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HADAMARD AND DRAGOMIR–AGARWAL INEQUALITIES, HIGHER–ORDER CONVEXITY AND THE EULER FORMULA

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ABSTRACT. We obtain bounds relating to Euler's formula for the case of a function with higher-order convexity properties. These are used to derive estimates of the error involved in the use of the trapezoidal formula for integrating such a function.

1. Introduction

One of the cornerstones of real analysis is the Hadamard inequality, which states that if $[a, b]$ $(a < b)$ is a real interval and $f : [a, b] \to \mathbf{R}$ a convex function, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}.$$

Over the last decade this pair of inequalities has been improved and extended in a number of ways, including the derivation of estimates of the differences between the two sides of each inequality.

Recently Dragomir and Agarwal [3] have made use of the latter to derive bounds for the error term in the trapezoidal formula for the numerical integration of an integrable function f such that $\left|f'\right|^q$ is convex for some $q \geq 1$. Some improvements to their results have been derived in [4]. In particular, the following basic result was obtained for the difference between the two sides of the right–hand Hadamard inequality.

THEOREM A. *Suppose $f : I^0 \to R$ is a differentiable mapping on $I^0 \subseteq R$, $a, b \in I^0$ with $a < b$. If $q \geq 1$ and the mapping $\left|f'\right|^q$ is convex*

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on $[a, b]$, *then*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| \le \frac{b-a}{4}\left[\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right]^{1/q}.$$

Such results suggest the possibility of obtaining similar bounds for functions known to have higher–order convexity properties. In this article we address this question. We adopt the terminology that f is $(j+2)$–convex if $f^{(j)}$ is convex, so ordinary convexity is two–convexity. A corresponding definition applies for $(j+2)$–concavity.

In the following section we derive some basic results for the Euler formula, which is important in a variety of applications. In Section 3 we give as an application the derivation of error estimates with use of the trapezoidal formula. This estimates an integral $\int_a^{a+nh} f(x)dx$ by

$$T(f;h) := h\left[\frac{1}{2}f(a) + \sum_{k=1}^{n-1} f(a+kh) + \frac{1}{2}f(a+nh)\right].$$

Our starting point, the Euler formula, states that provided f can be differentiated a sufficient number of times,

$$\begin{aligned}\int_a^{a+nh} f(x)dx &= T(f;h) - \sum_{k=1}^{r-1}\frac{B_{2k}h^{2k}}{(2k)!}\left[f^{(2k-1)}(a+nh) - f^{(2k-1)}(a)\right] \\ &\quad + h^{2r+1}\sum_{k=0}^{n-1}\int_0^1 P_{2r}(t)\, f^{(2r)}(a+h(t+k))\,dt\end{aligned}$$

(see [2, p. 275]). Here $B_n(\cdot)$ is the n–th Bernoulli polynomial and $P_n(t) = [B_n(t) - B_n]/n!$, where $B_n = B_n(0)$ is the n–th Bernoulli number. The empty sum is interpreted as zero for $r = 1$. This relation provides an estimate of the integral of a function f over an interval in terms of the trapezoid term T, values of derivatives of f at the end points of the interval and the somewhat inaccessible term involving P_{2r}.

It is convenient to use a variant form of this result [2, p. 274] which states that

$$\begin{aligned}\int_a^b f(x)dx &= \frac{b-a}{2}[f(a)+f(b)] \\ &- \sum_{k=1}^{r-1} \frac{(b-a)^{2k} B_{2k}}{(2k)!}\left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right] \\ &+ (b-a)^{2r+1}\int_0^1 P_{2r}(t)\, f^{(2r)}(a+t(b-a))\,dt.\end{aligned} \tag{1.1}$$

The merit of this version is that the term in P_{2r} is more manageable. The thrust of Section 2 is putting bounds on this term.

The relevant key properties of the Bernoulli polynomials are

$$\begin{aligned}&B_n'(t) = nB_{n-1}(t), \quad (n \geq 1)\\ &B_n(1+t) - B_n(t) = nt^{n-1}, \quad (n \geq 0)\\ &B_n(1-t) = (-1)^n B_n(t), \quad (n \geq 0)\\ &B_{2i} = (-1)^{i+1}\frac{2(2i)!}{(2i)^{2i}}\sum_{k=1}^{\infty}\frac{1}{k^i}\end{aligned}$$

(see, for example [1, Chapter 23]). For convenience we set $b_{2r} = B_{2r}/(2r)!$ $(r \geq 0)$. We note that $P_{2n}(t)$ does not change sign on (0,1). We readily verify that

$$\int_0^1 P_{2r}(t)dt = \frac{B_{2r+1}(1) - B_{2r+1}}{(2r+1)!} - \frac{B_{2r}}{(2r)!} = -b_{2r}. \tag{1.2}$$

Since $(-1)^{r+1}B_{2r} > 0$, we have $(-1)^r P_{2r}(t) > 0$.

2. Basic results for the Euler formula

In the remainder of the paper we shall use the notation

$$\begin{aligned}I_r :=(-1)^r\Bigg\{\int_a^b f(x)dx - \frac{b-a}{2}[f(a)+f(b)]\\ +\sum_{k=1}^{r-1}\frac{B_{2k}(b-a)^{2k}}{(2k)!}\left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right]\Bigg\}.\end{aligned}$$

As above, the empty sum for $r = 1$ is interpreted as zero.

We begin with a result of Hadamard type.

THEOREM 1. *If $f : [a,b] \to R$ is a $(2r+2)$–convex function ($r \geq 1$), then*

$$
\begin{aligned}
&(b-a)^{2r+1}\frac{|B_{2r}|}{(2r)!}f^{(2r)}\left(\frac{a+b}{2}\right) \\
&\leq I_r \leq (b-a)^{2r+1}\frac{|B_{2r}|}{(2r)!}\frac{f^{(2r)}(a)+f^{(2r)}(b)}{2}.
\end{aligned} \tag{2.1}
$$

If f is a $(2r+2)$–concave function, then the reverse inequality applies.

Proof. By (1.2),

$$
\begin{aligned}
&\int_0^1 tP_{2r}(t)dt + \int_0^1 (1-t)P_{2r}(t)dt \\
&= \int_0^1 P_{2r}(t)dt = -b_{2r}.
\end{aligned}
$$

Also $B_{2r}(1-t) = B_{2r}(t)$ and so $P_{2r}(1-t) = P_{2r}(t)$, whence

$$
\int_0^1 tP_{2r}(t)dt = \int_0^1 (1-t)P_{2r}(t)dt = -b_{2r}/2.
$$

If f is $(2r+2)$–convex, then $f^{(2r)}$ is convex, so that

$$
\begin{aligned}
f^{(2r)}(a+t(b-a)) &= f^{(2r)}[(1-t)a+tb] \\
&\leq tf^{(2r)}(b) + (1-t)f^{(2r)}(a).
\end{aligned}
$$

Hence from (1.1) we have

$$
\begin{aligned}
I_r &\leq (b-a)^{2r+1}\int_a^b (-1)^r P_{2r}(t)\left[tf^{(2r)}(b)+(1-t)f^{(2r)}(a)\right]dt \\
&= (-1)^{r+1}(b-a)^{2r+1}b_{2r}\frac{f^{(2r)}(a)+f^{(2r)}(b)}{2}.
\end{aligned}
$$

On the other hand, we have from Jensen's theorem and the convexity of $f^{(2r)}$ that

$$\begin{aligned} I_r &= (b-a)^{2r+1}\int_0^1 (-1)^r P_{2r}(t) f^{(2r)}(a+t(b-a))dt \\ &\geq (b-a)^{2r+1}\left(\int_0^1 (-1)^r P_{2r}(t)dt\right) \times \\ &\quad f^{(2r)}\left(\frac{\int_0^1(-1)^r P_{2r}(t)(a+t(b-a))dt}{\int_0^1(-1)^r P_{2r}(t)dt}\right) \\ &= (b-a)^{2r+1}\frac{|B_{2r}|}{(2r)!} f^{(2r)}\left(\frac{a+b}{2}\right). \end{aligned}$$

The result for the convex case follows. Obvious modifications to the proof provide the corresponding concavity result. □

REMARK 1. If f is a 4-convex function, then since $B_2 = 1/6$, we may substitute for I_1 in (2.1) to derive

$$\begin{aligned} &\frac{(b-a)^3}{12} f''\left(\frac{a+b}{2}\right) \\ &\leq \frac{b-a}{2}[f(a)+f(b)] - \int_a^b f(x)dx \\ &\leq \frac{(b-a)^3}{12}\frac{f''(a)+f''(b)}{2}. \end{aligned}$$

If f is convex, then the leftmost expression in this inequality is nonnegative. We thus have two-sided nonnegative bounds for the central expression in this inequality if f happens to be both 4-convex and convex.

THEOREM 2. *Suppose $f : [a,b] \to R$ is a real-valued $(2r)$-times differentiable function such that $\left|f^{(2r)}\right|^q$ is convex for some $q \geq 1$. Then*

$$|I_r| \leq (b-a)^{2r+1}\frac{|B_{2r}|}{(2r)!}\left[\frac{\left|f^{(2r)}(a)\right|^q + \left|f^{(2r)}(b)\right|^q}{2}\right]^{1/q}.$$

If $\left|f^{(2r)}\right|^q$ is concave, then

$$|I_r| \leq (b-a)^{2r+1}\frac{|B_{2r}|}{(2r)!}\left|f^{(2r)}\left(\frac{a+b}{2}\right)\right|. \tag{2.2}$$

Proof. From (1.1), we have

$$\begin{aligned}
&|I_r| \\
&\le (b-a)^{2r+1} \int_0^1 |P_{2r}(t)| \left| f^{(2r)}(tb+(1-t)a) \right| dt \\
&\le (b-a)^{2r+1} \left[\int_0^1 |P_{2r}(t)|\, dt \right]^{1-1/q} \\
&\qquad \times \left[\int_0^1 |P_{2r}(t)| \left| f^{(2r)}(tb+(1-t)a) \right| dt \right]^{1/q},
\end{aligned}$$

by the power–mean inequality. Hence by Jensen's inequality

$$\begin{aligned}
|I_r| &\le (b-a)^{2r+1} \left(\int_0^1 |P_{2r}(t)|\, dt \right)^{1-1/q} \\
&\quad \times \left(\int_0^1 |P_{2r}(t)| \left[t \left| f^{(2r)}(b) \right|^q + (1-t) \left| f^{(2r)}(a) \right|^q \right] dt \right)^{1/q} \\
&= (b-a)^{2r+1} \left| \int_0^1 P_{2r}(t) dt \right|^{1-1/q} \\
&\quad \times \left(\left| f^{(2r)}(b) \right|^q \left| \int_0^1 t P_{2r}(t) dt \right| + \left| f^{(2r)}(a) \right|^q \left| \int_0^1 (1-t) P_{2r}(t) dt \right| \right)^{1/q}.
\end{aligned}$$

If $\left| f^{(2r)} \right|^q$ is concave, then

$$\begin{aligned}
|I_r| &\le (b-a)^{2r+1} \int_0^1 |P_{2r}(t)| \cdot \left| f^{(2r)}(a+t(b-a)) \right| dt \\
&\le (b-a)^{2r+1} \left(\int_0^1 |P_{2r}(t)|\, dt \right) \\
&\quad \times \left[\left| f^{(2r)} \left(\frac{\int_0^1 |P_{2r}(t)|\, (a+t(b-a)) dt}{\int_0^1 |P_{2r}(t)|\, dt} \right) \right|^q \right]^{1/q} \\
&= (b-a)^{2r+1} \frac{|B_{2r}|}{(2r)!} \left| f^{(2r)} \left(\frac{a+b}{2} \right) \right|,
\end{aligned}$$

and the theorem is proved. □

REMARK 2. For (2.2) to be satisfied it is enough to suppose that $\left| f^{(2r)} \right|$ is a concave function. For if $|g|^q$ is concave on $[a, b]$ for some

$q \geq 1$, then for $x, y \in [a,b]$ and $\lambda \in [0,1]$

$$\begin{aligned} |g(\lambda x + (1-\lambda)y)|^q &\geq \lambda\, |g(x)|^q + (1-\lambda)\, |g(y)|^q \\ &\geq (\lambda\, |g(x)| + (1-\lambda)\, |g(y)|)^q , \end{aligned}$$

by the power–mean inequality. Therefore $|g|$ is also concave on $[a,b]$.

3. The trapezoidal formula

To obtain estimates of the error in the trapezoidal formula, we apply the results of the previous section on each interval from the subdivision

$$[a, a+b], [a+h, a+2h], ..., [a+(n-1)h, a+nh].$$

We define

$$J_r := \int_a^{a+nh} f(x)dx - T(f;h) + \sum_{k=1}^{r-1} \frac{B_{2k}h^{2k}}{(2k)!} \left[f^{(2k-1)}(a+nh) - f^{(2k-1)}(a)\right]$$

and

$$M(f;h) := h \sum_{k=1}^{n-1} f\left(a + kh + \frac{1}{2}h\right).$$

THEOREM 3. *If $f : [a, a+nh] \to R$ is a $(2r+2)$–convex function, then*

$$h^{2r}\frac{|B_{2r}|}{(2r)!} M(f^{(2r)};h) \leq (-1)^r J_r \leq h^{2r}\frac{|B_{2r}|}{(2r)!} T(f^{(2r)};h).$$

If f is $(2r+2)$–concave, the inequality is reversed.

Proof. The desired result follows from Theorem 1 since

$$\begin{aligned} J_r = \sum_{m=1}^{n} \Big\{ &\int_{a+(m-1)h}^{a+mh} f(x)dx \\ &- \frac{h}{2}\Big[f(a+(m-1)h) + f(a+mh)\Big] \\ &+ \sum_{k=1}^{r-1} \frac{B_{2k}h^{2k}}{(2k)!} \\ &\times \Big[f^{(2k-1)}(a+mh) - f^{(2n-1)}(a+(m-1)h)\Big] \Big\}. \ \square \end{aligned} \tag{3.1}$$

THEOREM 4. *Suppose $f : [a, a+nh] \to R$ is a $(2r)$–times differentiable function ($r \geq 1$) such that $\left|f^{(2r)}\right|^q$ is convex for some $q \geq 1$. Then*

$$
\begin{aligned}
|J_r| &\leq h^{2r+1} \frac{|B_{2r}|}{(2r)!} \sum_{k=1}^{n} \left[\frac{\left|f^{(2r)}(a+nh)\right|^q + \left|f^{(2r)}(a)\right|^q}{2} \right]^{1/q} \\
&\leq h^{2r+1} \frac{|B_{2r}|}{(2r)!} \max\left\{ \left|f^{(2r)}(a)\right|, \left|f^{(2r)}(a+nh)\right| \right\}.
\end{aligned}
$$

If $\left|f^{(2r)}\right|$ is concave, then

$$
|J_r| \leq h^{2r} \frac{|B_{2r}|}{(2r)!} M\left(\left|f^{(2r)}\right| ; h \right).
$$

Proof. From (3.1), we have

$$
\begin{aligned}
|J_r| \leq \sum_{m=1}^{n} \Bigg| \int_{a+(m-1)h}^{a+mh} & f(x)dx - \frac{h}{2}\left[f(a+(m-1)h) + f(a+mh) \right] \\
&+ \sum_{k=1}^{r-1} \frac{B_{2k} h^{2k}}{(2k)!} \left[f^{(2k-1)}(a+mh) - f^{(2k-1)}(a+(m-1)h) \right] \Bigg| \\
\leq \sum_{m=1}^{n} h^{2r+1} \frac{|B_{2r}|}{(2r)!} & \left(\frac{\left|f^{(2r)}(a+mh)\right|^q + \left|f^{(2r)}(a+(m-1)h)\right|^q}{2} \right)^{1/q},
\end{aligned}
$$

by Theorem 2. Hence by the means inequality

$$
|J_r| \leq h^{2r+1} \frac{|B_{2r}|}{(2r)!} \sum_{m=1}^{n} \max\left\{ \left|f^{(2r)}(a+mh)\right|, \left|f^{(2r)}(a+(m-1)h)\right| \right\}.
$$

The first part of the theorem now follows from the convexity of $\left|f^{(2r)}\right|^q$. The second part follows similarly. □

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