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An Investigation of Cognitive-Metacognitive Characteristics in Problem Solving Behavior[1]

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This paper reports an investigation of problem solving activities of students at university level students. The study focused on the cognitive-metacognitive and affective activities appeared in problem solving process. The cognitive-metacognitive framework was used to analyze and categorize the written response and free response of interviews probing the students' cognitive-metacognitive activities. Affective factors were assessed by administering the problem solving survey (Carlson, The emergence of students' problem solving behavior, 1999).
This study provide an insight for the design of problem solving instruction by identifying cognitive, metacognitive and affective characteristics of the students' problem solving behaviors. The results report that the metacognitive factors were significantly related to problem solving performance interacting with both cognitive and affective factors.

1. INTRODUCTION

National Council of Teachers of Mathematics (1989) states that developing effective problem solvers should be a primary goal of introduction. Recent investigations have examined problem solving behaviors of university level students (Schoenfeld, 1998; Carlson, 1997). In a framework developed by Schoenfeld (1998), problem solving behavior and knowledge were classified into four categories: *resources, heuristics, control*, and *belief.*

Resources are the mathematical facts and procedures potentially accessible to the problem solver.

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Heuristics are the broad range of general problem solving techniques (e.g., working backwards, drawing figures, etc.).

Control refers to the global decisions regarding the selection and implementation of resources and strategies that determine the efficiency which facts, techniques, and strategies are exploited (e.g., planning, monitoring, decision making, etc.).

Schoenfeld (1992) view a problem as a 'problem';

> ... a problem is only a problem if you don't know how to go about solving it. A problem that has no 'surprise' in store, and can be solved comfortably by routine of familiar procedures (no matter how difficult!) is an exercise.

The problems posed will be satisfied with this description for the purpose of this study in this paper.

The nature of problem solving was explained by information based activity (Massaro and Cowan, 1993). From this viewpoint four major component were selected for attention:

(a) information gathering,
(b) information representation,
(c) information processing, and
(d) information validation.

These processes are not sequential, but interactive and cyclical.

Although the importance of conscious self regulation or executive control in the problem solving process has been examined (Garofalo and Lester, 1985; Schoenfeld 1987a, 1987b; Tanner and Jones, 1994), less attention has been given to problem solving behavior resulting from such discussion, referred to as 'metacognition'. 'Metacognition' refer to one's knowledge concerning one's own cognitive processes and products or anything related to them;

> ... metacognition refers, among other things, to the active monitoring and consequent regulation and orchestration of these processes in relation to the cognitive objects on which they bear, usually in service of some concrete goal or objective. (Flavell, 1976)

Metacognitive knowledge has been defined to consist primarily of knowledge and belief about which factors acts and interact to influence the course and outcome of cognitive enterprises. The framework of Garofalo and Lester (1985) is designed to identify key points where metacognitive decisions are likely to influence cognitive actions and consists of four categories of behavior within distinctive metacognitive activities are defined, namely *orientation, organization, execution,* and *verification.*

This study aims to gain insights into cognitive-metacognitive behaviors utilized when confronted with a problem, including one's monitoring, planning and control during problem solving. More specifically, this study will investigate the following research questions:

1. How are metacognitive activities exerted and manifested during problem solving?
2. How do metacognitive behaviors interact with cognitive processing during problem solving?
3. How do affective factors interact with problem solving behavior?

2. THEORETICAL FRAMEWORK

The framework will be utilized to facilitate the analysis of metacognitive aspects of cognitive processing in problem solving. Interviews will be analyzed to determine the presence of the following problem solving components.

2.1. Cognitive components:

(a) Initial cognitive engagement:
- Establishing meaning of problem.
- Internal mental representation of the problem.
- Intuitive thinking.
- Goals and given conditions are represented using symbols, diagrams, tables, etc.

(b) Cognitive resources:
- Facts, definitions, algorithm and conceptual understandings.
- Algorithmic and routine procedures.
- Informal and intuitive knowledge about the domain.

2.2. Metacognitive behaviors:

(a) Planning:
- Selection of strategies for developing and executing plans.
- Determination of a course of action aimed at achieving goals and subgoals.

(b) Control:

- Selection of potentially fruitful solution pathway.
- Coordination of the transition between phases of problem solving.
- Decision making to access known resourse/mathematical knowledge/strategies.
- Relate problem to other problems with experience.
- Recognition of solution pathway lead to dead ends.
- Modify the wrong pathway.
- Monitoring progress during execution.

(c) Heuristic strategies:
- Organizing information or data.
- Executing plans (i.e., work backward, looks for symmetries, etc.)
- Checking special or general case in the solution attempts.
- Substitute numbers.
- Set up a table, chart or diagram.

(d) Verification:
- Evaluation of orientation and organization.
- Self monitoring of execution activities.
- Evaluation of performance of final result.

2.3. Affective components:

(a) Beliefs.

(b) Attitudes: Persistence, Motivation.

(c) Emotions: Frustration, Confidence, Anxiety, Enjoyment.

3. METHOD

The subjects for this study comprised 127 students just completing calculus with written exam and questionnaire from three different universities located in Seoul, Korea. Forty-five students, fifteen from each university, were selected to participate in follow-up interviews and observation. This study was conducted across the three universities with different type of problems, related to specific contents or non-routine problems for the purpose of this study. Univ. X has been known as a typical so-called 'Celebrated University' which aims to accept students who are strong in both research and teaching. Univ. Y is a typical natural science and engineering oriented university known for their quality and Univ. Z is a typical teaching oriented university. The students were prompted to verbally describe their written responses and think aloud during problem solving, providing cognitive-metacognitive for their approach.

A structured interview protocol (Appendix B) was devised for categories consistent with cognitive-metacognitive framework. To facilitate further analysis of the problem solution protocols, the transcripts, the audio tapes of the interview and the written scripts were coded to distinguish and record specific key points in developing problem solving process where metacognitive decisions were likely to influence cognitive action. The problem solving survey (PSS: see Appendix C) was administered by the open and axial coding techniques of Strauss and Corbin (1988) and participated observations were employed to check reliability of the affective components of problem solving behaviors.

4. RESULTS

4.1. Written response, participated observation and interview results

Because of the large amount of data collected, selected data are presented here to reveal common student responses. The results described in this section have been taken from written scripts, students' responses of participated observation and interviews at the end of the two semester calculus courses.

Problem 1 (see Appendix A). Suppose it costs $c(x) = x^3 - 6x^2 + 15x$ dollars to produce x stoves and the manufactory is currently producing 10 stoves a day. About how much extra will it cost to produce one more stove a day?

Only three of the 127 students (two of Univ. X and one of Univ. Y) completed this problem with little hesitation, providing the argument of derivative concept $\Delta c = c(x+1) - c(x) \approx c'(x)$ for their justification. The remaining students were slow to devise a solution approach. They could not understand the marginal cost and have many difficulties in understanding the limit value of 1, even though they knew the limit value of Δx when $\Delta x \to 0$ generally. Thirty two students proceeded to attempt an algebraic solution to the problem by substituting 11 for x.

A common feature of the students who never arrived at the correct solution was that selection of inappropriate formulae, irrelevant data was followed in every case by constructing the links between the data. And they tend to recall the problem experienced instead of finding the pathways from given data.

During the problem solving process, almost students express frustration or show any indication that they were thinking of abandoning their attempts. However, successful three students had strong persistence and high confidence that facilitate success in constructing their solution.

Problem 3 (see Appendix A) has been translated by the figures below. And this figure was well enough formed to activate appropriate plan and strategies. Almost of all students easily described the given condition and interpreted it as relevant number. Twenty students took constant value (for example, a) for the length of the side of the square (Figure 1) and they employed *Pythagorean relationship* and proceeded to evaluate the area to obtain $h = \frac{1}{\sqrt{3}} a$. Thirty-one students took $\sqrt{3}$ for the length of the side of the square (Figure 2).

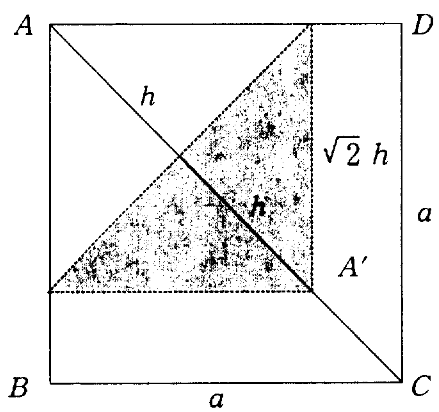

Figure 1. Square with side length a

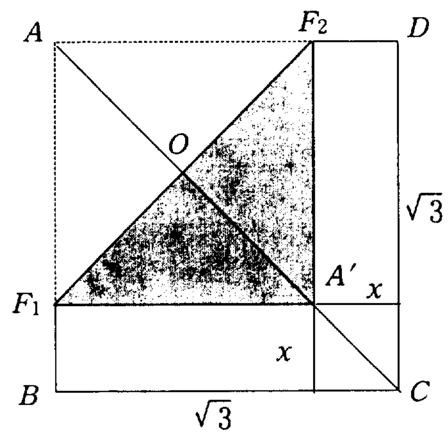

Figure 2. Square with side length $\sqrt{3}$

Thirteen students of Univ. X, five students of Univ. Y and three students of Univ. Z have obtained the correct answer with no hesitation. One student of Univ. X interpreted Problem 3 as Figure 2. To find the area from given condition, he used the calculation:

$$\begin{aligned} 2\sqrt{3}x - x^2 &= (\sqrt{3} - x)^2 \cdot \frac{1}{2} \\ \Rightarrow\ & x^2 - 2\sqrt{3}x + 1 = 0 \\ \Rightarrow\ & x = \sqrt{3} - \sqrt{2} \\ \Rightarrow\ & \overline{F_2A'} = \sqrt{2} \\ \Rightarrow\ & \overline{OA'} = 1. \end{aligned}$$

Difficulties are encountered due to incompleteness of representation and the error is not detected due to poor monitoring and the students pursue only one option of selecting the length a.

The remaining students tried to re-read problem, to re-organize problem data into figure and to work with a subset of problem information. As a consequence, puzzlement are linked with exploratory heuristics and bewilderment lead to frustration and a negative trend to anxiety, fear or despair.

Problem 4 (see Appendix A). Two numbers are "mirrors" if one can be obtained by reversing the other of the digits (i.e., 123 & 321 are mirrors). Can you find

a. two mirrors whose product is 92565?
b. two mirrors whose sum is 8768?

In solving this problem, some subjects established and represented it using two different ways:

a. $$\begin{array}{r} a\ b\ c \\ \times\ \underline{c\ b\ a} \\ 92565 \end{array}$$ and $$(a \cdot 10^2 + b \cdot 10^1 + c \cdot 10^0) \times (c \cdot 10^2 + b \cdot 10^1 + a \cdot 10^0) = 92565$$

b. $$\begin{array}{r} abc \\ +\ \underline{cba} \\ 8768 \end{array}$$ and $$(a \cdot 10^2 + b \cdot 10^1 + c \cdot 10^0) + (c \cdot 10^2 + b \cdot 10^1 + a \cdot 10^0) = 8768$$

It appeared that they were able to recall a similar problem after reading this problem and evaluate the numerical information learned from the similar problem. They also adapt the solution procedure that is similar to the efficient control behavior exhibited on the similar problem. In reviewing this procedure, the subjects checked decisions and numerical evaluation taken on the problem in connection with the progress. Using the two representations above, twenty of subjects (12 of Univ. X, 6 of Univ. Y and 2 of Univ. Z) solved the problem correctly. The remaining students chosen an approach that is not fruitful in solving problem and then it could be abandoned with a global belief about mathematics appears to be an inefficient thing. In the end, the remaining subjects could not find all cues and was not able to solve the problem. During the interview one subject said that "I don't have certain things at any finger tips and I have no factual information accessible, so it was frustrating and time consuming to solve the mathematical problem, but sometimes I was able to see it after this".

4.2. PSS Results for Affective Components across Three Universities

Taxonomy for the Problem Solving Survey (PSS: see Appendix C) is as follows:

Beliefs:	1, 7
Attitudes:	2, 3, 4, 5, 9, 10, 14, 15, 17, 19
Emotions:	6, 8, 12, 13, 16, 18

The PSS response of 43 students of Univ. X were compared with the responses of 42 students of Univ. Y and 42 students of Univ. Z at the end of their calculus courses.

Table 1. Item 1 of PSS and its profiles

For me, solving a problem that involves mathematical reasoning is:
a. an enjoyable experience.
b. a frustrating experience.

Expert view: options 1–2, Mixed view: option 3, Folk view: options 4–5

Using PSS profile classification strategy, each of these three groups was classified into one of three PSS profiles: experts, mixed and folk. Comparing the relative profiles of these three groups revealed that 78% of student in Univ. X were classified as having an expert profile, while 21% of students in Univ. Y and only 8% of students in Univ. Z had this profile.

The students of Univ. X had fewer classifications into the two lower profiles than the other two groups.

PSS response revealed that the students of Univ. X report much persistence and greater enjoyment when confronting a complex problem.

Student responses on individual PSS items were investigated to identify the PSS items where the differences in the response of the three groups were the greatest.

Comparison of these students' PSS responses revealed that the students of Univ. X report much greater persistence and greater confidence when confronting a complex problem than the remaining students of Univ. Y and university Z.

Bar chart view on Item 12 of PSS for students of Univ. X, Univ. Y and Univ. Z is shown in Figure 3.

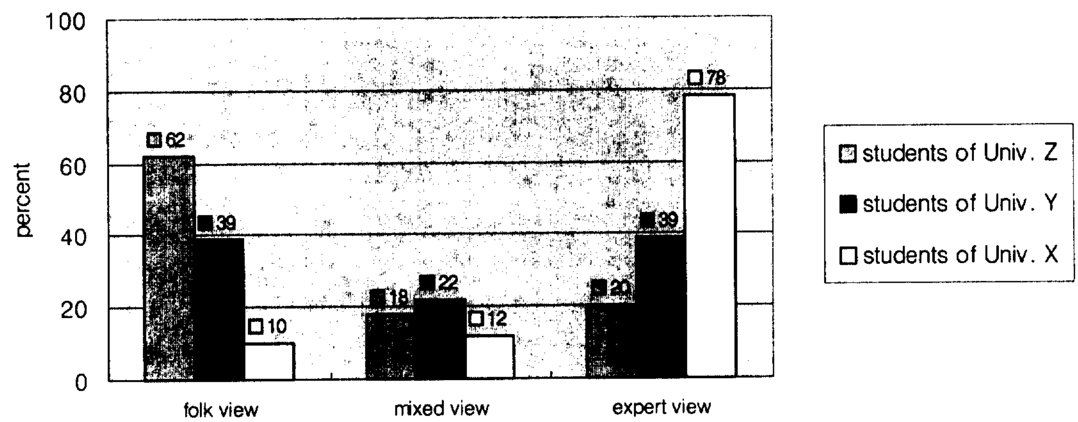

Figure 3. Bar chart view on Item 12 of PSS

5. DISCUSSION AND CONCLUSION

These results suggest that metacognitive activities were involved in all phases of the solution process identifiable in terms of the theoretical framework.

In the phase of initial cognitive engagement, the successful subjects tended to focus attention on the problem statement and the problem representation selecting appropriate cues and recalling relevant fact and procedural knowledge from memory. However, in the responses of successful students, students displayed an absence of strategies for establishing and executing plans, believing planning to be unnecessary.

The participated observation report that the low-performing student tended to consider cues of the problem within experience rather than finding the cues from given problem at initial engagement.

In this study, decision makings associated with the use of intuitive thinking were displayed by high-performing students confronting complex problems (e.g., Problems 3, 4 and 5).

Inefficient problem solvers were very restricted in using metacognitive knowledge for monitoring progress during execution, selecting fruitful solution pathway and coordinating the transitions between phases of problem solving.

Almost of subjects of this study utilized useful strategies of setting up a table, chart or diagram and substituting numbers. On the other hand, verification strategies received little attention from all subjects. An implication of results and observation of this study reported that students' decision making about resources is crucial to their success for solving specific-content problems (e.g., Problems 1 and 2) and available decision making about metacognitive strategies influenced and linked to students' success for solving unfamiliar problems (e.g., Problems 3, 4 and 5). This study has provided all the evidence that students' belief, confidence of problem solving abilities and persistence influence their success during problem solving. Noteworthy observation was the reporting from interview subject that they need to have not only knowledge and deep understanding of the content but also strategies for regulating feeling of anxiety and frustration, and mechanism in which the metacognitive strategies are consciously brought to bear. As a consequence, students need instruction that give experience to be aware of using metacognitive strategies confronting complex problems.

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Appendix A: Problem Solving Assessment

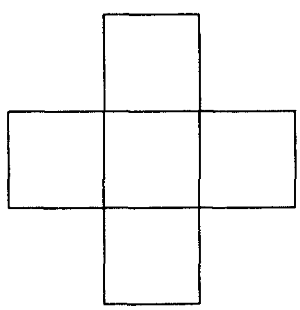
1. Suppose it costs $c(x) = x^3 - 6x^2 + 15x$ dollars to produce x stoves and the manufactory is currently producing 10 stoves a day. About how much extra will it cost to produce one more stove a day?

2. If a bicycle accelerating from zero takes 3 sec to go 18ft with the equation $s(x) = x^3 - 6x^2 + 15x$, its average velocity for the 3-sec interval is $18/3 = 6$ ft/sec. At what point during the acceleration the speedometer must read exactly 6 ft/sec?

3. A square piece of paper $ABCD$ is white on the front side and black on the back side and has an area of 3 square inches. Corner A is folded over to point A' which lie on the diagonal AC such that the total visible area is $1/2$ white and $1/2$ black. How far is A' from the fold line?

4. Two numbers are "mirrors" if one can be obtained by reversing the order of the digits (i.e., 123 and 321 are mirrors). Can you find
   a. two mirrors whose product is 92565?
   b. two mirrors whose sum is 8768?

5. Each side of the below figure is of equal length. One can cut this figure along a straight line into two pieces, then cut one of the pieces along a straight line into two pieces. The resulting three pieces can be fit together to make two identical side by side squares, that is a rectangle whose length is twice the width. Find the two necessary cuts.



Appendix B: Interview Protocol

1. Tell me about the meaning of the problem.
2. Had you seen a similar or related problem before?
3. Were you able to recall all the formulae you needed from memory?
4. Before you started working on the problem did you think it would be difficult to solve?
5. What techniques/strategies/behavior/beliefs most influence your success in solving problems?
6. How do you prioritize the actions you take during problem solving?
7. What techniques/strategies do you use so that you are able to keep persisting with a problem when you begin to feel frustrated?

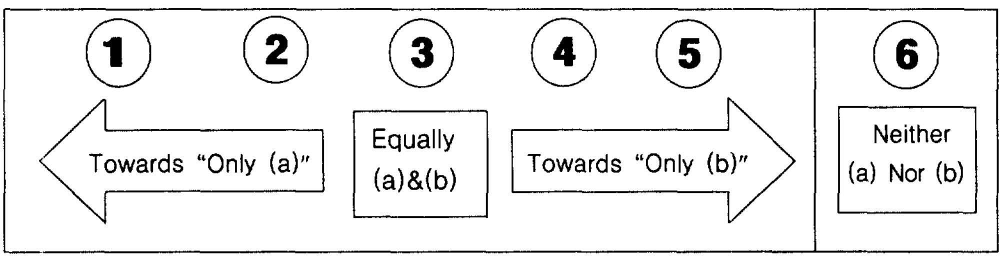
Appendix C: Problem Solving Survey (PSS)

Your participation is **voluntary**. The results will not affect your grade, even if you choose not to participate. All data are **confidential.** Your identity will not be disclosed to any party. Return of the survey materials will be considered your consent to participate.

On the first five items,

① indicates (a) response of (b) exclusively;
⑤ indicates (b) response of (a) exclusively;
③ indicates both (a) and (b);
⑥ indicates neither (a) nor (b).

A Response Choice Diagram



1. For me, solving a problem that involves mathematical reasoning is:
   (a) an enjoyable experience.
   (b) a frustrating experience.

   1 2 3 4 5 6

2. When an idea emerges regarding a possible approach to an unfamiliar problem:
   (a) I reflect on the potential merits of the approach.
   (b) I pursue that approach with little reflection.

   1 2 3 4 5 6

3. When solving a difficult mathematics problem, I prefer to work:
   (a) alone.
   (b). with colleagues.

   1 2 3 4 5 6

4. When I experience a difficulty while studying mathematics:
   (a) I try hard to figure it out on my own.
   (b) give up trying.

   1 2 3 4 5 6

5. In order to solve a mathematics problem, I need to:
   (a) apply general problem solving techniques.
   (b) have seen the solution to a similar problem before.

   1 2 3 4 5 6

6. When presented with a mathematical problem, I feel motivated to expend mental energy to "sort out" the problem.

   1. strongly agree 2. agree 3. neither 4. disagree strongly 5. disagree

7. I believe I can see most solvable mathematics problems.

   1. strongly agree 2. agree 3. neither 4. disagree strongly 5. disagree

8. Frustration and anxiety facilitate the process of working a complex problems.

   1. strongly agree 2. agree 3. neither 4. disagree strongly 5. disagree

9. The problem solving strategies I use when solving mathematics problems apply to other non-mathematics problems that I encounter.

   1. strongly agree 2. agree 3. neither 4. disagree strongly 5. disagree

10. When attempting a difficult problem, I engage in an internal dialogue with myself.

    1. strongly agree 2. agree 3. neither 4. disagree strongly 5. disagree

11. My motivation to "think hard" about a mathematics problem varies widely based on my interest in stated problem.

    1. strongly agree 2. agree 3. neither 4. disagree strongly 5. disagree

12. When confronted with a difficult problem, I persevere until I find a possible solution.

    1. strongly agree 2. agree 3. neither 4. disagree strongly 5. disagree

13. When I become frustrated while attempting a challenging mathematics problem, I stop trying.

    1. strongly agree 2. agree 3. neither 4. disagree strongly 5. disagree

14. When solving a problem, I periodically reflect on the efficiency of my approach.

    1. strongly agree 2. agree 3. neither 4. disagree strongly 5. disagree

15. When solving an unfamiliar problem I try to access the mathematics that I know to help me solve the problem.

    1. strongly agree 2. agree 3. neither 4. disagree strongly 5. disagree

16. Anxiety and tension are feelings that typically emerge when I am confronted with an unfamiliar problem.

    1. strongly agree 2. agree 3. neither 4. disagree strongly 5. disagree

17. Once I have decided on an approach, I am reluctant to consider other approaches.

    1. strongly agree 2. agree 3. neither 4. disagree strongly 5. disagree

18. I am confident in my mathematical abilities.

    1. strongly agree 2. agree 3. neither 4. disagree strongly 5. disagree

19. If I am unable to devise an approach to an unfamiliar problem (like the ones presented in this research), I usually give up within.

    1. 5 minutes
    2. 15 minutes
    3. 60 minutes
    4. 2 hours
    5. greater than 2 hours