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WELL-POSEDNESS FOR THE BENJAMIN EQUATIONS

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Abstract. We consider the time local well-posedness of the Benjamin equation. Like the result due to Kenig-Ponce-Vega [10], [12], we show that the initial value problem is time locally well posed in the Sobolev space $H^s(\mathbb{R})$ for $s > -3/4$.

1. Introduction

In this paper, we consider the following initial value problem.

$$\begin{cases} \partial_t u - \partial_x^3 u - \nu \mathcal{H}_x \partial_x^2 u + \partial_x(u^2) = 0 \quad x, t \in \mathbb{R}, \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

where $0 < \nu < 1$. $\mathcal{H}_x$ denotes the Hilbert transform defined by

$$\mathcal{H}_x f(x) = \text{p.v.}\frac{1}{\pi}\int \frac{f(y)}{y-x}dy = \mathcal{F}_\xi^{-1}(-i \cdot \text{sgn}(\xi)(\mathcal{F}_x f)(\xi))$$

and $\mathcal{F}_x$, $\mathcal{F}_\xi^{-1}$ denote Fourier and inverse Fourier transform respect the variable x and ξ.

Problem (1.1) was introduced by Benjamin [4] and describes the intermediate wave for the stratified fluid under the condition when the capillarity on the surface is not negligible.

The purpose of this paper is to consider the initial value problem in a weak function space, namely the Sobolev space $H^s(\mathbb{R})$ with negative index s and establish the well-posedness to the problem (1.1).

Related to the problem (1.1), the Cauchy problem for the Korteweg-de Vries equations:

$$\begin{cases} \partial_t v - \partial_x^3 v + \partial_x(v^2) = 0 \quad x, t \in \mathbb{R}, \\ v(x,0) = u_0(x) \end{cases} \tag{1.2}$$

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and the Benjamin-One equations:

$$(1.3) \qquad \begin{cases} \partial_t v - \nu \mathcal{H}_x \partial_x^2 v + \partial_x(v^2) = 0 \quad x, t \in \mathbb{R}, \\ v(x,0) = u_0(x) \end{cases}$$

are known and there are many research for them.

Comparing with those problems, our problem (1.1) has a common structures: the linear term of (1.1) consists of the linear terms appearing in (1.2) and (1.3) and has the same nonlinear term. Now for the well-posedness for the KdV equation, the recent works by Bourgain [4] and Kenig-Ponce-Vega [11] show that the well-posedness for those dispersive equation hold for the Sobolev spaces with the negative exponent.

Bourgain [4] obtained the global well-posedness for (1.2) in $L^2(\mathbb{R})$ and Kenig-Ponce-Vega [11], [12], [13] established the well-posedness in H^s, with $s > -3/4$ According to their estimate, the bilinear estimate for the KdV equation is valid only for $s > -3/4$ and there is a counter example when $s \leq -3/4$ (see [12] and [16]).

Concerning to the initial value problem for the Benjamin-Ono, Iorio Jr. [8] showed the $H^s(\mathbb{R})$ global well-posedness for $s > 3/2$ and later on, Ponce [17] improves his result to $H^{3/2}(\mathbb{R})$. For the Benjamin-Ono case, the bilinear argument does not work well. In fact, it is shown by Mlinet-Saut-Tzvetkov [15] that a similar bilinear estimate in the Fourier restriction space does not hold.

Another similar result was obtained by Takaoka [18] for the Hirota equation of KdV-nonlinear Schrödinger mixed type dispersive equations. It is also known that there exists a solution for the Benjamin equation (1.1) (see Angulo [2]).

Since the solution to the problem (1.1) has the following conservation laws

$$(1.4) \qquad I_1(u) = \frac{1}{2} \int_{-\infty}^{\infty} u^2(t) dx,$$

$$(1.5) \qquad I_2(u) = \int_{-\infty}^{\infty} \left(\frac{1}{2} |\partial_x u(t)|^2 - \frac{1}{2} u(t) \mathcal{H}_x \partial_x u(t) - \frac{1}{3} u^3(t) \right) dx,$$

it is possible to show the global well-posedness for (1.1) when the data is belonging to H^s for $s \geq 0$.

This global well-posedness result in $L^2(\mathbb{R})$ was obtained by Linares [14] by using the analogous argument to [6] and [11], and showed the L^2- conservation law.

Here we consider the well-posedness for (1.1) in H^s, where $-3/4 < s < 0$. Moreover since the existence time T can be taken independent of the parameter $\nu > 0$, we can show that as the limiting procedure of $\nu \to 0$ the solution of Benjamin equation converges to the solution of the KdV equation with the corresponding data u_0.

In the following sections, we define the Fourier restriction norm and prepare some useful lemmas in Section 2. In Section 3, we recall the linear estimate and in Section 4 we give the main bilinear estimates. Finally we give the proof of the well-posedness in the final section. We use the following notations. Let $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$ and $\langle D_x \rangle^s = (1 - \partial_x^2)^{s/2} = \mathcal{F}_\xi^{-1} \langle \xi \rangle^s \mathcal{F}_x$, $H^s(\mathbb{R}) = \langle D_x \rangle^{-s} L^2(\mathbb{R})$: Sobolev spaces with the norm $\| \cdot \|_{H^s} = \| \langle D_x \rangle^s \cdot \|_{L^2}$. $\chi(\cdot)$ denotes a characteristic function of interval $[0, 1]$. $f \sim g$ means that there exists constants C_0, C_1 such that $C_0 f \leq g \leq C_1 f$ holds. $\langle \cdot, \cdot \rangle_H$ is the inner product for the Hilbert space H. Various constants are simply denoted by C.

2. Results

We first define the space where the solution is constructed.

DEFINITION 1.1. Let the Hilbert space $X_\nu^{s,b}$ be the space of all function ω with the norm $\| \cdot \|_{X_\nu^{s,b}}$. Namely, $\omega \in X_\nu^{s,b}$, if

$$\|\omega\|_{X_\nu^{s,b}} = \left(\iint_{\mathbb{R}^2} \langle \tau - \phi_\nu(\xi) \rangle^{2b} \langle \xi \rangle^{2s} |\hat{\omega}(\xi, \tau)|^2 d\xi d\tau \right)^{1/2} < \infty, \tag{2.6}$$

where $\phi_\nu, \hat{\omega}$ stands for

$$\phi_\nu(\xi) = \nu|\xi|\xi - \xi^3 \quad \text{(especially} \quad \phi(\xi) \equiv \phi_1(\xi)),$$

and

$$\hat{\omega}(\xi, \tau) = \frac{1}{2\pi} \iint_{\mathbb{R}^2} e^{-i(x\xi + t\tau)} \omega(x, t) dx dt.$$

Our well-posedness result for the problem (1.1) is the following.

THEOREM 2.1. *Let $\nu \in \mathbb{R}$. For any $s \in (-3/4, 0]$ and $u_0 \in H^s(\mathbb{R})$ there exists $b \in (1/2, 1)$ and $T = T(\|u_0\|_{H^s}) > 0$ with $\lim_{a \to 0} T(a) = \infty$*

such that there exists a unique solution u of (1.1) on $[-T,T]$ with the following properties;

$$u \in C([-T,T], H^s(\mathbb{R})) \cap X_\nu^{s,b}, \tag{2.7}$$
$$\partial_x(u^2) \in X_\nu^{s,b-1}.$$

Moreover for any $T' \in (0,T)$, there exists $R = R(T') > 0$ such that the mapping $\{\tilde{u}_0 \in H^s(\mathbb{R}) : \|u_0 - \tilde{u}_0\|_{H^s} < R\} \ni \tilde{u}_0 \to \tilde{u} \in C([-T',T'], H^s(\mathbb{R})) \cap X_\nu^{s,b}$ is Lipschitz continuous.

REMARK. 1. Linares [14] showed that when $s = 0$, i.e., when $u_0 \in L^2$ the global well-posedness is valid for $T = \infty$. For the KdV equation (1.2), Kenig-Ponce-Vega [11] showed the similar result.

2. The local existence time $T = T(\|u_0\|_{H^s})$ can be taken uniformly for the parameter $\nu \in (0,1)$. In this sense, our theorem covers the previous result on the KdV equation (1.2) by Kenig-Ponce-Vega [12].

According to the above observation, we can prove the solution of Benjamin equation $u = u_\nu$ converges to the solution of the KdV equation v.

THEOREM 2.2. *For any $s \in (-3/4, 0]$ and $u_0 \in H^s(\mathbb{R})$ there exists $b \in (1/2, 1)$ and $T = T(\|u_0\|_{H^s}) > 0$ $(T(a) \to \infty$ $(a \to 0))$ such that for $t \in [-T,T]$, the unique solution of (1.1) u_ν obtained in Theorem 2.1 converges to $v \in X_0^{s,b}$ and v solves the initial value problem for the KdV equation;*

$$\begin{cases} \partial_t v - \partial_x^3 v + \partial_x(v^2) = 0, & x \in \mathbb{R}, \quad t \in [-T.T], \\ v(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{2.8}$$

3. Linear estimates

In this section, we consider the linear version of the equation (1.1)

$$\begin{cases} \partial_t v - \nu \mathcal{H}_x \partial_x^2 v - \partial_x^3 v = 0 \quad t, x \in \mathbb{R}, \\ v(x,0) = v_0(x). \end{cases} \tag{3.1}$$

According to the well known Stone theorem, there is a unitary group $\{W_\nu(t)\}_{-\infty}^{\infty}$ on L^2 such that the solution to (3.1) can be obtained by

$$v(x,t) = W_\nu(t)v_0(x) = S_t * v_0(x), \tag{3.2}$$
$$S_t(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} e^{it(\nu|\xi|\xi - \xi^3)} d\xi.$$

Introducing $\langle D_x \rangle^s$, $\langle D_t \rangle^b$ defined by

$$\langle D_x \rangle^s h = \mathcal{F}_\xi^{-1} \langle \xi \rangle^s (\mathcal{F}_x h)(\xi), \ \langle D_t \rangle^b g = \mathcal{F}_\tau^{-1} \langle \tau \rangle^b (\mathcal{F}_t g)(\xi),$$

we then see

$$\|f\|_{X_\nu^{s,b}} = \|\langle D_t \rangle^b \langle D_x \rangle^s W_\nu(-t) f\|_{L_t^2 L_x^2}. \tag{3.3}$$

Now let $\psi(t)$ be $\psi \in C_0^\infty(\mathbb{R})$, $\psi \equiv 1$ $(t \in [-1/2, 1/2])$, $\operatorname{supp} \psi \subset (-1,1)$ and we let $0 < \delta < 1$.

We use the following Sobolev inequality.

LEMMA 3.1 (Sobolev-Morrey). *For $f \in H^s(\mathbb{R})$ with $b > 1/2$, we have*

$$\|f\|_{L^\infty(\mathbb{R})} \le C\|f\|_{H^b(\mathbb{R})} \tag{3.4}$$

and $f \in C^{b-1/2}(\mathbb{R})$.

Here we summarize some linear estimates that is used for proof.

PROPOSITION 3.2 ([12]). (1) *For $1/2 < b \le 1$*

$$\|\psi(\delta^{-1}t)\omega\|_{X_\nu^{s,b}} \le C\delta^{(1-2b)/2}\|\omega\|_{X_\nu^{s,b}}. \tag{3.5}$$

(2) *For $0 < b < 1/2$,*

$$\|\psi(\delta^{-1}t)\omega\|_{X_\nu^{s,b}} \le C\|\omega\|_{X_\nu^{s,b}} \quad \text{for all} \quad h \in X_\nu^{s,b}, \ s \in \mathbb{R}. \tag{3.6}$$

(3) *For $0 \le a < \overline{a} < 1/2, \delta \in (0,1), s \in \mathbb{R}$*

$$\|\psi(\delta^{-1}t)\omega\|_{X_\nu^{s,-\overline{a}}} \le C\delta^{\overline{a}-a}\|\omega\|_{X_\nu^{s,-a}} \tag{3.7}$$

PROPOSITION 3.3 ([6], [10]). *For $1/2 < b \le 1$, we have*

$$\|\psi(\delta^{-1}t)W_\nu(t)u_0\|_{X_\nu^{s,b}} \le C\delta^{(1-2b)/2}\|u_0\|_{H^s} \tag{3.8}$$

for all $u_0 \in H^s$, $s \in \mathbb{R}$.

For $1/2 < b \leq 1$, *we have*

$$\left\|\psi(\delta^{-1}t)\int_0^t W_\nu(t-t')\omega(t')dt'\right\|_{X_\nu^{s,b}} \leq C\delta^{(1-2b)/2}\|\omega\|_{X_\nu^{s,b-1}}. \tag{3.9}$$

See for the proof, [6], [12], [7], and [3].

4. Nonlinear estimates

PROPOSITION 4.1. *For* $s \in (-3/4, 0]$, *there exists* $b \in (1/2, 1)$ *such that we have*

$$\|\partial_x(\omega^2)\|_{X_\nu^{s,b-1}} \leq C\|\omega\|^2_{X_\nu^{s,b}} \quad \text{for any } \omega \in X_\nu^{s,b}. \tag{4.1}$$

Let $s \in (-3/4, 0]$. For $\omega \in X_\nu^{s,b}$, we set

$$f(\xi,\tau) = \langle \tau - \phi_\nu(\xi)\rangle^b \langle \xi\rangle^s \hat{\omega}(\xi,\tau).$$

Then by the definition of $X_\nu^{s,b}$, $\|f\|_{L_\xi^2 L_\tau^2} = \|\omega\|_{X_\nu^{s,b}}$. While noting $\widehat{\partial_x(\omega^2)}(\xi,\tau) = C\xi(\hat{\omega}*\hat{\omega})(\xi,\tau)$,

$$\begin{aligned}
&\|\partial_x(\omega^2)\|_{X_\nu^{s,b-1}} \\
=\ & \|\langle\tau-\phi_\nu(\xi)\rangle^{b-1}\langle\xi\rangle^{-|s|}\widehat{\partial_x(\omega^2)}\|_{L_\xi^2L_\tau^2} \\
=\ & C\|\langle\tau-\phi_\nu(\xi)\rangle^{b-1}\langle\xi\rangle^{-|s|}\xi(\hat{\omega}*\hat{\omega})\|_{L_\xi^2L_\tau^2} \\
=\ & C\left\|\frac{\xi}{\langle\tau-\phi_\nu(\xi)\rangle^{1-b}\langle\xi\rangle^{|s|}}\right. \\
&\left.\times\iint_{\mathbb{R}^2}\frac{f(\xi_1,\tau_1)\langle\xi_1|\rangle^{|s|}}{\langle\tau_1-\phi_\nu(\xi_1)\rangle^b}\frac{f(\xi-\xi_1,\tau-\tau_1)\langle\xi-\xi_1\rangle^{|s|}}{\langle\tau-\tau_1-\phi_\nu(\xi-\xi_1)\rangle^b}d\xi_1d\tau_1\right\|_{L_\xi^2L_\tau^2}.
\end{aligned}$$

Hence Proposition 4.1 can be established once the following estimate is obtained.

PROPOSITION 4.2. *For* $s \in (-3/4, 1/2)$, *there exists* $b \in (1/2, 1)$ *such that for* $b - b' \leq$*min*$(|s| - 1/2, 1/4 - |s|/3)$ *with* $b' \in (1/2, b]$ *the following* *estimate holds*

$$\begin{aligned}(4.2)\quad &\Big\| \frac{\xi}{\langle \tau-\phi_\nu(\xi)\rangle^{1-b}\langle\xi\rangle^{|s|}} \\ &\times \iint_{\mathbb{R}^2} \frac{f(\xi_1,\tau_1)\langle\xi_1\rangle^{|s|}}{\langle \tau_1-\phi_\nu(\xi_1)\rangle^{b'}} \frac{f(\xi-\xi_1,\tau-\tau_1)\langle\xi-\xi_1\rangle^{|s|}}{\langle \tau-\tau_1-\phi_\nu(\xi-\xi_1)\rangle^{b'}} d\xi_1 d\tau_1 \Big\|_{L^2_\xi L^2_\tau} \\ &\le C\|f\|^2_{L^2_\xi L^2_\tau}.\end{aligned}$$

If $s=0$, then the same conclusion holds for $b\in(1/2,3/4), b'\in(1/2,b]$.

Proposition 4.1 is an immediate consequence for $b'=b$ in Proposition 4.2 under $s=0, s\in(-1/2,-3/4)$. The case for $s\in(-1/2,0)$ is similarly obtained.

We first prepare the following estimates.

LEMMA 4.3. *For $p,q>0$ and $r=\min(p,q)$ with $p+q>1+r$, there exists $C>0$ such that*

$$(4.3)\qquad \int_{-\infty}^{\infty} \frac{dx}{\langle x-a\rangle^p\langle x-b\rangle^q} \le \frac{C}{\langle a-b\rangle^r}.$$

For $r\in(1/2,1]$, there exists $C>0$ such that

$$(4.4)\qquad \int_{-\infty}^{\infty} \frac{dx}{\langle x\rangle^{2r}\sqrt{|a-x|}} \le \frac{C}{\langle a\rangle^{1/2}},$$

$$(4.5)\qquad \int_{|x|\le b} \frac{dx}{\langle x\rangle^{2(1-r)}\sqrt{|a-x|}} \le C\frac{(1+b)^{2(r-1/2)}}{\langle a\rangle^{1/2}}.$$

In order to prove Proposition 4.2, it suffices to show the following three lemmas. The first one states as follows.

LEMMA 4.4. *For any $b\in(1/2,3/4], b'\in(1/2,b]$, we have some $C=C(b,b')>0$ such that*

$$\begin{aligned}(4.6)\quad I_1 &\equiv \frac{|\xi|}{\langle\tau-\phi_\nu(\xi)\rangle^{1-b}} \\ &\times \left(\iint_{\mathbb{R}^2} \frac{d\tau_1 d\xi_1}{\langle\tau_1-\phi_\nu(\xi_1)\rangle^{2b'}\langle\tau-\tau_1-\phi_\nu(\xi-\xi_1)\rangle^{2b'}}\right)^{1/2} \\ &\le C.\end{aligned}$$

Proof. For the simplicity, we give the proof for $\nu = 1$. The other case is quite similar. Let $\phi(\xi) = \phi_1(\xi)$. We divide the integral region $\mathbb{R}^2$ into the four cases:

$$\text{(i)} \quad \{(\xi_1,\xi);\ \xi \geq \xi_1,\ \ \xi_1 \geq 0\}, \quad \text{(ii)} \quad \{(\xi_1,\xi);\ \xi \geq \xi_1,\ \ \xi_1 < 0\},$$

$$\text{(iii)} \quad \{(\xi_1,\xi);\ \xi < \xi_1,\ \ \xi_1 \geq 0\}, \quad \text{(iv)} \quad \{(\xi_1,\xi);\ \xi < \xi_1,\ \ \xi_1 < 0\}.$$

THE CASE (i). It follows by the condition on ξ and ξ_1, $\phi(\xi_1) = \xi_1^2 - \xi_1^3$ and $\phi(\xi - \xi_1) = (\xi - \xi_1)^2 - (\xi - \xi_1)^3$. By (4.6) of Lemma 4.3,

$$\begin{aligned}&\int_{-\infty}^{\infty} \frac{d\tau_1}{\langle \tau_1 - \phi(\xi_1)\rangle^{2b'} \langle \tau - \tau_1 - \phi(\xi - \xi_1)\rangle^{2b'}} \\ &\leq C \frac{1}{\langle \tau - \phi(\xi - \xi_1) - \phi(\xi_1)\rangle^{2b'}}.\end{aligned}$$

(a) Case $0 \leq \xi \leq 1$. Since $0 \leq \xi_1 \leq \xi \leq 1$,

$$\frac{C|\xi|}{\langle \tau - \phi(\xi)\rangle^{1-b}} \left(\int_0^1 \frac{d\xi_1}{\langle \tau - \phi(\xi - \xi_1) - \phi(\xi_1\rangle^{2b'}} \right)^{1/2} \leq C.$$

(b) Case $\xi \geq 1$. By change of variable such as

$$\mu \equiv \tau - \phi(\xi - \xi_1) - \phi(\xi_1) = \tau - \{\xi^2 - \xi^3 + (3\xi - 2)\xi_1(\xi - \xi_1)\},$$

we have

$$d\mu = (3\xi - 2)(2\xi_1 - \xi) d\xi_1, \quad \xi_1 = \frac{1}{2}\left\{ \xi \pm \sqrt{\frac{4\mu - \xi^3 + 2\xi^2 - 4\tau}{3\xi - 2}} \right\}$$

and

$$|(3\xi - 2)(2\xi_1 - \xi)| = |3\xi - 2|^{1/2} \sqrt{|4\tau + \xi^3 - 2\xi^2 - 4\mu|}.$$

Hence

$$\begin{aligned}&\int_{-\infty}^{\infty} \frac{d\xi_1}{\langle \tau - \phi(\xi - \xi_1) - \phi(\xi_1)\rangle^{2b'}} \\ \leq\ & C \int_{-\infty}^{\infty} \frac{d\mu}{|3\xi - 2|^{1/2} \langle \mu\rangle^{2b'} \sqrt{|4\tau + \xi^3 - 2\xi^2 - 4\mu|}} \\ \leq\ & C \frac{1}{|\xi|^{1/2}} \int_{-\infty}^{\infty} \frac{d\mu}{\langle \mu\rangle^{2b'} \sqrt{|4\tau + \xi^3 - 2\xi^2 - 4\mu|}}\end{aligned}$$

(using (4.4))

$$\leq \frac{C}{|\xi|^{1/2} \langle 4\tau + \xi^3 - 2\xi^2\rangle^{1/2}}.$$

Therefore we have

$$\begin{aligned} I_1 &\le \frac{C|\xi|}{\langle \tau - \phi(\xi)\rangle^{1-b}} \left(\frac{1}{|\xi|^{1/2}\langle 4\tau + \xi^3 - 2\xi^2\rangle^{1/2}} \right)^{1/2} \\ &\le \frac{C|\xi|^{3/4}}{\langle \tau - (|\xi|\xi - \xi^3)\rangle^{1-b}\langle 4\tau + \xi^3 - 2\xi^2\rangle^{1/4}} \\ &= \theta_1(\xi, \tau). \end{aligned}$$

For $b \le 3/4$, $\theta_1(\xi, \tau)$ is bounded in $\mathbb{R}_\xi \times \mathbb{R}_\tau$.

THE CASE (ii). Note that $\phi(\xi_1) = -\xi_1^2 - \xi_1^3$, $\phi(\xi - \xi_1) = (\xi - \xi_1)^2 - (\xi - \xi_1)^3$.

By changing the variable from ξ_1 to μ such as

$$\mu \equiv \tau - \phi(\xi - \xi_1) - \phi(\xi_1) = \tau - \{\xi^2 - \xi^3 - 2\xi\xi_1 + 3\xi\xi_1(\xi - \xi_1)\},$$

we have

$$d\mu = \xi(2 - 3\xi + 6\xi_1)d\xi_1,$$

$$\xi_1 = \frac{1}{6}\left\{ (3\xi - 2) \pm \sqrt{\frac{12\mu - 3\xi^3 + 4\xi - 12\tau}{\xi}} \right\},$$

$$|\xi(2 - 3\xi + 6\xi)| = |\xi|^{1/2}\sqrt{|12\tau + 3\xi^3 - 4\xi - 12\mu|}.$$

Hence

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{d\xi_1}{\langle \tau - \phi(\xi - \xi_1) - \phi(\xi_1)\rangle^{2b'}} \\ &\le C \int_{-\infty}^{\infty} \frac{d\mu}{|\xi|^{1/2}\langle \mu \rangle^{2b'}\sqrt{|12\tau + 3\xi^3 - 4\xi - 12\mu|}} \\ &\le \frac{C}{|\xi|^{1/2}\langle 12\tau + 3\xi^3 - 4\xi\rangle^{1/2}}. \end{aligned} \tag{4.7}$$

Therefore,

$$\begin{aligned} I_1 &\le C\frac{|\xi|}{\langle \tau - \phi(\xi)\rangle^{1-b}} \left(\frac{1}{|\xi|^{1/2}\langle 12\tau + 3\xi^3 - 4\xi\rangle^{1/2}} \right)^{1/2} \\ &\le C\frac{|\xi|^{3/4}}{\langle \tau - (|\xi|\xi - \xi^3)\rangle^{1-b}\langle 12\tau + 3\xi^3 - 4\xi\rangle^{1/4}} \\ &= \theta_2(\xi, \tau) \le C, \end{aligned}$$

uniformly on $\mathbb{R}_\xi \times \mathbb{R}_\tau$ for $b \le 3/4$.

The case (iii) follows from a similar way of (ii). The case (iv) also follows similarly to (i). This proves the Lemma. □

Next we give the crucial part of the estimate. We divide the integral region for (ξ_1, τ_1) into the three cases.

Let A, B and B^* defined as

$$\begin{aligned} A =&A(\xi,\tau) \\ =&\Big\{(\xi_1,\tau_1)\in\mathbb{R}^2 \;\Big|\; |\xi_1|\ge 1, |\xi-\xi_1|\ge 1, \\ &\qquad |\tau-\tau_1-\phi(\xi-\xi_1)|\le|\tau_1-\phi(\xi_1)| \\ &\qquad\qquad \le|\tau-\phi(\xi)|\Big\}, \end{aligned} \tag{4.8}$$

$$\begin{aligned} B =&B(\xi_1,\tau_1) \\ =&\Big\{(\xi_1,\tau_1)\in\mathbb{R}^2 \;\Big|\; |\xi-\xi_1|\ge 1, |\xi_1|\ge 1, \\ &\qquad |\tau-\phi(\xi)|\le|\tau_1-\phi(\xi_1)|, \\ &\qquad |\tau-\tau_1-\phi(\xi-\xi_1)|\le|\tau_1-\phi(\xi_1)|\Big\}, \end{aligned} \tag{4.9}$$

$$\begin{aligned} B^* =&B^*(\xi,\tau) \\ =&\Big\{(\xi,\tau)\in\mathbb{R}^2 \;\Big|\; |\xi-\xi_1|\ge 1, |\xi_1|\ge 1, \\ &\qquad |\tau-\phi(\xi)|\le|\tau_1-\phi(\xi_1)|, \\ &\qquad |\tau-\tau_1-\phi(\xi-\xi_1)|\le|\tau_1-\phi(\xi_1)|\Big\}. \end{aligned} \tag{4.10}$$

LEMMA 4.5. *If* $s\in(-3/4,\ -1/2)$, $b\in(1/2,\ 3/4+s/3]$, $b'\in(1/2,\ b]$, *then for some* $C>0$,

(4.11)

$$\begin{aligned} I_2 \equiv& \frac{|\xi|}{\langle\tau-\phi_\nu(\xi)\rangle^{1-b}\langle\xi\rangle^{|s|}} \\ &\times\left(\iint_A \frac{|\xi_1(\xi-\xi_1)|^{2|s|}}{\langle\tau_1-\phi_\nu(\xi_1)\rangle^{2b'}\langle\tau-\tau_1-\phi_\nu(\xi-\xi_1)\rangle^{2b'}}d\tau_1 d\xi_1\right)^{1/2} \\ \le& C, \end{aligned}$$

where

$$\begin{aligned} A &= A(\xi,\tau) \\ &= \Big\{(\xi_1,\tau_1)\in\mathbb{R}^2 \;\Big|\; |\xi_1|\ge 1, |\xi-\xi_1|\ge 1, \\ &\qquad |\tau-\tau_1-\phi(\xi-\xi_1)| \le |\tau_1-\phi(\xi_1)| \\ &\qquad\qquad \le |\tau-\phi(\xi)|\Big\}. \end{aligned}$$

Proof. Similarly before we only consider the case $\nu = 1$. Since $|\tau - \phi(\xi-\xi_1) - \phi(\xi_1)| \le 2|\tau-\phi(\xi)|$ over A, by using (4.6),

$$\begin{aligned} &\int \frac{d\tau_1}{\langle \tau_1-\phi(\xi_1)\rangle^{2b'}\langle \tau-\tau_1-\phi(\xi-\xi_1)\rangle^{2b'}} \\ &\le C\frac{\chi\left(\frac{|\tau-\phi(\xi-\xi_1)-\phi(\xi_1)|}{2|\tau-\phi(\xi)|}\le 1\right)}{\langle \tau-\phi(\xi-\xi_1)-\phi(\xi_1)\rangle^{2b'}}. \end{aligned}$$

Thus

$$\begin{aligned} &\iint_A \frac{|\xi_1(\xi-\xi_1)|^{2|s|}}{\langle \tau_1-\phi(\xi_1)\rangle^{2b'}\langle \tau-\tau_1-\phi(\xi-\xi_1)\rangle^{2b'}}d\tau_1 d\xi_1 \\ &\le C\int \frac{|\xi_1(\xi-\xi_1)|^{2|s|}}{\langle \tau-\phi(\xi-\xi_1)-\phi(\xi_1)\rangle^{2b'}} \\ &\quad\times\chi\left(\left\{\frac{|\tau-\phi(\xi-\xi_1)-\phi(\xi_1)|}{2|\tau-\phi(\xi)|}\le 1\right\}\right)d\xi_1 \\ &\equiv \tilde{I}_2. \end{aligned} \tag{4.12}$$

Then similarly before, we divide the region A into four cases

(i) $\{(\xi_1,\xi);\ \xi\ge\xi_1,\ \ \xi_1\ge 0\}$, (ii) $\{(\xi_1,\xi);\ \xi\ge\xi_1,\ \ \xi_1< 0\}$,

(iii) $\{(\xi_1,\xi);\ \xi<\xi_1,\ \ \xi_1\ge 0\}$, (iv) $\{(\xi_1,\xi);\ \xi<\xi_1,\ \ \xi_1< 0\}$.

THE CASE (i). By changing

$$\mu \equiv \tau-\phi(\xi-\xi_1)-\phi(\xi_1) = \tau - \{\xi^2-\xi^3+(3\xi-2)\xi_1(\xi-\xi_1)\},$$

we have

$$\begin{aligned} \xi_1(\xi-\xi_1) &= \frac{\tau+\xi^3-\xi^2-\mu}{3\xi-2}, \\ |\xi_1(\xi-\xi_1)| &\le \frac{|\tau+\xi^3-\xi^2-\mu|}{|3\xi-2|} \le \frac{|\tau+\xi^3-\xi^2-\mu|}{|\xi|}. \end{aligned}$$

From (4.12),

$$\tilde{I}_2 \leq C \int_{|\mu| \leq 2|\tau-\phi(\xi)|} \frac{|\tau+\xi^3-\xi^2-\mu|^{2|s|}}{|\xi|^{2|s|+1/2}\langle\mu\rangle^{2b'}\sqrt{|4\tau+\xi^3-2\xi^2-4\mu|}}d\mu.$$

Since $\xi \geq 0$, $\phi(\xi)=\xi^2-\xi^3$, $|s|>1/2$, (4.5) implies

$$\tilde{I}_2 \leq C\frac{|\tau+\xi^3-\xi^2|^{2|s|}}{|\xi|^{2|s|+1/2}\langle 4\tau+\xi^3-2\xi^2\rangle^{1/2}},$$

and therefore

$$\begin{aligned} I_2 &\leq C\frac{|\xi|}{\langle\tau-\phi(\xi)\rangle^{1-b}\langle\xi\rangle^{|s|}}\left(\frac{|\tau+\xi^3-\xi^2|^{2|s|}}{|\xi|^{2|s|+1/2}\langle 4\tau+\xi^3-2\xi^2\rangle^{1/2}}\right)^{1/2} \\ &\leq C\frac{|\xi|^{3/4-|s|}\langle\tau+\xi^3-\xi^2\rangle^{|s|+b-1}}{\langle\xi\rangle^{|s|}\langle 4\tau+\xi^3-2\xi^2\rangle^{1/4}} \\ &= \Theta_1(\xi,\tau). \end{aligned}$$

Then $\Theta_1(\xi,\tau)$ is bounded when $3/4 - 2|s| + 3|s| + 3b - 3 = 3b + |s| - 9/4 \leq 0$ and this is fulfilled for some s, b, b' under the conditions of Lemma 4.5.

THE CASE (ii).

When (a) $\xi \geq 0$, setting

$$\begin{aligned} &\mu \equiv \tau-\phi(\xi-\xi_1)-\phi(\xi_1) = \tau-\{\xi^2-\xi^3-2\xi\xi_1+3\xi\xi_1(\xi-\xi_1)\}, \\ &\xi_1(\xi-\xi_1) = \frac{1}{3\xi}\{\tau+\xi^3-\xi^2-\mu+2\xi\xi_1\}, \\ &|\xi_1(\xi-\xi_1)| \leq C\left\{\frac{|\tau+\xi^3-\xi^2-\mu|}{|\xi|}+\frac{|\xi\xi_1|}{|\xi|}\right\}, \\ &\xi\xi_1 = \frac{\tau+\xi^3-\xi^2-\mu}{3(\xi-\xi_1)-2}, \\ &|\xi\xi_1| \leq \frac{|\tau+\xi^3-\xi^2-\mu|}{|3(\xi-\xi_1)-2|} \leq |\tau+\xi^3-\xi^2-\mu|, \\ &|\xi_1(\xi-\xi_1)| \leq C\frac{|\tau+\xi^3-\xi^2-\mu|}{|\xi|}. \end{aligned}$$

Thus by (4.12)

$$
\begin{aligned}
\tilde{I}_2 &\leq C\int_{|\mu|\leq 2|\tau-\phi(\xi)|}\frac{|\tau+\xi^3-\xi^2-\mu|^{2|s|}d\mu}{|\xi|^{2|s|+1/2}\langle\mu\rangle^{2b'}\sqrt{|12\tau+3\xi^3-4\xi-12\mu|}}\\
&\leq C\frac{|\tau+\xi^3-\xi^2|^{2|s|}}{|\xi|^{2|s|+1/2}\langle 12\tau+3\xi^3-4\xi\rangle^{1/2}}\\
I_2 &\leq C\frac{|\xi|}{\langle\tau-\phi(\xi)\rangle^{1-b}\langle\xi\rangle^{|s|}}\left(\frac{|\tau+\xi^3-\xi^2|^{2|s|}}{|\xi|^{2|s|+1/2}\langle 12\tau+3\xi^3-4\xi\rangle^{1/2}}\right)^{1/2}\\
&\leq C\frac{|\xi|^{3/4-|s|}\langle\tau+\xi^3-\xi^2\rangle^{|s|+b-1}}{\langle\xi\rangle^{|s|}\langle 12\tau+3\xi^3-4\xi\rangle^{1/4}}\\
&=\Theta_{2,a}(\xi,\tau).
\end{aligned}
$$

Hence $\Theta_{2,a}(\xi,\tau)$ is bounded under the conditions for s, b, b' of Lemma 4.5 on $\mathbb{R}_\xi\times\mathbb{R}_\tau$.

When (b) $\xi<0$, letting

$$
\begin{aligned}
\mu &\equiv \tau-\phi(\xi-\xi_1)-\phi(\xi_1)\\
&=\tau-\{\xi^2-\xi^3-2\xi\xi_1+3\xi\xi_1(\xi-\xi_1)\},\\
\xi_1(\xi-\xi_1) &= \frac{1}{3\xi}\{\tau+\xi^3+\xi^2-\mu-2\xi(\xi-\xi_1)\},\\
\xi(\xi-\xi_1) &= \frac{\tau+\xi^3+\xi^2-\mu}{3\xi_1+2},
\end{aligned}
$$

then combining

$$
\begin{aligned}
|\xi(\xi-\xi_1)| &\leq \frac{|\tau+\xi^3+\xi^2-\mu|}{|3\xi_1+2|}\leq|\tau+\xi^3+\xi^2-\mu|,\\
|\xi_1(\xi-\xi_1)| &\leq C\left\{\frac{|\tau+\xi^3+\xi^2-\mu|}{|\xi|}+\frac{|\xi(\xi-\xi_1)|}{|\xi|}\right\},
\end{aligned}
$$

we see

$$
|\xi(\xi-\xi_1)|\leq C\frac{|\tau+\xi^3+\xi^2-\mu|}{|\xi|},
$$

and it follows from (4.12)

$$
\begin{aligned}
\tilde{I}_2 &\leq C\int_{|\mu|\leq 2|\tau-\phi(\xi)|}\frac{|\tau+\xi^3+\xi^2-\mu|^{2|s|}}{|\xi|^{2|s|+1/2}\langle\mu\rangle^{2b'}\sqrt{|12\tau+3\xi^3-4\xi-12\mu|}}\\
&\leq C\frac{|\tau+\xi^3+\xi^2|^{2|s|}}{|\xi|^{2|s|+1/2}\langle 12\tau+3\xi^3-4\xi\rangle^{1/2}}.
\end{aligned}
$$

Thus

$$\begin{aligned} I_2 &\le C\frac{|\xi|}{\langle\tau-\phi(\xi)\rangle^{1-b}\langle\xi\rangle^{|s|}}\left(\frac{|\tau+\xi^3+\xi^2|^{2|s|}}{|\xi|^{2|s|+1/2}\langle 12\tau+3\xi^3-4\xi\rangle^{1/2}}\right)^{1/2} \\ &\le C\frac{|\xi|^{3/4-|s|}\langle\tau+\xi^3+\xi^2\rangle^{|s|+b-1}}{\langle\xi\rangle^{|s|}\langle 12\tau+3\xi^3-4\xi\rangle^{1/4}} \\ &= \Theta_{2,b}(\xi,\tau). \end{aligned}$$

Hence $\Theta_{2,b}(\xi,\tau)$ is bounded under the conditions of Lemma 4.5 on $\mathbb{R}_\xi \times \mathbb{R}_\tau$.

The case (iii) follows from analogous argument of (ii). The case (iv) also follows similarly to (i). □

LEMMA 4.6. *For any $s \in (-3/4,-1/2)$, there exists $b \in (1/2,1)$ such that for all b' with $b-b' \le \min(|s|-1/2, 1/4-|s|/3)$ we have the following inequality for some $C>0$.*

$$\begin{aligned} &\frac{1}{\langle\tau_1-\phi_\nu(\xi_1)\rangle^{b'}} \\ &\times\left(\int\!\!\int_B \frac{|\xi|^{2(1-|s|)}|\xi\xi_1(\xi-\xi_1)|^{2|s|}d\tau d\xi}{\langle\xi\rangle^{2|s|}\langle\tau-\phi_\nu(\xi)\rangle^{2(1-b)}\langle\tau-\tau_1-\phi_\nu(\xi-\xi_1)\rangle^{2b'}}\right)^{1/2} \\ &\le C, \end{aligned} \tag{4.13}$$

where

$$\begin{aligned} B &= B(\xi_1,\tau_1) \\ &= \Big\{(\xi,\tau)\in\mathbb{R}^2 \;\Big|\; |\xi-\xi_1|\ge 1, |\xi_1|\ge 1, \\ &\qquad |\tau-\phi(\xi)| \le |\tau_1-\phi(\xi_1)|, \\ &\qquad |\tau-\tau_1-\phi(\xi-\xi_1)| \le |\tau_1-\phi(\xi_1)|\Big\}. \end{aligned}$$

Proof. As before, we set $\phi(\xi)=\phi_1(\xi)$. Over B, it holds $|\tau_1+\phi(\xi-\xi_1)-\phi(\xi)| \le 2|\tau_1-\phi(\xi_1)|$. Therefore from (4.5),

$$\begin{aligned} &\int \frac{d\tau}{\langle\tau-\phi(\xi)\rangle^{2(1-b)}\langle\tau-\tau_1-\phi(\xi-\xi_1)\rangle^{2b'}} \\ &\le C\frac{1}{\langle\tau_1+\phi(\xi-\xi_1)-\phi(\xi)\rangle^{2(1-b)}}. \end{aligned}$$

Then it suffices to show that $I(B') \le C$, where

$$I(D) \equiv \frac{1}{\langle \tau_1 - \phi(\xi_1)\rangle^{b'}} \left(\int_D \frac{|\xi|^{2(1-|s|)}|\xi\xi_1(\xi-\xi_1)|^{2|s|}}{\langle \xi \rangle^{2|s|}\langle \tau_1 + \phi(\xi-\xi_1) - \phi(\xi)\rangle^{2(1-b)}} d\xi \right)^{1/2}$$

and

$$\begin{aligned} B' &= B'(\xi_1, \tau_1) \\ &= \{\xi \in \mathbb{R} \mid |\xi - \xi_1| \ge 1, |\xi_1| \ge 1, \\ &\qquad |\tau_1 + \phi(\xi-\xi_1) - \phi(\xi)| \le 2|\tau_1 - \phi(\xi_1)|\}. \end{aligned}$$

Then similarly before, we divide the region B' into four cases

$$\text{(i)} \quad \{(\xi_1,\xi);\ \xi \ge \xi_1,\ \ \xi_1 \ge 0\}, \quad \text{(ii)} \quad \{(\xi_1,\xi);\ \xi \ge \xi_1,\ \ \xi_1 < 0\},$$

$$\text{(iii)} \quad \{(\xi_1,\xi);\ \xi < \xi_1,\ \ \xi_1 \ge 0\}, \quad \text{(iv)} \quad \{(\xi_1,\xi);\ \xi < \xi_1,\ \ \xi_1 < 0\}.$$

THE CASE (i). $B'_1 = \{\xi \in B' \mid \ \xi \ge \xi_1, \xi \ge 0\}$,

$$\tau_1 + \phi(\xi - \xi_1) - \phi(\xi) = \tau_1 - \{{\xi_1}^2 - {\xi_1}^3 + (2 - 3\xi)\xi_1(\xi - \xi_1)\}.$$

Here we divide $B'_1 = \{\xi \in B';\ \ |\xi| \ge \xi_1,\ \xi \ge 0\}$ into the cases $B'_{1+} \equiv B'_1 \cap \{\xi_1 < 0\} = B'^1_{1+} \cup B'^2_{1+}$ and $B'_{1-} \equiv B'_1 \cap \{\xi_1 < 0\} = B'^1_{1-} \cup B'^2_{1-}$, where

$$\begin{aligned} B'^1_{1+} &= \{\xi \in B'_1 |\ 0 \le \xi_1, \\ &\qquad |(2-3\xi)\xi_1(\xi-\xi_1)| \le \frac{1}{2}|\tau_1 - ({\xi_1}^2 - {\xi_1}^3)|\}, \\ B'^2_{1+} &= \{\xi \in B'_1 |\ 0 \le \xi_1, \\ &\qquad \frac{1}{2}|\tau_1 - ({\xi_1}^2 - {\xi_1}^3)| \le |(2-3\xi)\xi_1(\xi-\xi_1)| \\ &\qquad\qquad \le 3|\tau_1 - ({\xi_1}^2 - {\xi_1}^3)|\}, \\ B'^1_{1-} &= \{\xi \in B'_1 \mid 0 > \xi_1, \\ &\qquad |2\xi\xi_1 - 3\xi\xi_1(\xi-\xi_1)| \le \frac{1}{2}|\tau_1 - ({\xi_1}^3 + {\xi_1}^2)|\ \}, \\ B'^2_{1-} &= \{\xi \in B'_1 \mid\ 0 > \xi_1, \\ &\qquad \frac{1}{2}|\tau_1 + {\xi_1}^2 + {\xi_1}^2)| \le |2\xi\xi_1 - 3\xi\xi_1(\xi-\xi_1)| \\ &\qquad\qquad \le 3|\tau_1 + {\xi_1}^2 + {\xi_1}^3|\}. \end{aligned} \tag{4.14}$$

On $B_{1\pm}^{'1}$, we see that

$$|\xi\xi_1(\xi-\xi_1)| \le |(2-3\xi)\xi_1(\xi-\xi_1)| \le \frac{1}{2}|\tau_1 \mp \xi_1^2 + \xi_1^3)|, \tag{4.15}$$

$$\frac{1}{2}|\tau_1 + \xi_1^3 \mp \xi_1^2| \le |\tau_1 - \{\xi_1^2 - \xi_1^3 + (2-3\xi)\xi_1(\xi-\xi_1)\}|, \tag{4.16}$$

$$|\xi| \le |(2-3\xi)\xi_1(\xi-\xi_1)| \le \frac{1}{2}|\tau_1 \mp \xi_1^2 + \xi_1^3)|, \tag{4.17}$$

$$\begin{aligned}
& I(B_{1+}^{'1}) \\
& \le \frac{1}{\langle \tau_1 - (\xi_1^2 - \xi_1^3)\rangle^{b'}} \\
& \quad \times \left(\frac{|\xi|^{2(1-|s|)} |\xi\xi_1(\xi-\xi_1)|^{2|s|}}{\langle\xi\rangle^{2|s|} \langle \tau_1 - \{\xi_1^2 - \xi_1^3 + (2-3\xi)\xi_1(\xi-\xi_1)\}\rangle^{2(1-b)}} \right)^{1/2} \\
& \le C\langle \tau_1 - (\xi_1^2 - \xi_1^3)\rangle^{-b'+|s|+b-1} \left(\int_{|\xi| \le \frac{1}{2}|\tau_1 - (\xi_1^2 - \xi_1^3)|} \frac{|\xi|^{2(1-|s|)}}{\langle\xi\rangle^{2|s|}} d\xi \right)^{1/2} \\
& \le C\langle \tau_1 - (\xi_1^2 - \xi_1^3)\rangle^{-b'+|s|+b-1-2|s|+3/2}.
\end{aligned}$$

The last function is bounded since $-b' + |s| + b - 1 - 2|s| + 3/2 = b - b' - |s| + 1/2 \le 0$.

Similarly,

$$\begin{aligned}
& I(B_{1-}^{'1}) \\
& \le C\langle \tau_1 + \xi_1^3 + \xi_1^2\rangle^{-b'+|s|+b-1} \left(\int_{|\xi| \le \frac{1}{2}|\tau_1 + \xi_1^3 + \xi_1^2|} \frac{|\xi|^{2(1-|s|)}}{\langle\xi\rangle^{2|s|}} d\xi \right)^{1/2} \\
& \le C\langle \tau_1 + \xi_1^3 + \xi_1^2\rangle^{-b'+b-|s|+1/2} \le C,
\end{aligned}$$

since $-b'+b-|s|+1/2 \le 0$. Next we split $B_{1\pm}^{'2}$ into $B_{1\pm}^{'21}$, $B_{1\pm}^{'22}$ and $B_{1\pm}^{'23}$, where

$$\begin{aligned}
B_{1\pm}^{'21} &= \{\xi \in B_{1\pm}^{'2} \mid |\xi|/4 \le |\xi_1| \le 100|\xi| \}, \\
B_{1\pm}^{'22} &= \{\xi \in B_{1\pm}^{'2} \mid 1 \le |\xi_1| \le |\xi|/4 \}, \\
B_{1\pm}^{'23} &= \{\xi \in B_{1\pm}^{'2} \mid 100|\xi| \le |\xi_1| \}.
\end{aligned}$$

For the estimate on $B_{1\pm}^{'2}$,

$$|\xi\xi_1(\xi-\xi_1)| \le |(2-3\xi)\xi_1(\xi-\xi_1)| \le 3|\tau_1 - (\xi_1^2 - \xi_1^3)|. \tag{4.18}$$

While over $B_{1\pm}^{'21}$,

$$C\langle \tau_1 \pm \xi_1^2 + \xi_1^3)\rangle \le C|\xi_1|^3 \sim C|\xi|^3 \tag{4.19}$$

we see

$$
\begin{aligned}
&I(B'^{21}_{1\pm})\\
&\le \langle \tau_1 \mp {\xi_1}^2 + {\xi_1}^3\rangle^{-b'}\\
&\quad\times\left(\int \frac{|\xi|^{2(1-2|s|)}|\xi\xi_1(\xi-\xi_1)|^{2|s|}d\xi}{\langle \tau_1 \mp {\xi_1}^2 + {\xi_1}^3 - (2-3\xi)\xi_1(\xi-\xi_1)\rangle^{2(1-b)}}\right)^{1/2}\\
&\quad(\text{by setting } \mu_1 = \tau_1 \mp {\xi_1}^2 + {\xi_1}^3 - (2-3\xi)\xi_1(\xi-\xi_1))\\
&\le C\langle \tau_1 \mp {\xi_1}^2 + {\xi_1}^3\rangle^{-b'+(1-2|s|)/3+|s|}\\
&\quad\times\left(\int_{|\mu_1|\le 3|\tau_1\mp{\xi_1}^2+{\xi_1}^3|} \frac{d\mu_1}{|\xi_1|^{\frac12}\langle\mu_1\rangle^{2(1-b)}\sqrt{|12\tau_1+3{\xi_1}^3-4\xi_1-12\mu_1|}}\right)^{\frac12}\\
&\le C\langle \tau_1 \mp {\xi_1}^2 + {\xi_1}^3\rangle^{-b'+(1-2|s|)/3+|s|-1/12}\left(\frac{\langle \tau_1 \mp {\xi_1}^2 + {\xi_1}^3\rangle^{2(b-1/2)}}{\langle 12\tau_1+3{\xi_1}^3-4\xi_1\rangle^{1/2}}\right)^{\frac12}\\
&\le C\frac{\langle \tau_1 \mp {\xi_1}^2 + {\xi_1}^3\rangle^{-b'+(1-2|s|)/3+|s|-1/12+b-1/2}}{\langle 12\tau_1+3{\xi_1}^3-4\xi_1\rangle^{1/4}}.
\end{aligned}
$$

This is bounded since $-b' + (1-2|s|)/3 + |s| - 1/12 + b - 1/2 = -b' + b + |s|/3 - 1/4 \le 0$.

When $B'^{22}_{1+} \cup B'^{23}_{1+}$

$$|6\xi - 3\xi_1 - 2| \sim |\xi| \sim |\xi - \xi_1|, \tag{4.20}$$

$$\frac{1}{2}\langle \tau_1 - ({\xi_1}^2 - {\xi_1}^3)\rangle \le |(2-3\xi)\xi_1(\xi-\xi_1)| \sim |\xi|^2|\xi_1| \le C|\xi|^3, \tag{4.21}$$

setting $\mu_1 = \tau_1 - \{{\xi_1}^2 - {\xi_1}^3 + (2-3\xi)\xi_1(\xi-\xi_1)\}$, we see

$$
\begin{aligned}
&I(B'^{22}_{1+} \cup B'^{23}_{1+})\\
&\le \langle \tau_1 - ({\xi_1}^2 - {\xi_1}^3)\rangle^{-b'}\\
&\quad\times\left(\int \frac{|\xi|^{2(1-2|s|)}|\xi\xi_1(\xi-\xi_1)|^{2|s|}d\xi}{\langle \tau_1 - \{{\xi_1}^2 - {\xi_1}^3 + (2-3\xi)\xi_1(\xi-\xi_1)\}\rangle^{2(1-b)}}\right)^{1/2}\\
&\le C\langle \tau_1 - ({\xi_1}^2 - {\xi_1}^3)\rangle^{-b'+(1-2|s|)/3+|s|}\\
&\quad\times\left(\int_{|\mu_1|\le 2|\tau_1-({\xi_1}^2-{\xi_1}^3)|} \frac{d\mu_1}{\langle\mu_1\rangle^{2(1-b)}|\xi_1||6\xi-3\xi_1-2|}\right)^{1/2}
\end{aligned}
$$

(by using (4.20))

$$
\begin{aligned}
&\leq C\langle \tau_1 - ({\xi_1}^2 - {\xi_1}^3)\rangle^{-b'+(1-2|s|)/3+|s|} \\
&\quad \times \left(\int_{|\mu_1| \leq 2|\tau_1 - ({\xi_1}^2 - {\xi_1}^3)|} \frac{d\mu_1}{\langle \mu_1 \rangle^{2(1-b)} |\xi_1|^{1/2} (|\xi_1|^{1/2} |\xi|)} \right)^{1/2}
\end{aligned}
$$

(by using (4.21))

$$
\begin{aligned}
&\leq C\langle \tau_1 - ({\xi_1}^2 - {\xi_1}^3)\rangle^{-b'+(1-2|s|)/3+|s|} \\
&\quad \times \left(\int_{|\mu_1| \leq 2|\tau_1 - ({\xi_1}^2 - {\xi_1}^3)|} \frac{d\mu_1}{\langle \mu_1 \rangle^{2(1-b)} \langle \tau_1 - ({\xi_1}^2 - {\xi_1}^3)\rangle^{1/2}} \right)^{1/2} \\
&\leq C\langle \tau_1 - ({\xi_1}^2 - {\xi_1}^3)\rangle^{-b'+(1-2|s|)/3+|s|} \\
&\quad \times \left(\int_{|\mu_1| \leq 2|\tau_1 - ({\xi_1}^2 - {\xi_1}^3)|} \frac{d\mu_1}{\langle \mu_1 \rangle^{2(1-b)}} \right)^{1/2} \\
&\leq C\langle \tau_1 - ({\xi_1}^2 - {\xi_1}^3)\rangle^{-b'+(1-2|s|)/3+|s|-1/4+b-1+1/2}.
\end{aligned}
$$

This is also bounded since $-b' + (1-2|s|)/3 + |s| - 1/4 + b - 1 + 1/2 = -b' + b + |s|/3 - 5/12 \leq 0$.

Now on $B_{1-}^{\prime 22}$,

$$
\begin{aligned}
&|6\xi - 3\xi_1 - 2| \sim |\xi| \sim |\xi - \xi_1| \\
&C\langle \tau_1 + {\xi_1}^3 + {\xi_1}^2 \rangle \leq |2\xi\xi_1 - 3\xi\xi_1(\xi - \xi_1)| \sim |\xi|^2 |\xi_1| \leq C|\xi|^3
\end{aligned} \tag{4.22}
$$

$$
\begin{aligned}
&I(B_{1-}^{\prime 22})\\
&\le C\langle \tau_1+{\xi_1}^3+{\xi_1}^2\rangle^{-b'}\\
&\quad\times\left(\int_{|\mu_1|\le 2|\tau_1+{\xi_1}^3+{\xi_1}^2|}\frac{|\xi|^{2(1-2|s|)}|\xi\xi_1(\xi-\xi_1)|^{2|s|}d\mu_1}{|\xi_1||2-3\xi_1+6\xi|\langle\mu_1\rangle^{2(1-b)}}\right)^{1/2}\\
&\le C\langle \tau_1+{\xi_1}^3+{\xi_1}^2\rangle^{-b'+(1-2|s|)/3+|s|}\\
&\quad\times\left(\int_{|\mu_1|\le 2|\tau_1+{\xi_1}^3+{\xi_1}^2|}\frac{d\mu_1}{\langle\mu_1\rangle^{2(1-b)}|\xi_1|^{1/2}\langle\tau_1+{\xi_1}^3+{\xi_1}^2\rangle^{1/2}}\right)^{1/2}\\
&\le C\langle \tau_1+{\xi_1}^3+{\xi_1}^2\rangle^{-b'+(1-2|s|)/3+|s|-1/4}\\
&\quad\times\left(\int_{|\mu_1|\le 2|\tau_1+{\xi_1}^3+{\xi_1}^2|}\frac{d\mu_1}{\langle\mu_1\rangle^{2(1-b)}}\right)^{1/2}\\
&\le C\langle \tau_1+{\xi_1}^3+{\xi_1}^2\rangle^{-b'+(1-2|s|)/3+|s|-1/4+b-1+1/2},
\end{aligned}
$$

which is bounded since $-b'+(1-2|s|)/3+|s|-1/4+b-1+1/2=-b'+b+|s|/3-5/12\le 0$.

On $B_{1-}^{\prime 23}$, we can see

$$|\tau_1+{\xi_1}^3+{\xi_1}^2|\le C|\xi_1|^3, \tag{4.23}$$

$$|12\tau_1+3{\xi_1}^3-4\xi_1|\sim|\xi_1|^3. \tag{4.24}$$

Therefore

$$
\begin{aligned}
&I(B_{1-}^{\prime 23})\\
&\le C\langle \tau_1+{\xi_1}^3+{\xi_1}^2\rangle^{-b'+|s|}\\
&\quad\times\left(\int_{|\mu_1|\le 2|\tau_1+{\xi_1}^3+{\xi_1}^2|}\frac{d\mu_1}{|\xi_1|^{\frac12}\langle\mu_1\rangle^{2(1-b)}\sqrt{|12\tau_1+3{\xi_1}^3-4\xi_1-12\mu_1|}}\right)^{\frac12}\\
&\le C\langle \tau_1+{\xi_1}^3+{\xi_1}^2\rangle^{-b'+|s|-1/12}\left(\frac{\langle\tau_1+{\xi_1}^3+{\xi_1}^2\rangle^{2(b-1/2)}}{\langle 12\tau_1+3{\xi_1}^3-4\xi_1\rangle^{\frac12}}\right)^{\frac12}\\
&\le C\langle \tau_1+{\xi_1}^3+{\xi_1}^2\rangle^{-b'+|s|-1/12+b-1/2-1/4}
\end{aligned}
$$

which is bounded since $-b'+|s|-1/12+b-1/2-1/4=-b'+b+|s|-5/6\le 0$.

CASE (ii). As before we set $B_2' = \{\xi \in B' \mid \xi \geq \xi_1,\ \xi_1 < 0\}$. Then we observe that on B_2', $\mu_1 = \tau_1 + \phi(\xi - \xi_1) - \phi(\xi_1) = \tau_1 + {\xi_1}^3 + {\xi_1}^2 + (3\xi_1 - 2)\xi(\xi - \xi_1)$,

$$d\mu_1 = (3\xi_1 + 2)(2\xi - \xi_1)d\xi_1,$$

$$|\xi| = \frac{1}{2}\left\{\xi_1 \pm \sqrt{\frac{4\mu_1 - {\xi_1}^3 - 2{\xi_1}^2 - 4\tau_1}{3\xi_1 + 2}}\right\},$$

$$|(3\xi_1 + 2)(2\xi - \xi_1)| = |(3\xi_1 + 2)|^{1/2}\sqrt{|4\tau_1 + {\xi_1}^3 + 2{\xi_1}^2 - 4\mu_1|}.$$

Let $B_2' = B_2'^1 \cup B_2'^2$, where

$$B_2'^1 = \{\xi \in B_2' \mid\ |(3\xi_1 + 2)\xi(\xi - \xi_1)| \leq \frac{1}{2}|\tau_1 + {\xi_1}^3 + {\xi_1}^2|\},$$

$$\begin{aligned} B_2'^2 = \{\xi \in B_2' \mid\ \frac{1}{2}|\tau_1 + {\xi_1}^3 + {\xi_1}^2| &\leq |(3\xi_1 + 2)\xi(\xi - \xi_1)| \\ &\leq 3|\tau_1 + {\xi_1}^3 + {\xi_1}^2|\}. \end{aligned}$$

We have over $B_2'^1$, we have

$$\begin{aligned} &\frac{1}{2}|\tau_1 + {\xi_1}^3 + {\xi_1}^2| \leq |\tau_1 + {\xi_1}^3 + {\xi_1}^2 + (3\xi_1 - 2)\xi(\xi - \xi_1)\ |, \\ &|\xi| \leq |(3\xi_1 + 2)\xi(\xi - \xi_1)| \leq \frac{1}{2}|\tau_1 + {\xi_1}^3 + {\xi_1}^2|, \qquad (4.25) \\ &|\xi\xi_1(\xi - \xi_1)| \leq \frac{1}{2}|\tau_1 + {\xi_1}^3 + {\xi_1}^2|. \end{aligned}$$

Therefore

$$\begin{aligned} I(B_2'^1) &\leq C\langle \tau_1 + {\xi_1}^3 + {\xi_1}^2 \rangle^{-b' + |s| + b - 1} \\ &\quad \times \left(\int_{|\xi| \leq \frac{1}{2}|\tau_1 + {\xi_1}^3 + {\xi_1}^2|} \frac{|\xi|^{2(1-|s|)}}{\langle \xi \rangle^{2|s|}} d\xi \right)^{1/2} \qquad (4.26) \\ &\leq C\langle \tau_1 + {\xi_1}^3 + {\xi_1}^2 \rangle^{-b' + |s| + b - 1 - 2|s| + 2/3}. \end{aligned}$$

Now we note that $-b' + |s| + b - 1 - 2|s| + 2/3 = b - b' + 1/2 - |s| \leq 0$. Hence (4.26) is bounded by a constant.

Let $B_2'^2 = B_2'^{21} \cup B_2'^{22}$, where

$$B_2'^{21} = \{\xi \in B_2'^2 \mid\ |\xi| \leq |\xi_1| \leq 100|\xi|\},$$

$$B_2'^{22} = \{\xi \in B_2'^2 \mid\ 100|\xi| \leq |\xi_1|\}.$$

Over $B_2'^{21}$, we have

$$C|\tau_1+\xi_1{}^3+\xi_1{}^2| \le |\xi_1|^3 \sim |\xi|^3 \tag{4.27}$$

and

$$\begin{aligned}
&I(B_2'^{21})\\
&\le C\langle\tau_1+\xi_1{}^3+\xi_1{}^2\rangle^{-b'+|s|+(1-2|s|)/3}\\
&\quad\times\left(\int_{|\mu_1|\le 2|\tau_1+\xi_1{}^3+\xi_1{}^2|}\frac{d\mu_1}{\langle\mu_1\rangle^{2(1-b)}|3\xi_1+2|^{\frac12}\sqrt{|4\tau_1+\xi_1{}^3+2\xi_1{}^2-4\mu_1|}}\right)^{\frac12}\\
&\le C\langle\tau_1+\xi_1{}^3+\xi_1{}^2\rangle^{-b'+|s|+(1-2|s|)/3-1/12}\\
&\quad\times\left(\frac{\langle\tau_1+\xi_1{}^3+\xi_1{}^2\rangle^{2(b-1/2)}}{\langle 4\tau_1+\xi_1{}^3+2\xi_1{}^2\rangle^{1/2}}\right)^{1/2}\\
&\le C\frac{\langle\tau_1+\xi_1{}^3+\xi_1{}^2\rangle^{b-b'+|s|/3-1/4}}{\langle 4\tau_1+\xi_1{}^3+2\xi_1{}^2\rangle^{1/4}}.
\end{aligned}$$

This is bounded since $b-b'+|s|/3-1/4\le 0$.

On $B_2'^{22}$, we have

$$|\tau_1+\xi_1{}^3+\xi_1{}^2|\le C|\xi_1|^3, \tag{4.28}$$

$$|4\tau_1+\xi_1{}^3+2\xi_1{}^2|\sim|\xi_1|^3 \tag{4.29}$$

and then

$$\begin{aligned}
&I(B_2'^{22})\\
&\le C\langle\tau_1+\xi_1{}^3+\xi_1{}^2\rangle^{-b'+|s|}\\
&\quad\times\left(\int_{|\mu_1|\le 2|\tau_1+\xi_1{}^3+\xi_1{}^2|}\right.\\
&\qquad\left.\times\frac{d\mu_1}{\langle\mu_1\rangle^{2(1-b)}|3\xi_1+2|^{1/2}\sqrt{|4\tau_1+\xi_1{}^3+2\xi_1{}^2-4\mu_1|}}\right)^{1/2}\\
&\le C\langle\tau_1+\xi_1{}^3+\xi_1{}^2\rangle^{-b'+|s|-1/12}\left(\frac{\langle\tau_1+\xi_1{}^3+\xi_1{}^2\rangle^{2(b-1/2)}}{\langle 4\tau_1+\xi_1{}^3+2\xi_1{}^2\rangle^{1/2}}\right)^{1/2}\\
&\le C\langle\tau_1+\xi_1{}^3+\xi_1{}^2\rangle^{-b'+|s|-1/12-1/4+b-1/2}.
\end{aligned}$$

which is bounded since $-b'+|s|-1/12-1/4+b-1/2=b-b'+|s|-5/6\le 0$.

The case (iii) follows from analogous argument of (ii). The case (iv) also follows similarly to (i). □

We now stand the place for proving Proposition 4.2.

Proof for Proposition 4.2.

When $s = 0$, from Lemma 4.4 and Schwartz's inequality,

$$\begin{aligned}
&\left\| \frac{\xi}{\langle \tau - \phi_\nu(\xi)\rangle^{1-b}} \right. \\
&\qquad \left. \times \iint \frac{f(\xi_1, \tau_1)}{\langle \tau_1 - \phi_\nu(\xi_1)\rangle^{b'}} \frac{f(\xi - \xi_1, \tau - \tau_1)}{\langle \tau - \tau_1 - \phi_\nu(\xi - \xi_1)\rangle^{b'}} d\xi_1 d\tau_1 \right\|_{L^2_\xi L^2_\tau} \\
&\le \left\| \frac{\xi}{\langle \tau - \phi_\nu(\xi)\rangle^{1-b}} \right. \\
&\qquad \left. \times \left(\iint \frac{d\xi_1 d\tau_1}{\langle \tau_1 - \phi_\nu(\xi_1)\rangle^{2b'} \langle \tau - \tau_1 - \phi_\nu(\xi - \xi_1)\rangle^{2b'}} \right)^{1/2} \right\|_{L^\infty_\xi L^\infty_\tau} \\
&\quad \times \left\| \left(\iint |f(\xi_1, \tau_1)|^2 |f(\xi - \xi_1, \tau - \tau_1)|^2 d\xi_1 d\tau_1 \right)^{1/2} \right\|_{L^2_\xi L^2_\tau} \\
&\le C \|f\|^2_{L^2_\xi L^2_\tau}.
\end{aligned}$$

Next for $|s| = -s \in (1/2, 3/4)$, when $|\xi_1| \le 1$ or $|\xi - \xi_1| \le 1$, it holds

$$\langle \xi_1 \rangle^{|s|} \langle \xi - \xi_1 \rangle^{|s|} \le C \langle \xi \rangle^{|s|}$$

and this case is reduced when $|s| = 0$.

Now the other case, $|\xi_1| \ge 1$ B+ D(J $|\xi - \xi_1| \ge 1$, by the symmetry, we may assume that $|\tau - \tau_1 - \phi_\nu(\xi - \xi_1)| \le |\tau_1 - \phi_\nu(\xi_1)|$ without losing the generality.

We divide the integral region into two parts:
(1) $|\tau_1 - \phi_\nu(\xi_1)| \le |\tau - \phi_\nu(\xi)|$ and
(2) $|\tau_1 - \phi_\nu(\xi_1)| \ge |\tau - \phi_\nu(\xi)|$.

THE CASE (1). Since $(\xi_1,\tau_1) \in A$, by applying Lemma 4.5 and Schwartz inequality, it follows

$$\begin{aligned}
&\Bigg\| \frac{\xi}{\langle \tau - \phi_\nu(\xi)\rangle^{1-b}\langle \xi\rangle^{|s|}} \\
&\qquad \times \iint \frac{f(\xi_1,\tau_1)\langle \xi_1\rangle^{|s|}}{\langle \tau_1 - \phi_\nu(\xi_1)\rangle^{b'}} \frac{f(\xi-\xi_1,\tau-\tau_1)\langle \xi-\xi_1\rangle^{|s|}}{\langle \tau-\tau_1-\phi_\nu(\xi-\xi_1)\rangle^{b'}} d\xi_1 d\tau_1 \Bigg\|_{L^2_\xi L^2_\tau} \\
&\le \Bigg\| \frac{\xi}{\langle \tau - \phi_\nu(\xi)\rangle^{1-b}\langle \xi-\xi_1\rangle^{|s|}} \\
&\qquad \times \left(\iint_A \frac{|\xi_1(\xi-\xi_1)|^{2|s|} d\xi_1 d\tau_1}{\langle \tau_1 - \phi_\nu(\xi_1)\rangle^{2b'}\langle \tau-\tau_1-\phi_\nu(\xi-\xi_1)\rangle^{2b'}} \right)^{1/2} \Bigg\|_{L^\infty_\xi L^\infty_\tau} \\
&\quad \times \left\| \left(\iint |f(\xi_1,\tau_1)|^2 |f(\xi-\xi_1,\tau-\tau_1)|^2 d\xi_1 d\tau_1 \right)^{1/2} \right\|_{L^2_\xi L^2_\tau} \\
&\le C\|f\|^2_{L^2_\xi L^2_\tau}.
\end{aligned}$$

For the case (2),

$$\begin{aligned}
h(\xi,\tau) &= \frac{\xi}{\langle \tau - \phi_\nu(\xi)\rangle^{1-b}\langle \xi\rangle^{|s|}} \\
&\qquad \times \iint \chi_B \frac{f(\xi_1,\tau_1)\langle \xi_1\rangle^{|s|}}{\langle \tau_1 - \phi_\nu(\xi_1)\rangle^{b'}} \frac{f(\xi-\xi_1,\tau-\tau_1)\langle \xi-\xi_1\rangle^{|s|}}{\langle \tau-\tau_1-\phi_\nu(\xi-\xi_1)\rangle^{b'}} d\xi_1 d\tau_1,
\end{aligned}$$

$$\chi_B = \chi_B(\xi_1,\tau_1) = \begin{cases} 1 & \text{if } (\xi_1,\tau_1) \in B(\tau,\xi), \\ 0 & \text{otherwise.} \end{cases}$$

Let $g(\xi,\tau) \in L^2_\xi L^2_\tau$. By using Lemma 4.6,

$$\begin{aligned}
&\langle g,h\rangle_{L^2_\xi L^2_\tau} \\
&\le C \iint_{\mathbb{R}^2} \frac{|f(\xi_1,\tau_1)|}{\langle \tau_1 - \phi_\nu(\xi_1)\rangle^{b'}} \\
&\quad \times \Bigg(\iint_{\mathbb{R}^2} \chi_B(\tau_1,\xi_1) \frac{|g(\xi,\tau)||f(\xi-\xi_1,\tau-\tau_1)||\xi|^{1-|s|}|\xi\xi_1(\xi-\xi_1)|^{|s|}}{\langle \xi\rangle^{|s|}\langle \tau-\phi_\nu(\xi)\rangle^{1-b}\langle \tau-\tau_1-\phi_\nu(\xi-\xi_1)\rangle^{b'}} \\
&\qquad \times d\xi d\tau \Bigg) d\xi_1 d\tau_1
\end{aligned}$$

$$
\begin{aligned}
&\le C \iint_{\mathbb{R}^2} |f(\xi_1,\tau_1)| \left(\iint_{\mathbb{R}^2} |g(\xi,\tau)|^2 |f(\xi-\xi_1,\tau-\tau_1|^2 d\xi d\tau \right)^{1/2} \\
&\quad \times \frac{1}{\langle \tau_1 - \phi_\nu(\xi_1)\rangle^{b'}} \\
&\quad \times \left(\iint_B \frac{|\xi|^{2(1-|s|)} |\xi\xi_1(\xi-\xi_1)|^{2|s|} d\xi d\tau}{\langle\xi\rangle^{2|s|} \langle \tau-\phi_\nu(\xi)\rangle^{2(1-b)} \langle \tau-\tau_1-\phi_\nu(\xi-\xi_1)\rangle^{2b'}} \right)^{1/2} d\xi_1 d\tau_1 \\
&\le C \iint_{\mathbb{R}^2} |f(\xi_1,\tau_1)| \left(\iint_{\mathbb{R}^2} |g(\xi,\tau)|^2 |f(\xi-\xi_1,\tau-\tau_1|^2 d\xi d\tau \right)^{1/2} d\xi_1 d\tau_1 \\
&\le C \left(\iint_{\mathbb{R}^2} |f(\xi_1,\tau_1)|^2 d\xi_1 d\tau_1 \right)^{1/2} \left(\iint_{\mathbb{R}^2} |g|^2 * |f|^2(\xi_1,\tau_1) d\xi_1 d\tau_1 \right)^{1/2} \\
&\le C \|g\|_{L^2_\xi L^2_\tau} \|f\|^2_{L^2_\xi L^2_\tau}.
\end{aligned}
$$

Therefore by duality, we conclude that

$$
\|h\|_{L^2_\xi L^2_\tau} \le C \|f\|^2_{L^2_\xi L^2_\tau}
$$

and the proof is now complete. □

5. Proof of the well-posedness

By the Duhamel principle, we consider the integral equation

$$
u(t) = W_\nu(t)u_0 - \int_0^t W_\nu(t-t')\partial_x(u^2(t'))dt' , \tag{5.1}
$$

which is equivalent to the Cauchy problem. Hence we show the existence of the solution u to the integral equation (5.1).

Let

$$
\Phi(u) = \psi(t)W_\nu(t)u_0 - \psi(t)\int_0^t W_\nu(t-t')\psi(\delta^{-1}t')\partial_x(u^2(t'))dt'
$$

and suppose that ω is in

$$
\mathcal{B} = \{ u \in X_\nu^{s,b} |\ \|u\|_{X_\nu^{s,b}} \le 2C\|u_0\|_{H^s} \}.
$$

According to Proposition 3.3, Φ can be bounded as follows:

$$
\begin{aligned}
\|\Phi(u)\|_{X_\nu^{s,b}} &\le \|\psi(\delta^{-1}t)W(t)u_0\|_{X_\nu^{s,b}} \\
&\quad + \left\| \psi(t)\int_0^t W(t-t')\psi(\delta^{-1}t')\partial_x(u^2(t'))dt' \right\|_{X_\nu^{s,b}} \\
&\le C\|u_0\|_{H^s} + C\|\psi(\delta^{-1}\cdot)\partial_x(u^2(\cdot))\|_{X_\nu^{s,b-1}}.
\end{aligned}
$$

Here we choose $\bar{b}$ such that $b \le \bar{b} \le 3/4, \bar{b}-b \le \min\{-s-1/2, 1/4+s/3\}$. Then since $\omega \in \mathcal{B}$, Proposition 3.2 (3.7) and Proposition 4.1 yield that

$$\begin{aligned}\|\Phi(u)\|_{X_\nu^{s,b}} &\le C\|u_0\|_{H^s} + C\delta^{\bar{b}-b}\|\partial_x(u^2)\|_{X_\nu^{s,\bar{b}}} \\ &\le C\|u_0\|_{H^s} + C\delta^{\bar{b}-b}\|u\|^2_{X_\nu^{s,b}} \\ &\le C\|u_0\|_{H^s} + 4C^3\delta^{\bar{b}-b}\|u_0\|^2_{H^s}.\end{aligned}$$

Hence by choosing δ small such that $4C^2\delta^{\bar{b}-b}\|u_0\|^2_{H^s} \le 1/2$, we see $\Phi(u) \in \mathcal{B}$. Also for $u, \tilde{u} \in \mathcal{B}$,

$$\begin{aligned}\|\Phi(\tilde{u}) - \Phi(u)\|_{X_\nu^{s,b}} &\le C\delta^{\bar{b}-b}\|\tilde{u}+u\|_{X_\nu^{s,b}}\|\tilde{u}-u\|_{X_\nu^{s,b}} \\ &\le 4C^2\delta^{\bar{b}-b}\|\tilde{u}-u\|_{X_\nu^{s,b}} \\ &\le \frac{1}{2}\|\tilde{u}-u\|_{X_\nu^{s,b}}.\end{aligned}$$

Therefore Φ is a contraction mapping on $\mathcal{B}$. By the Banach fixed point theorem, there exists a solution $u \in \mathcal{B}$ such that

$$u(t) = \psi(t)\left(W_\nu(t)u_0 - \int_0^t W_\nu(t-t')\psi(\delta^{-1}t')\partial_x(u^2(t'))dt'\right).$$

If we choose $T < \min\{\delta/2, 1/2\}$, then $u(t)$ can be regarded as a solution of (5.1) over $t \in [-T, T]$. This and similar argument found in [3] shows the existence and uniqueness of the local solution to (1.1).

The continuity for the solution in H^s directly follows from the Sobolev imbedding $H^b(\mathbb{R}_t; H^s) \subset C(\mathbb{R}_t; H^s)$ The continuous dependence for the initial data also follows from a similar argument.

Next we show the uniqueness of the solution in the above class. For simplicity, we show the uniqueness of the solution to (1.2) with $\gamma = 0$. The general case follows similar argument. We introduce the following auxiliary norms. For $T > 0$, we let

$$\|u\|_{X_T} = \inf_w\{\|w\|_{X_\nu^{s,b}} : w \in X_\nu^{s,b} \quad \text{such that} \quad u(t) = w(t) \quad t \in [0,T] \text{ in } H^s\}. \tag{5.2}$$

Obviously, if $\|u_1 - u_2\|_{X_T} = 0$, we have $u_1(t) = u_2(t)$ in H^s for $t \in [0, T]$.

Let u_1 be the solution obtained above and u_2 be a solution of the integral equation with the same initial data u_0. We assume that for some $M > 0$,

$$\|u_1\|_{X_\nu^{s,b}}, \|\psi u_2\|_{X_\nu^{s,b}} \le M. \tag{5.3}$$

Without loss of generality, we may assume that $1 < M$ and $T < 1$. For some $T^* < T$ which will be fixed later, we have

$$\psi u_2(t) = \psi(t)W_\nu(t)u_0 - \psi(t)\int_0^t W_\nu(t-t')\psi_{T^*}(t')\psi^2(t')\partial_x(u_2(t'))^2 dt' \tag{5.4}$$

for $t \in [0, T^*]$.

Consider the difference $u_1 - \psi u_2$. For any $\varepsilon > 0$, there exists $\omega \in X_\nu^{s,b}$ such that for $t \in [0, T^*]$,

$$\omega(t) = u_1(t) - \psi(t)u_2(t), \tag{5.5}$$

and

$$\|\omega\|_{X_\nu^{s,b}} \leq \|u_1 - \psi u_2\|_{X_{T^*}} + \varepsilon. \tag{5.6}$$

Set $\tilde{\omega}$ satisfying

$$\tilde{\omega}(t) = -\psi(t)\int_0^t W_\nu(t-t')\psi_{T^*}\partial_x\{\omega(t')u_1(t') + \psi(t')\omega(t')u_2(t')\}dt'. \tag{5.7}$$

By (4.5) we have $\tilde{\omega}(t) = \omega(t) = u_1 - \psi(t)u_2(t)$ for $t \in [0, T^*]$.

Then according to Proposition 3.2, Proposition 3.3 and Proposition 4.1, we have for $b < b' < 3/4$ and $\mu = (b' - b)/4b'$,

$$\begin{aligned} \|u_1 - \psi u_2\|_{X_{T^*}} &\leq \|\tilde{\omega}\|_{X_\nu^{s,b}} \\ &\leq C\|\psi_{T^*}\{\omega u_1 + \psi\omega u_2\}\|_{X_\nu^{s,b'-1}} \\ &\leq C_3 T^{*\mu}(\|\omega\|_{X_\nu^{s,b}}\|u_1\|_{X_\nu^{s,b}} + \|\omega\|_{X_\nu^{s,b}}\|u_2\|_{X_\nu^{s,b}}) \\ &\leq C_3 T^{*\mu}\|\omega\|_{X_\nu^{s,b}}. \end{aligned} \tag{5.8}$$

If $T^{*\mu} \leq \frac{1}{2C_3 M}$, we have

$$\|u_1 - \psi u_2\|_{X_{T^*}} \leq \frac{1}{2}\|\omega\|_{X_\nu^{s,b}}.$$

By (5.6), we conclude

$$\|u_1 - \psi u_2\|_{X_{T^*}} \leq 2\varepsilon.$$

This proves $u_1 = u_2$ on $[0, T^*]$. Repeating this procedure, we obtain the uniqueness result for any existence interval. □

6. The limiting problem - proof of Theorem 2.2

In this section we prove Theorem 2.2. To show our theorem, we first consider the regular solutions u_ν for (1.3) and the regular solutions u for (1.4) with the same initial data $u_0 \in H^3$. By an approximation procedure the conclusion follows from the well-posedness results already established.

Proof of Theorem 2.2. Let u_ν be a unique solution of (1.1) in $C([0,\infty); H^3)$ and let v be a solution of (1.2) in the same space both with initial data $u_0 \in H^3$. Without loss of generality, we may assume that $0 < \nu < 1/2$.

We fix the time interval $[0,T]$ so that each of the solutions satisfy the following

$$\|u_\nu\|_{X_\nu^{s,b}}, \quad \|v\|_{X_0^{s,b}} \le M \quad \text{for all } \nu \in [0,1/2]. \tag{6.1}$$

We note that u_ν has enough regularity to satisfy the equation (1.1) in the strong sense, so we have the solution to the integral equation associated with (1.2)

$$u_\nu(t) = W_0(t)u_0 - \int_0^t W_0(t-t')\left(\partial_x u_\nu^2(t') - \nu\mathcal{H}_x\partial_x^2(u_\nu(t'))\right)dt',$$

and we have the following formular for a solution v of (1.2),

$$v(t) = W_0(t)u_0 - \int_0^t W_0(t-t')\partial_x v^2(t')dt'.$$

Then the difference satisfies

$$u_\nu(t) - v(t) = -\psi\int_0^t W_0(t-t')\left\{\partial_x(u_\nu(t')^2 - v(t')^2) - \nu\mathcal{H}_x\partial^2 u_\nu\right\}dt'.$$

Then by taking the norms $\|\cdot\|_{X_0^{s,b}}$, we have by (2.4) in Proposition 3.2 and 3.3 and Proposition 4.1 with $\nu = 0$ that

$$\begin{aligned}
&\|u_\nu - v\|_{X_0^{s,b}} \\
&\le \Big\|\psi\int_0^t W_0(t-t') \\
&\qquad \times \left\{(u_\nu(t')+v(t'))(u_\nu(t')-v(t')) - \nu\mathcal{H}_x\partial^2 u_\nu(t')\right\}dt'\Big\|_{X_0^{s,b}} \\
&\le C\delta^\mu(\|u_\nu+v\|_{X_0^{s,b}}\|u_\nu - v\|_{X_0^{s,b}} + |\nu|\|u_\nu\|_{X_0^{s+2,b-1}}) \\
&\le C\delta^\mu(M\|u_\nu - v\|_{X_0^{s,b}} + |\nu|\|u_\nu\|_{L^\infty(I;H^2)}).
\end{aligned} \tag{6.2}$$

Here we should note that all the estimates from Proposition 3.2 Proposition 3.3 and Proposition 4.1 is independent of the parameter ν. Hence if

$$\delta^{\mu} \leq \frac{1}{2CM},$$

then

$$\|u_\nu - v\|_{X_0^{s,b}} \leq \frac{|\nu|}{2}\|u_\nu\|_{X_0^{s+2,b-1}} \leq \frac{|\nu|}{2}\|u_\nu\|_{L^\infty(I;H^2)}. \tag{6.3}$$

By (6.3), we have

$$\|u_\nu - v\|_{X_0^{s,b}} \to 0$$

as $\nu \to 0$.

Now we prove the general case. For any initial data $u_0 \in H^s$ we choose a sequence $u_0^n \in H^3$ such that

$$u_0^n \to u_0 \qquad \text{in} \quad H^s. \tag{6.4}$$

Let $\tilde{u}_\nu$ is the corresponding solution to the Benjamin equation (1.1) and $\tilde{v}$ is the solution to the KdV equation (1.2). Since both solutions $\tilde{u}_\nu$ and $\tilde{v}$ can be found in $C([0,T_*];H^3)$, by combining Theorem 2.1 and the above conclusion, we have that

$$\begin{aligned} \|u_\nu - v\|_{X_0^{s,b}} &\leq \|u_\nu - \tilde{u}_\nu\|_{X_0^{s,b}} + \|\tilde{u}_\nu - \tilde{v}\|_{X_0^{s,b}} + \|\tilde{v} - v\|_{X_0^{s,b}} \\ &\leq C(T)\big(\|u_0^n - u_0\|_{H_x^s} + \|u_0^n - u_0\|_{H_x^s}\big) \\ &\quad + |\nu|\|\mathcal{H}_x\partial_x^2\tilde{u}_\nu\|_{L_t^\infty(H_x^2)}. \end{aligned} \tag{6.5}$$

By letting $\nu \to 0$

$$\limsup_{\nu\to 0}\|u_\nu - v\|_{X_0^{s,b}} \leq C\|u_0^n - u_0\|_{H^s}$$

and passing $n \to \infty$ we have that $u_0^n \to u_0$ and

$$u_\nu^n \to u \qquad \text{in} \quad X_0^{s,b}. \tag{6.6}$$

Repeating this procedure, we have the conclusion for the desired time interval. □

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References

[1] L. Abdelouhab, J. L. Bona, M. Felland, and J.-C. Saut, *Nonlocal models for nonlinear, dispersive waves*, Physica D **40** (1989), 360–392.

[2] J. Angulo, *Existence and stability of solitary wave solutions of the Benjamin equation*, J. Differential Equations **152** (1999), 136–159.

[3] D. Bekiranov, T. Ogawa, and G. Ponce, *Interaction Equations for Short and Long Dispersive Waves*, J. Funct. Anal. **158** (1998), no. 2, 357–388.

[4] T. B. Benjamin, *Internal waves of permanent form in fluids of great depth*, J. Fluid. Mech. **29** (1967), no. 2, 559–592.

[5] ———, *A new kind of solitary waves*, J. Fluid Mech. **245** (1992), 401–411.

[6] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear equations. II. The KdV equation*, GAFA. **3** (1993), 209–262.

[7] J. Ginibre, Y. Tsutsumi, and G. Velo, *On the Cauchy problem for the Zakharov system*, J. Funct. Anal. (to appear)

[8] R. J. Iorio Jr., *On the Cauchy problem for the Benjamin-Ono equation*, Comm. Partial Differential Equations **11(10)** (1986), 1031–1081.

[9] C. E. Kenig, G. Ponce, and L. Vega, *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction mapping principle*, Comm. Pure Appl. Math. **46** (1993), 527–620.

[10] ———, *The Cauchy problem for the Korteweg-de Vries equation in Sobolev spaces of negative indices*, Duke Math. J. **71** (1993), 1–21.

[11] ———, *On the generalized Benjamin-Ono equation*, Trans. Ameri. Math. Soc. **342** (1994), 155–172.

[12] ———, *A bilinear estimate with applications to the KdV equation*, J. Amer. Math. Soc. **9** (1996), 573–603.

[13] ———, *Smoothing effects and local existence theory for the generalized nonlinear Schrödinger equations*, Invent. Math. **134** (1998), no. 3, 489–545.

[14] F. Linares, *L^2 global well-posedness of the initial value problem associated to the Benjamin equation*, J. Differential Equations **152** (1999), 377–393.

[15] L. Molinet, J. C. Saut, and N. Tzvetkov, *Ill-posedness issues for the Benjamin-Ono and related equations*, preprint, Université de Paris-Sud, Orsay, 2001.

[16] K. Nakanishi, H. Takaoka, and Y. Tsutsumi, *Counter examples to bilinear estimates related with the KdV equation and the nonilnear Schrödinger equation*, preprint, Tohoku University.

[17] G. Ponce, *On the global well-posedness of the Benjamin-Ono equation*, Diff. Integral Equations **4** (1991), 527–542.

[18] H. Takaoka, *Well-posedness for the higher order nonlinear Schrödinger equation*, Advances. Math. Sci. Appl. **10** (2000), 149–171.

[19] ———, *The Cauchy problem for the Korteweg-de Vries equation with measure as initial data*, SIAM J. Math. Anal **20** (1989), 582–588.

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