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ON A FUNCTIONAL EQUATION FOR QUADRATIC INVARIANT CURVES

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ABSTRACT. Quadratic invariant curve is one of the simplest nonlinear invariant curves and was considered by C. T. Ng and the author in order to study the one-dimensional nonlinear dynamics displayed by a second order delay differential equation with piecewise constant argument. In this paper a functional equation derived from the problem of invariant curves is discussed. Using a different method from what C. T. Ng and the author once used, we define solutions piecewisely and give results in the remaining difficult case left in C. T. Ng and the author's work. A problem of analytic extension given in their work is also answered negatively.

1. Introduction

There have been found many results (e.g. [1, 2, 3, 4, 12]) on existence, periodicity and oscillation of solutions of the equation

$$(\mathrm{EPCA}_k) \qquad \frac{d^k}{dt^k}x(t) + g(x([t])) = 0, \quad t \in \mathbf{R}, x \in \mathbf{R},$$

with a piecewise constant argument, where $[t]$ denotes the greatest integer less than or equal to t and $g : \mathbf{R} \to \mathbf{R}$ continuously or at least piecewise continuously, (EPCA_k) is a special functional differential equation with variable delay and sometimes is called EPCA for short. With orbits in an infinite-dimensional phase space (EPCA_k) may display a complicated dynamics. Even some differential equations with constant delays possess chaotic behaviors [9, 11]. Observe that for $g(x) = x^2 + (1-\mu)x$ with $\mu > 3.75$ equation (EPCA_1) is Li-Yorke chaotic. In fact, a logistic mapping $x_{n+1} = \mu x_n(1 - x_n)$ is deduced by letting $x_n := x(n) =$

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$x([t])$, $t \in [n, n+1)$, and by integrating (EPCA_1) on $[n, n+1)$. When $\mu > 3.75$ the Li-Yorke chaos arises from *Feigenbaum cascade*, as indicated in Chapter 9 of [10]. Being a striking contrast, a modified (EPCA_2), i.e.,

$$\frac{d^2}{dt^2}x(t) + g(x(t)) = 0, \quad t \in \mathbf{R}, x \in \mathbf{R}, \tag{1.1}$$

where $[t]$ in (EPCA_2) is replaced by t, is a Hamiltonian system displaying simple dynamical behaviors with closed orbits and saddle loops but certainly no chaos. This suggests researching nonlinear dynamics of (EPCA_k) for $k \geq 2$.

As in [1, 2, 3, 4, 12], a function $x(t)$ is called a *solution* of (EPCA_k) on an interval $I \subset \mathbf{R}$ if (i) $x(t)$ is C^{k-1} (i.e., the $(k-1)$-th continuously differentiable), (ii) $x(t)$ is k-order differentiable on $I \backslash \mathbf{Z}$ and k-th one-sided derivatives of $x(t)$ are defined at each integer in I, and (iii) $x(t)$ satisfies (EPCA_k) on each interval $[n, n+1) \subset I$ for integer n. In consistence with Walther's idea [11] we first derive a mapping from (EPCA_2), which reflects the dynamics of (EPCA_2). More generally, let

$$x_n := x(n) = x([t]), \quad x_n^{(i)} := x^{(i)}(n) = x^{(i)}([t]), \tag{1.2}$$

for $t \in [n, n+1)$, $i = 1, \cdots, k-1$. By integrating (EPCA_k) from n to t we have

$$x^{(k-1)}(t) = -g(x_n)(t-n) + x_n^{(k-1)}. \tag{1.3}$$

We also get by integrating (1.3) that

$$x^{(k-2)}(t) = -\frac{1}{2}g(x_n)(t-n)^2 + x_n^{(k-1)}(t-n) + x_n^{(k-2)}. \tag{1.4}$$

By induction we have

$$x^{(k-i)}(t) = -\frac{1}{i!}g(x_n)(t-n)^i + \sum_{s=1}^{i} \frac{1}{(i-s)!}x_n^{(k-s)}(t-n)^{(i-s)} \tag{1.5}$$

for $i \leq k$. Letting $t \to (n+1)_-$ in (1.5), i.e., letting t tend to $n+1$ from left, we derive a mapping $F_k : \mathbf{R}^k \to \mathbf{R}^k$ defined by $(y_0, y_1, \cdots, y_{k-1}) \mapsto (y_0', y_1', \cdots, y_{k-1}')$ where

$$y_i' = -\frac{1}{(k-i)!}g(y_0) + \sum_{s=1}^{k-i} \frac{1}{(k-i-s)!}y_{k-s}, \quad i = 0, 1, \cdots, k-1. \tag{1.6}$$

Clearly this mapping describes how the sequence $\{(x_n, x_n^{(1)}, \cdots x_n^{(k-1)}) : n = 0, 1, 2, \cdots\}$ evolves and how the dynamics of (EPCA_k) displays. Especially, for (EPCA_2) the derived mapping (1.6) is a planar one F_2 :

$(x,y) \mapsto (-\frac{1}{2}g(x)+x+y, -g(x)+y)$. Define $G_2 : \mathbf{R}^2 \to \mathbf{R}^2$ by $(x,y) \mapsto (y, 2y-x-\frac{1}{2}(g(x)+g(y)))$. It is clear that F_2 is topologically conjugate to G_2, because $h : (x,y) \mapsto (x, -\frac{1}{2}g(x)+x+y)$ is a homeomorphism of $\mathbf{R}^2$ onto itself and $G_2 \circ h = h \circ F_2$. Hence we can equivalently discuss G_2 in stead.

Invariant curves of G_2 of the form $\Gamma : y = f(x)$ can be obtained by solving the functional equation

$$\text{(FE)} \qquad f(f(x)) = 2f(x) - x - \frac{1}{2}(g(f(x)) + g(x)), \qquad x \in \mathbf{R}.$$

In particular, as in [7] we want to know what choice of $g(x)$ guarantees the mapping G_2 to have a quadratic invariant curve Γ where

$$f(x) = ax^2 + bx + c, \quad a \neq 0. \tag{1.7}$$

In 1997 C. T. Ng and the author [7] proved that the functional equation (FE) has continuous or piecewisely continuous or even locally analytic function g when f is taken in the form of (1.7) for $C := \frac{1}{4}(2b-b^2)+ac$ in $[0,+\infty)$ but has no solution for $C \in (-\infty, -3/4)$. The remaining case of $C \in [-3/4, 0)$ looked difficult and was discussed later in [8] in a manner of series. Another problem whether the local analyticity of g can be extended is not solved yet.

Based on [7], in this paper we discuss equation (FE) with quadratic f in (1.7), constructing its solutions by defining piecewisely and giving results in the remaining difficult case left in [7] by use of a different method from [7] and [8]. We also answer negatively the problem of analytic extension in [7].

2. Existence and construction of solutions

In this section we discuss the cases where there exists the quadratic invariant curve, i.e., where (FE) has solutions. Our method is different from [7] although these cases were discussed there.

THEOREM 2.1. *For given constants a, b, c with $a \neq 0$,*

$$ac \geq -\frac{1}{4}, \quad \text{and} \quad 1 - \sqrt{1+4ac} \leq b \leq 1 + \sqrt{1+4ac}, \tag{2.8}$$

there is a piecewise continuous function $g : \mathbf{R} \to \mathbf{R}$ such that equation (FE) *has a solution of the form* (1.7).

Without proof the following lemma is basic and useful.

LEMMA 2.1. *Suppose that $I \subset [0,+\infty)$ is an interval and that $f : I \to I$ is C^1 and strictly increasing. The following statements hold:*

(i) *If $f(x) \geq x$ (resp. $f(x) \leq x$), $\forall x \in I$ then for any $x_0 \in I$ the sequence $\{f^n(x_0)\}$ of iterates of f is increasing (resp. decreasing).*
(ii) *If f has a fixed point $\tilde{x} \in I$ and $f(x) > x$ on the left (resp. on the right) of $\tilde{x}$, then $\tilde{x}$ attracts its a left-half neighborhood (resp. repels its a right-half neighborhood).*
(iii) *If f has a fixed point $\tilde{x} \in I$ and $f(x) < x$ on the left (resp. on the right) of $\tilde{x}$, then $\tilde{x}$ repels its a left-half neighborhood (resp. attracts its a right-half neighborhood).*

Proof of Theorem 2.1. Observe that with f in the form of (1.7) equation (FE) is equivalent to

$$\begin{aligned}\frac{1}{2}g(f(x)) + \frac{1}{2}g(x) &= -f(f(x)) + 2f(x) - x \\ &= -a(f(x))^2 - (b-2)f(x) - x - c, \end{aligned} \tag{2.9}$$

that is,

$$\begin{aligned}&\frac{1}{2}g(f(x)) + a(f(x))^2 + (b-3)f(x) + \frac{1}{2}g(x) + ax^2 + (b-3)x \\ &\quad = -4x - 2c,\end{aligned} \tag{2.10}$$

and thus we obtain that

$$h(f(x)) + h(x) = x, \tag{2.11}$$

where f is defined in (1.7) and $h(x) = -\frac{1}{4}(\frac{1}{2}g(x) + ax^2 + (b-3)x + c)$.

Since $f(x) = a(x + \frac{b}{2a})^2 - \frac{\Delta}{4a}$ where $\Delta = b^2 - 4ac$, by the change of variables $y = a(x + \frac{b}{2a})$ in (2.11) we have

$$h(\frac{1}{a}y^2 - \frac{\Delta}{4a}) + h(\frac{1}{a}y - \frac{b}{2a}) = \frac{1}{a}y - \frac{b}{2a}. \tag{2.12}$$

Let $h_1(x) = ah(\frac{1}{a}x - \frac{\Delta}{4a}) + \frac{b}{4}$. It follows that

$$h_1(x^2) + h_1(x + \frac{\Delta - 2b}{4}) = x, \tag{2.13}$$

that is,

$$H(x^2 - r) = x - H(x), \tag{2.14}$$

where $H(x) = h_1(x+r)$ and

$$(2.15) \qquad r = \frac{\Delta - 2b}{4} = \frac{b^2 - 2b - 4ac}{4}.$$

Clearly the condition (2.8) is equivalent to that $r \le 0$. For convenience let $p(x) = x^2 - r$. Our problem is reduced to the existence of continuous or piecewise continuous function H such that

$$(2.16) \qquad H(p(x)) + H(x) = x$$

for $r \le 0$. In what follows we turn to study (2.16) for $r \le 0$ case-by-case.

In the case (a) where $r < -\frac{1}{4}$, p has no fixed points and $p(x) > x$ on $[0, +\infty)$. Taking notations $\alpha_n = p^n(0)$, $n = 0, 1, 2, \cdots$, where p^n denotes the n-th iterate of p, by lemma 2.1 we see that $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n < \alpha_{n+1} < \cdots \to +\infty$ and that for each n the map $p : [\alpha_n, \alpha_{n+1}) \to [\alpha_{n+1}, \alpha_{n+2})$ is an orientation-preserving homeomorphism. With an arbitrary choice of continuous function $H_0(x)$ on $[0, -r)$ such that

$$(2.17) \qquad \lim_{x \to -r} H_0(x) = -H_0(0),$$

we define

$$(2.18) \quad H(x) = \begin{cases} H_0(x), & x \in [0, -r) = [\alpha_0, \alpha_1), \\ p^{-1}(x) - H(p^{-1}(x)), & x \in [\alpha_n, \alpha_{n+1}), \quad n = 1, 2, \cdots \end{cases}$$

H is continuous on $[0, +\infty)$. In fact, H is continuous at each points α_n because $\lim_{x \to \alpha_{n+1}} H(x) = \lim_{y \to \alpha_n}(y - H(y)) = \alpha_n - \lim_{y \to \alpha_n} H(y)$ for $x \in [\alpha_n, \alpha_{n+1})$ and $y = p^{-1}(x) \in [\alpha_{n-1}, \alpha_n)$. To further give a continuation in the whole real axis, define

$$(2.19) \qquad H(x) = x - H(p(x)), \quad x \in (-\infty, 0).$$

Obviously $p(x) = x^2 - r > x^2 + \frac{1}{4} > 0$ in $(-\infty, 0)$ and $\lim_{x \to 0_-} H(x) = 0 - \lim_{x \to 0_-} H(x^2 - r) = -H(-r) = H_0(0)$, so $H(x)$ is defined reasonably and continuous.

In the case (b) where $r = -\frac{1}{4}$, p has exactly a fixed point $x_0 = \frac{1}{2}$ and $p(x) \ge x$ on $[0, +\infty)$. Taking notations $\alpha_0 = 0, \alpha_n = p^n(0)$, $n = 1, 2, \cdots$ and $\beta_0 = 1, \beta_n = p^n(1)$, $n = \pm 1, \pm 2, \cdots$, where p^n denotes the n-th iterate of p, by lemma 2.1 we see that $\lim_{n \to +\infty} \alpha_n = x_0$, $\lim_{n \to -\infty} \beta_n = x_0$ and $\lim_{n \to +\infty} \beta_n \to +\infty$, and that for each n both the map $p : [\alpha_n, \alpha_{n+1}) \to [\alpha_{n+1}, \alpha_{n+2})$ and the map $p : [\beta_n, \beta_{n+1}) \to [\beta_{n+1}, \beta_{n+2})$ are orientation-preserving homeomorphisms. For arbitrarily chosen continuous functions $H_1(x)$ on $[0, -r)$ and $H_2(x)$ on $[1, 1-r)$

with

$$\lim_{x \to -r} H_1(x) = -H_1(0), \quad \lim_{x \to 1-r} H_2(x) = 1 - H_2(1), \tag{2.20}$$

we define

$$H(x) = \begin{cases} H_1(x), & x \in [0, -r) = [\alpha_0, \alpha_1), \\ p^{-1}(x) - H(p^{-1}(x)), & x \in [\alpha_n, \alpha_{n+1}), \quad n \geq 1, \\ \frac{1}{4}, & x = x_0, \\ H_2(x), & x \in [1, 1-r) = [\beta_0, \beta_1), \\ x - H(p(x)), & x \in [\beta_n, \beta_{n+1}), \quad n \leq -1, \\ p^{-1}(x) - H(p^{-1}(x)), & x \in [\beta_n, \beta_{n+1}), \quad n \geq 1. \end{cases} \tag{2.21}$$

Using the same arguments as in the case (a) we can prove the continuity of H separately on $[0, \frac{1}{2})$ and $(\frac{1}{2} + \infty)$. Furthermore, on $(-\infty, 0)$ let $H(x) = x - H(p(x))$, which is determined by H on $[0, +\infty)$ because $p(x) = x^2 + \frac{1}{4} > 0$ for $x < 0$, so a continuation of H in the whole real axis is given.

In the case (c) where $-\frac{1}{4} < r < 0$, p has two fixed points $x_1 = \frac{1-\sqrt{1+4r}}{2}$, $x_2 = \frac{1+\sqrt{1+4r}}{2}$ such that $0 < x_1 < \frac{1}{2} < x_2$ and $p(x) > x$ on $[0, x_1) \cup (x_2, +\infty)$ and $p(x) < x$ on (x_1, x_2). By lemma 2.1, x_1 attracts both $[0, x_1)$ and (x_1, x_2) under the iteration of p; x_2 attracts both (x_1, x_2) and $(x_2, +\infty)$ under the iteration of p^{-1}. Using the same method as above we can inductively define $H(x)$ on separate intervals $[0, x_1), (x_1, x_2)$ and $(x_2, +\infty)$ and prove its piecewise continuity on $[0, +\infty)$. Finally H can be extended on the whole real axis by defining $H(x) = x - H(p(x))$ on $(-\infty, 0)$, which is determined by H on $[0, +\infty)$.

Finally in the case (d) where $r = 0$, p has two fixed points $x_1 = 0, x_2 = 1$ and $p(x) > x$ on $(1, +\infty)$ and $p(x) < x$ on $(0, 1)$. The proof is similar to the case (c). Thus the proof is complete. □

3. Analytic extension

As above, Theorem 2.1 is only related to continuity. We can also give results with analyticity. For example, in Section 2 of [7] it is proved that there is a unique analytic function

$$g(x) = -2x - 2x^2 - 8 \sum_{j=1}^{\infty} (-1)^j x^{2^j} \tag{3.22}$$

on $(-1,1)$ such that (FE) has a quadratic solution $f(x) = x^2$. Meanwhile, a question whether the function g on $(-1,1)$ can be extended analytically is mentioned.

Thanks to the so-called "high indices theorem" in [5], which says, under the hypotheses that $\{\lambda_n\}$ is an increasing sequence of positive numbers satisfying Hadamard's gap condition

$$\liminf_{n\to\infty}(\lambda_{n+1}/\lambda_n) > 1$$

and that the series

$$\xi(x) = \sum_{n=1}^{\infty} a_n x^{\lambda_n},$$

where all a_n $(n = 1, 2, \cdots)$ are real numbers, converges for $0 \le x < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges if $\xi(x)$ tends to a finite limit as $x \to 1$, we know easily that

$$\lim_{x\to x_-} \sum_{j=1}^{\infty}(-1)^j x^{2^j}$$

does not exist. Hence the answer to the question is negative.

4. Nonexistence of solutions

Previously our problem is reduced to the existence of continuous or piecewise continuous solutions of equation (2.16). In this section we continue to discuss (2.16) for $r > 0$. We will see that sometimes those quadratic invariant curves like (1.7) disappear through a bifurcation of periodic orbits.

THEOREM 4.1. *Equation* (FE) *does not have a solution of the form* (1.7) *such that* $a \neq 0$ *and*

$$\begin{array}{lll} a\,c < -1 & \text{or} & \\ a\,c \ge -1, & \text{and } b < 1 - 2\sqrt{1+ac} & \text{or } b > 1 + 2\sqrt{1+ac}. \end{array} \tag{4.23}$$

LEMMA 4.1. *Equation* (2.16) *has no solution if* p *has a period 2 orbit.*

Proof. Assume that $\xi \neq \eta$ and $p(\xi) = \eta$, $p(\eta) = \xi$. From (2.16) we have $H(\eta) + H(\xi) = \xi$ and $H(\xi) + H(\eta) = \eta$. The two equivalences are contradicting. □

Proof of Theorem 4.1. From (2.15) the condition (4.23) tells equivalently that $r > \frac{3}{4}$. It suffices to prove that there is no solution of (2.16) in the case where $r > \frac{3}{4}$.

By solving the equation $p(p(x)) = x$, i.e.,

$$x^4 - 2rx^2 - x + r^2 - r = (x^2 - x - r)(x^2 + x + 1 - r) = 0, \tag{4.24}$$

we see the roots of the factor $x^2 - x - r$ are just the two fixed points of p and the roots of the factor $x^2 + x + 1 - r$ are given by

$$\xi = \frac{-1 - \sqrt{-3 + 4r}}{2}, \quad \eta = \frac{-1 + \sqrt{-3 + 4r}}{2}, \tag{4.25}$$

which are real when $r \geq \frac{3}{4}$. At $r = \frac{3}{4}$ the two roots ξ and η coincide at x_1, i.e., $\xi = \eta = x_1$, and there is no period 2 point. However, when $r \geq \frac{3}{4}$ there are just two period 2 points ξ and η which is bifurcated from the lower fixed point x_1 and get a period 2 orbit ξ, η. By Lemma 4.1 the claimed result is proved. □

5. The case $0 < r < \frac{3}{4}$

Both the case $0 < r < \frac{3}{4}$ and the case $r = \frac{3}{4}$ to be discussed in next section are just what we did not solve in Section 3 of [7], where $C \in [-\frac{3}{4}, 0)$ and orbits of p are complicated.

In the case $0 < r < \frac{3}{4}$, the mapping p has an asymptotically stable fixed point $x_1 < 0$ and an unstable fixed point $x_2 > 0$. The asymptotic stability is determined by the fact whether the absolute value of the derivative of p at the fixed point is less than 1. Observing that $p(0) = -r$, for simplicity, let

$$s = p(-r) \tag{5.26}$$

and z_1 and z_2 be two zeros of p with $z_1 < 0 < z_2$. Obviously $z_1 < -r < x_1 < s < 0 < z_2 < x_2$. By taking a continuous function $H_0 : (s, 0] \to \mathbf{R}$ arbitrarily as an initial function, we will construct the solution H of (2.16) piecewisely.

Step 1. *Construction on* $[-r, 0]\backslash\{x_1\}$. Note that the fixed point x_1 attracts the interval $[-r, 0]$. In this case p is decreasing and the iterative sequence $\{p^n(x)\}$ for $x < 0$ is not monotone, but $p^2(x)$ is increasing. Thus we consider the monotone iterative sequence of p^2 instead. It is easy to see

$$0 > s = p(-r) = p^2(0) > \cdots > p^{2n}(0) > p^{2n+2}(0) > \cdots \to x_1 \tag{5.27}$$

$$p(0) < p^3(0) = p(s) < \cdots < p^{2n-1}(0) < p^{2n+1}(0) < \cdots \to x_1. \tag{5.28}$$

Moreover,

$$p^2 : (p^{2n}(0), p^{2n-2}(0)] \to (p^{2n+2}(0), p^{2n}(0)], \tag{5.29}$$

$$p^2 : [p^{2n-1}(0), p^{2n+1}(0)) \to [p^{2n+1}(0), p^{2n+3}(0)) \tag{5.30}$$

homeomorphically. On the other hand, from (2.16) we have

$$H(p^2(x)) + H(p(x)) = p(x). \tag{5.31}$$

Eliminating $H(p(x))$ from (2.16) and (5.31) we get

$$H(p^2(x)) - H(x) = p(x) - x. \tag{5.32}$$

Then we can define a piecewise continuous function $H : [-r, 0]\backslash\{x_1\} \to \mathbf{R}$ by

(5.33)

$$H(x) = \begin{cases} H_0(x), & x \in (s, 0] = (p^2(0), p^0(0)], \\ H(p^{-2}(x)) + p^{-1}(x) - p^{-2}(x), & x \in (p^{2n+2}(0), p^{2n}(0)],\ n \in \mathbf{Z}_+, \\ H_0(p^{-1}(x)), & x \in [-r, p(s)) = [p(0), p^3(0)), \\ H(p^{-2}(x)) + p^{-1}(x) - p^{-2}(x), & x \in [p^{2n+1}(0), p^{2n+3}(0)),\ n \in \mathbf{Z}_+. \end{cases}$$

Step 2. *Construction at x_1*. Define

$$H(x_1) = x_1/2, \tag{5.34}$$

because $2H(x_1) = H(x_1) + H(x_1) = x_1$ in (2.16).

Step 3. *Construction on $[z_1, -r)$*. Since $p(z_1) = 0$, $p(-r) = s$ and p maps $[z_1, -r)$ onto $(s, 0]$ homeomorphically, we can define a continuous function H on $[z_1, -r)$ by

$$H(x) = x - H_0(p(x)). \tag{5.35}$$

Step 4. *Construction on $(0, x_2)$*. Since p is increasing for $x > 0$ and as above the function H is well defined on $[-r, 0]$, as in Section 3, we can define a piecewise continuous function H on $(0, x_2)$ by

$$H(x) = x - H(p(x)) \tag{5.36}$$

step-by-step.

Step 5. *Construction on $[x_2, +\infty)$*. Similar to **Step 4** and proof in Section 3.

Step 6. *Construction on $(-\infty, z_1)$*. Obviously $p(x) > 0$, $\forall x \in (-\infty, z_1)$. Then $p : (-\infty, z_1) \to (0, +\infty)$ is one-to-one and monotone. Hence we can define a piecewise continuous function H by (5.36) and by functions of H well defined as above on $(0, +\infty)$.

Finally a piecewise continuous solution H of the equation (2.16) on $\mathbf{R}$ is well defined by all functions defined piecewisely as above. This provides a proof for the following result.

THEOREM 5.1. *For given constants a, b, c with $a \neq 0$,*

$$-1 \le a\,c < -1/4 \ \text{ and } \ 1 - 2\sqrt{1+ac} < b < 1 + 2\sqrt{1+ac}, \tag{5.37}$$

or

$$\begin{array}{lll} a\,c \ge -1/4 & \text{and} & \\ & \text{or} & \begin{array}{l} 1 - 2\sqrt{1+ac} < \ b < 1 - \sqrt{1+4ac} \\ 1 + \sqrt{1+4ac} < \ b < 1 + 2\sqrt{1+ac}, \end{array} \end{array} \tag{5.38}$$

there is a piecewise continuous function $g : \mathbf{R} \to \mathbf{R}$ such that equation (FE) *has a solution of the quadratic polynomial form* (1.7).

Note that condition (5.37) together with (5.38) says equivalently that $0 < r < \frac{3}{4}$. In fact, the following inequalities

$$\begin{array}{ll} a\,c < -1/4, & \text{or} \\ a\,c \ge -1/4, & \text{and } b < 1 - \sqrt{1+4ac} \ \text{ or } \ b > 1 + \sqrt{1+4ac}, \end{array} \tag{5.39}$$

and

$$a\,c \ge -1 \text{ and } \ 1 - 2\sqrt{1+ac} < b < 1 + 2\sqrt{1+ac}, \tag{5.40}$$

are valid. Comparing with (2.8) and (4.23) we easily deduce our conclusions.

6. The case $r = \frac{3}{4}$

In this case the system of p, where

$$p(x) = x^2 - \frac{3}{4}, \tag{6.41}$$

is placed at the bifurcation point and the fixed point x_1 is not asymptotically stable. Its stability needs to be determined further. As shown in section 4, the loss of stability causes something more difficult.

Obviously in $(-\infty, 0)$ the system (6.41) has a fixed point $x_1 = -1/2$ and a zero point $z_1 = -\sqrt{3}/2$. $z_1 < -r < x_1 < 0$.

LEMMA 6.1. *x_1 is a stable fixed point of the system* (6.41).

Proof. Clearly we have

$$p(x) - x_1 = -(x - x_1) + (x - x_1)^2, \quad \forall x \in \mathbf{R}. \tag{6.42}$$

Thus

$$\begin{aligned} p^2(x) - x_1 &= -(p(x) - x_1) + (p(x) - x_1)^2 \\ &= (x - x_1) + (x - \frac{3}{2})(x + \frac{1}{2})^3. \end{aligned} \tag{6.43}$$

When $x_1 < x \le 0$ we see that $p^2(x) - x_1 > 0$ and $x - x_1 > 0$, because p^2 is strictly increasing, and that $(x - \frac{3}{2})(x + \frac{1}{2})^3 < 0$. It follows that $p^2(x) - x_1 < x - x_1$. On the other hand, when $x < x_1$, for the same reason $p^2(x) - x_1 < 0$, $x - x_1 < 0$ and $(x - \frac{3}{2})(x + \frac{1}{2})^3 > 0$, so $x_1 - p^2(x) < x_1 - x$. Hence

$$|p^2(x) - x_1| < |x - x_1|, \quad \forall x \in [-r, 0] \backslash \{x_1\}, \tag{6.44}$$

and clearly x_1 is stable. □

(6.44) implies that x_1 attracts the interval $[-r, 0]$. In fact, for every $x \in [-r, 0] \backslash \{x_1\}$ the sequence $\{p^{2n}(x)\}$ is monotone and bounded and thus it has a limit denoted by $\tilde{x}$. The continuity implies that $p^2(\tilde{x}) = \tilde{x}$. However, if $\tilde{x} \ne x_1$ substituting x with $\tilde{x}$ in the inequality (6.44) we deduce a contradiction. Therefore, as in step 1 of the last section we can consider the monotone iterative sequence of p^2 similarly and prove the follows by piecewise construction.

THEOREM 6.1. *For given constants a, b, c with $a \ne 0$,*

$$a\,c \ge -1 \tag{6.45}$$

and

$$b = 1 - 2\sqrt{1 + ac} \quad or \quad b = 1 + 2\sqrt{1 + ac}, \tag{6.46}$$

there is a piecewise continuous function $g : \mathbf{R} \to \mathbf{R}$ such that equation (FE) *has a solution of the quadratic polynomial form* (1.7).

So far the remaining case, regarded as difficult in Section 3 of [7], is discussed in Theorems 5.1 and 6.1.

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