$

Equation (1.3) also yields

$$(\alpha)_{m+n} = (\alpha)_m(\alpha+m)_n, \tag{1.12}$$

which,in conjunction with (1.11), gives

$$(\alpha)_{n-k} = \frac{(-1)^k\,(\alpha)_n}{(1-\alpha-n)_k} \quad (n,\, k \in \mathbb{N}_0;\, 0 \leq k \leq n). \tag{1.13}$$

For $\alpha = 1$, we have

$$(n-k)! = \frac{(-1)^k \, n!}{(-n)_k} \qquad (n,\, k \in \mathbb{N}_0;\ 0 \leq k \leq n)\,. \tag{1.14}$$

In view of the definition (1.2), it is not difficult to show that

$$(\alpha)_{2n} = 2^{2n} \left(\frac{1}{2}\alpha\right)_n \left(\frac{1}{2}\alpha + \frac{1}{2}\right)_n \qquad (n \in \mathbb{N}_0), \tag{1.15}$$

which follows also from Legendre's duplication formula for the Gamma function:

$$\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\,\Gamma(z)\,\Gamma\left(z + \frac{1}{2}\right) \qquad \left(z \neq 0,\ -\frac{1}{2},\ -1,\ -\frac{3}{2},\ \cdots\right). \tag{1.16}$$

The Legendre polynomial $P_n(x)$ is defined by the generating function

$$(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n, \tag{1.17}$$

in which $(1 - 2xt + t^2)^{-\frac{1}{2}}$ denotes the particular branch which converges to 1 as $t \to 0$. It is easy to get from (1.17) that

$$P_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k \, (\frac{1}{2})_{n-k} \, (2x)^{n-2k}}{k!\,(n-2k)!}, \tag{1.18}$$

from which it follows that $P_n(x)$ is a polynomial of degree precisely n in x.

Here we aim at giving a transformation formula for a finite series involving ${}_3F_2$ and some identities associated with the binomial coefficients by making main use of the theory of Legendre polynomials $P_n(x)$ and some summation theorems for the hypergeometric functions ${}_pF_q$. Some integral formulas are also considered.

2. A Transformation Formula

Consider the following integral (cf. Poole [2]):

$$I := \int_{-1}^{1} (1+x)^{m+n}\, P_m(x)\, P_n(x)\, dx, \tag{2.1}$$

where $P_n(x)$ denotes the Legendre polynomial defined by (1.17). By substituting the Rodrigues' formula for $P_n(x)$ (cf. Rainville [3, p. 162]):

$$P_n(x) = \frac{1}{2^n\, n!}\, D_x^n \left[(x-1)^n\,(x+1)^n\right] \qquad \left(D_x := \frac{d}{dx}\right) \tag{2.2}$$

for the integrand in (2.1) and letting $x = 2y - 1$ in the resulting equation, we obtain

$$I = \frac{2^{m+n+1}}{m!\,n!}\, J, \tag{2.3}$$

where, for convenience,

$$J := \int_0^1 y^{m+n} \{D_y^n (y-1)^n y^n\} \{D_y^m (y-1)^m y^m\} \, dy \qquad \left(D_y := \frac{d}{dy}\right). \tag{2.4}$$

Integrating m times by parts in (2.4), we have

$$J = \int_0^1 y^m (1-y)^m D_y^m \{y^{m+n} D_y^n (y-1)^n y^n\} \, dy. \tag{2.5}$$

Applying Leibniz's rule for differentiation to the integrand in (2.4), we obtain

$$J = (-1)^{m+n} m!\, n! \sum_{k=0}^{m} (-1)^k \binom{m}{k}^2 \sum_{j=0}^{n} (-1)^j \binom{n}{j}^2 \int_0^1 y^{m+n+k+j} (1-y)^{m+n-k-j} \, dy,$$

which, upon considering (1.5) and (1.6) for the integral, yields

$$\begin{aligned} J = (-1)^{m+n} m!\, n! \\ \cdot \sum_{k=0}^{m} (-1)^k \binom{m}{k}^2 \sum_{j=0}^{n} (-1)^j \binom{n}{j}^2 \frac{(m+n+k+j)!\,(m+n-k-j)!}{(2m+2n+1)!}. \end{aligned} \tag{2.6}$$

By making use of (1.7) through (1.15), we readily obtain

$$\begin{aligned} J = \frac{(-1)^{m+n} m!\, n!\, (1)_{m+n}}{2^{2(m+n)} \left(\frac{3}{2}\right)_{m+n}} \\ \cdot \sum_{k=0}^{m} \binom{m}{k}^2 \frac{(m+n+1)_k}{(-m-n)_k} \, {}_3F_2 \left[\begin{matrix} -n, \, -n, \, m+n+k+1\,; \\ 1, \quad k-m-n\,; \end{matrix} \; 1 \right]. \end{aligned} \tag{2.7}$$

On the other hand, by first applying Leibniz's rule to the innermost differentiation of the integrand in (2.5), we get

$$J = n! \int_0^1 y^m (1-y)^m \sum_{j=0}^{n} \binom{n}{j}^2 \{D_y^m (y-1)^{m+2n-j} y^j\} \, dy,$$

which, upon appealing again the Leibniz's rule to the integrand, gives

$$\begin{aligned} J = n! \sum_{j=0}^{n} \binom{n}{j}^2 \sum_{k=0}^{m} (-1)^{j-k} \binom{m}{k} \frac{(m+2n-j)!}{(2n+k-j)!} \frac{j!}{(j-k)!} \\ \cdot \int_0^1 (1-y)^{m+j-k} y^{m+2n+k-j} \, dy. \end{aligned} \tag{2.8}$$

If we employ (1.5) and (1.6) for the integral in (2.8), similarly as in getting (2.7), we find that

$$J = \frac{m!\,n!\,(1+2n)_m\,(1+m+n)_n}{2^{2(m+n)}\,(\frac{3}{2})_{m+n}} \cdot \sum_{j=0}^{n} \binom{n}{j}^2 \frac{(1+m)_j\,(-2n)_j}{\{(-m-2n)_j\}^2}\,{}_3F_2\left[\begin{matrix} -j,\, -m,\, m+2n-j+1\,; \\ 2n-j+1,\, -m-j\,; \end{matrix}\, 1\right]. \tag{2.9}$$

Finally, equating (2.7) and (2.9), we obtain a transformation formula for the series involving ${}_3F_2$:

$$\begin{aligned} &\sum_{k=0}^{m} \binom{m}{k}^2 \frac{(m+n+1)_k}{(-m-n)_k}\,{}_3F_2\left[\begin{matrix} -n,\, -n,\, m+n+k+1\,; \\ 1,\quad k-m-n\,; \end{matrix}\, 1\right] \\ &= (-1)^{m+n}\,\frac{(2n+1)_m\,(m+n+1)_n}{(m+n)!} \\ &\quad \cdot \sum_{k=0}^{n} \binom{n}{k}^2 \frac{(m+1)_k\,(-2n)_k}{\{(-m-2n)_k\}^2}\,{}_3F_2\left[\begin{matrix} -k,\, -m,\, m+2n-k+1\,; \\ 2n-k+1,\, -m-k\,; \end{matrix}\, 1\right] \\ &\qquad (m,\, n \in \mathbb{N}_0). \end{aligned} \tag{2.10}$$

3. Finite Series involving Binomial Coefficients

By applying Leibniz's rule for differentiation to (2.2), we get

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 \left(\frac{x-1}{2}\right)^{n-k} \left(\frac{x+1}{2}\right)^{k}. \tag{3.1}$$

The following integral

$$\int_{-1}^{1} (1+x)^{\alpha-1}\,(1-x)^{\beta-1}\,dx = 2^{\alpha+\beta-1}\,B(\alpha,\,\beta) \quad (\Re(\alpha) > 0;\, \Re(\beta) > 0) \tag{3.2}$$

can be evaluated by setting $x = 2t - 1$ and using the definition (1.5) of $B(\alpha,\,\beta)$.

From the definition of $P_n(x)$ in (1.17), we have

$$\begin{aligned} (1-2xt+t^2)^{-\frac{1}{2}} &= [(1-t)^2 - 2t\,(x-1)]^{-\frac{1}{2}} \\ &= (1-t)^{-1}\left[1 - \tfrac{2t(x-1)}{(1-t)^2}\right]^{-\frac{1}{2}}, \end{aligned}$$

which, upon employing the binomial theorem:

$$(1-z)^{-\alpha} = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!}\,z^k \quad (|z| < 1;\, \alpha \in \mathbb{C}), \tag{3.3}$$

yields

$$\tag{3.4}\begin{aligned}\sum_{n=0}^{\infty} P_n(x)\, t^n &= \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k\, 2^k\, t^k\, (x-1)^k}{k!\,(1-t)^{2k+1}} \\ &= \sum_{n,\,k=0}^{\infty} \frac{(\frac{1}{2})_k\,(2k+1)_n\, 2^k\,(x-1)^k\, t^{n+k}}{k!\,n!}.\end{aligned}$$

The last part of (3.4) can be written as follows:

$$\sum_{n=0}^{\infty} P_n(x)\, t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^k\,(-n)_k\,(n+1)_k\,(x-1)^k}{2^k\,(k!)^2}\, t^n,$$

which, upon equating the coefficients of t^n, gives

$$\tag{3.5} P_n(x) = {}_2F_1\left[\begin{matrix}-n,\; n+1\,; \\ 1\,;\end{matrix}\; \frac{1-x}{2}\right].$$

By using the definition (1.1) of ${}_pF_q$ and (3.2), we obtain an integral formula involving ${}_{p+1}F_p$:

$$\tag{3.6}\begin{aligned}&\int_{-1}^{1} (1+x)^{\alpha-1}\,(1-x)^{\beta-1}\; {}_{p+1}F_p\left[\begin{matrix}\alpha_1, \cdots, \alpha_{p+1}\,; \\ \beta_1, \cdots, \beta_p\,;\end{matrix}\; \frac{1-x}{2}\right] dx \\ &\qquad = 2^{\alpha+\beta-1}\, B(\alpha,\beta)\; {}_{p+2}F_{p+1}\left[\begin{matrix}\beta, \alpha_1, \cdots, \alpha_{p+1}\,; \\ \alpha+\beta, \beta_1, \cdots, \beta_p\,;\end{matrix}\; 1\right] \\ &\qquad\qquad (p \in \mathbb{N}_0;\; \Re(\alpha) > 0;\; \Re(\beta) > 0),\end{aligned}$$

which, upon setting $p = 1$, $\alpha_1 = -n$, $\alpha_2 = n+1$, $\beta_1 = 1$, and considering (3.5), immediately yields (cf. Rainville [3, p. 184]):

$$\tag{3.7}\begin{aligned}&\int_{-1}^{1} (1+x)^{\alpha-1}\,(1-x)^{\beta-1}\, P_n(x)\, dx = 2^{\alpha-\beta-1}\, B(\alpha,\beta)\; {}_3F_2\left[\begin{matrix}-n,\; n+1,\; \beta\,; \\ 1,\; \alpha+\beta\,;\end{matrix}\; 1\right] \\ &\qquad\qquad (n \in \mathbb{N}_0;\; \Re(\alpha) > 0;\; \Re(\beta) > 0).\end{aligned}$$

On the other hand, similarly, upon applying the expression for $P_n(x)$ in (3.1) to the integrand in (3.7), we obtain an identity associated with the binomial coefficients:

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}^{2}\Gamma(\alpha+k)\,\Gamma(\beta+n-k) = \Gamma(\alpha+\beta+n)\,B(\alpha,\beta)\;{}_3F_2\left[\begin{matrix}-n,\, n+1,\, \beta\,; \\ 1,\, \alpha+\beta\,;\end{matrix}\; 1\right] \tag{3.8}$$

$$(n\in\mathbb{N}_0;\ \Re(\alpha)>0;\ \Re(\beta)>0).$$

If we set $\beta = 1$ in (3.8) and use the Gauss's summation theorem (cf. Slater [4, p. 243]):

$${}_2F_1\left[\begin{matrix}a,\, b\,; \\ c\,;\end{matrix}\; 1\right] = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \quad (\Re(c-a-b)>0;\ c\neq 0,\, -1,\, -2,\, \cdots), \tag{3.9}$$

we find that

$$\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}^{2}(n-k)!\,(\alpha)_k = \frac{(-1)^n\,\Gamma(\alpha)}{\Gamma(\alpha-n)} \quad (n\in\mathbb{N}_0;\ \Re(\alpha)>0). \tag{3.10}$$

The formula (3.10) can be seen to be a special case ($b = \alpha$ and $c = 1$) of the following formula:

$$\sum_{k=0}^{n}(-1)^{k}\,k!\binom{n}{k}\binom{-b}{k}\binom{-c}{k}^{-1} = \frac{(c-b)_n}{(c)_n} \tag{3.11}$$

$$(n\in\mathbb{N}_0;\ \Re(c-b)>-n;\ c\neq 0,\, -1,\, -2,\, \cdots),$$

which incidentally is equivalent to Chu (1303)-Vandermonde (1735-1796) convolution theorem (cf. Srivastava and Manocha [5, p. 31]):

$$\sum_{k=0}^{n}\binom{\lambda}{k}\binom{\mu}{n-k} = \binom{\lambda+\mu}{n} \quad (n\in\mathbb{N}_0), \tag{3.12}$$

λ and μ being any complex numbers.

The special case of (3.10) when $\alpha = 1$ leads to the familiar identity involving binomial coefficients:

$$\sum_{k=0}^{n}(-1)^{k}\binom{n}{k} = 0 \quad (n\in\mathbb{N}), \tag{3.13}$$

which is a special case of the binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k\,y^{n-k} \quad (n\in\mathbb{N}_0). \tag{3.14}$$

The formula (3.10) can also be obtained by setting $\alpha = 1$ in (3.8) and using Saalschütz's theorem (cf. Slater [4, p. 243]):

$$ {}_3F_2\left[\begin{matrix} -n,\ a,\ b\ ; \\ c,\quad d\ ; \end{matrix}\ 1\right] = \frac{(c-a)_n\,(c-b)_n}{(c)_n\,(c-a-b)_n} \qquad (c+d=a+b-n+1;\ n\in\mathbb{N}_0) \tag{3.15} $$

for ${}_3F_2(a=n+1,\ b=\beta,$ and $c=1)$ in the resulting equation, with various identities in Section 1.

The special case of (3.8) when $\alpha = 1$ and $\beta = n+1$, similarly, yields an identity involving binomial coefficients:

$$ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(n+1)_k}{k!} = (-1)^n \{(n+2)_n\}^2 \qquad (n\in\mathbb{N}_0). \tag{3.16} $$

We conclude this paper by presenting a transformation formula ${}_3F_2$, which is deducible from (3.8), in the sense of parameters:

$$ {}_3F_2\left[\begin{matrix} -n,\ -n,\ \alpha\ ; \\ 1,\ 1-\beta-n\ ; \end{matrix}\ 1\right] = (-1)^n \frac{(\alpha+\beta)_n}{(\beta)_n}\ {}_3F_2\left[\begin{matrix} -n,\ n+1,\ \beta\ ; \\ 1,\ \alpha+\beta\ ; \end{matrix}\ 1\right] \tag{3.17} $$
$$ (n\in\mathbb{N}_0;\ \Re(\alpha)>0;\ \Re(\beta)>0). $$

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