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COXETER GROUPS AND BRANCHED COVERINGS OF LENS SPACES

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ABSTRACT. The groups generated by reflections in faces of Coxeter polyhedra in three-dimensional Thurston's spaces are considered. We develop a method for finding of finite index subgroups of Coxeter groups which uniformize three-dimensional manifolds obtained as two-fold branched coverings of manifolds of Heegaard genus one, that are lens spaces $L(p,q)$ and the space $S^2 \times S^1$.

1. Introduction

An useful approach to obtain and study a three-dimensional manifold is to describe it as a branched covering of some other known manifold.

The classical Alexander theorem states that every closed orientable three-dimensional manifold M^3 can be obtained as a three-fold branched covering of the three-dimensional sphere S^3 [1, 10, 16]. This theorem is topological and doesn't say anything about a geometrical structure on the manifold M^3.

Here we are interested in covering properties of three-dimensional geometrical manifolds. In particular we will give the generalization of hyperelliptic manifolds admitting geometrical structures. Moreover, we describe such manifolds uniformized by subgroups of Coxeter groups.

Let $\mathbb{X}^3$ be one of three-dimensional geometries $\mathbb{E}^3$, $\mathbb{H}^3$, $\mathbb{S}^3$, $\mathbb{H}^2 \times \mathbb{E}^1$ or $\mathbb{S}^2 \times \mathbb{E}^1$ [18]. In this paper we consider three-dimensional manifolds M^3 which are quotient-spaces $M^3 = \mathbb{X}^3/\Gamma$, where Γ is a discrete group of isometries acting on $\mathbb{X}^3$ without fixed points.

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A manifold M^3 is said to be *hyperelliptic*, if it can be obtained as a two-fold branched covering of S^3. A three-dimensional hyperelliptic manifold can be considered as a natural analog of a hyperelliptic Riemann surface [3]. It was shown in [11] that in each of eight three-dimensional Thurston's geometries there exist hyperelliptic manifolds. Examples of manifolds admitting few hyperelliptic involutions can be found in [14]

In [12, 13] authors developed the method to glue hyperelliptic manifolds from a finite number of copies of a Coxeter polyhedron. It is interesting, that a sufficient condition for the existence of such construction is the existence of a hamiltonian cycle on the polyhedron.

Otherwise, it is well-known that not every three-dimensional manifold can be obtained as a two-fold branched covering of S^3. For example, it is not possible for the three-dimensional torus $T^3 = S^1 \times S^1 \times S^1$.

There are two natural directions for generalizations of the conception of a hyperelliptic three-manifold.

On the one hand, a two-fold branched covering can be considered as a particular case of an n-fold cyclic branched covering of S^3. Some results about geometrical properties of such manifolds and the structure of their fundamental groups can be found in [2, 5–9, 15, 21]. In particular, in this direction the Fibonacci manifolds and their generalizations are obtained.

On the other hand, in the theory of Riemann surfaces the important role is playing by one-hyperelliptic surfaces which are two-fold branched coverings of the torus, that is the unique Riemann surface of genus one [3]. Therefore, another natural direction for the generalization of two-fold branched coverings of S^3, that is a unique manifold of Heegaard genus zero, is to consider two-fold branched coverings of three-dimensional manifolds of Heegaard genus one, which are exactly lens spaces $L(p,q)$ and the space $S^2 \times S^1$.

Remark that the method for constructing of three-dimensional manifolds which are two-fold branched coverings of the projective space $P^3 = L(2,1)$ was developed in [12].

In the present paper, generalizing the methods from [12, 13], we will give a constructive description of two-fold branched coverings of $L(p,q)$ (see Theorem 1) and $S^2 \times S^1$ (see Theorem 2) in term of subgroups of Coxeter groups acting in three-dimensional spaces.

2. Basic definitions

A manifold $M^3 = \mathbb{X}^3/\Gamma$ is said to be *g-hyperelliptic*, if there exists an isometric involution τ such that the quotient-space $N^3 = M^3/\langle\tau\rangle$ is homeomorphic to a three-dimensional manifold of Heegaard genus g. Thus, 0-hyperellipic manifold is a two-fold branched covering of S^3 and is hyperelliptic in classical sense. In particular case $g = 1$, if N^3 is homeomorphic to a lens space $L(p, q)$, we will say that M^3 is a *$L(p, q)$-hyperelliptic* manifold, and if N^3 is homeomorphic to $S^2 \times S^1$, then M^3 is said to be a *$S^2 \times S^1$-hyperelliptic* manifold.

Here we shall use terminology and basic facts from the theory of three-dimensional orbifolds [20], chapter 13.

Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be orbifolds with the universal covering space $\mathcal{O}$ and orbifold fundamental groups Γ_1 and Γ_2 respectively. Thus, the following canonical covering maps hold: $\pi_1 : \mathcal{O} \to \mathcal{O}_1 = \mathcal{O}/\Gamma_1$ and $\pi_2 : \mathcal{O} \to \mathcal{O}_2 = \mathcal{O}/\Gamma_2$. Assume that $\Gamma_1 \subset \Gamma_2$ and denote by $\pi : \mathcal{O}_1 \to \mathcal{O}_2$ the orbifold covering induced by this group inclusion. The *branching index* of the covering π in a point $x \in \mathcal{O}_1$ is the number $|(\Gamma_2)_{\tilde{x}} : (\Gamma_1)_{\tilde{x}}|$, where $\tilde{x} = \pi_1^{-1}(x)$ and $(\Gamma_i)_{\tilde{x}}$ denotes the stabilizer of the point $\tilde{x}$ in the group Γ_i. In particular, if the covering π is unbranched in the point $x \in \mathcal{O}_1$, then we get $|(\Gamma_2)_{\tilde{x}} : (\Gamma_1)_{\tilde{x}}| = 1$ and $(\Gamma_1)_{\tilde{x}} = (\Gamma_2)_{\tilde{x}}$.

Let X and Y be metric spaces. For the covering $\pi : X \to Y$ we consider the group $\mathrm{Cov}_\pi(X, Y) = \{h \in \mathrm{Homeo}(X) : \pi \circ h = \pi\}$. The covering π is said to be a *regular G-covering branched over the weighted graph $T \subset Y$* if the following three conditions are satisfied:

(i) $\mathrm{Cov}_\pi(X, Y) \cong G$;

(ii) the map $\overline{\pi} : X \setminus \pi^{-1}(T) \to Y \setminus T$ induced by the map π, is an unbranched covering;

(iii) the quotient space $X/\mathrm{Cov}_\pi(X, Y)$ is an orbifold with the underlying space Y and the singular set T.

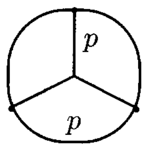
Two coverings $\pi : X \to Y$ and $\pi' : X' \to Y$ are said to be *equivalent* if there exists a homeomorphism $h : X \to X'$ such that $\pi = \pi' \circ h$.

Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be orbifolds with underlying spaces X and Y respectively, and let T be a weighted graph in Y. An orbifold covering $\pi : \mathcal{O}_1 \to \mathcal{O}_2$ is said to be *regular G-covering branched over T* if the associated covering $\pi_X : X \to Y$ of underlying spaces does satisfy the above conditions (i), (ii), and (iii).

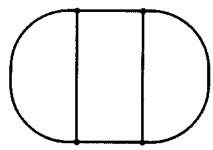
Let P be a Coxeter polyhedron (i.e. a polyhedron with dihedral angles of the form π/n, where $n \geq 2$ is integer) in one of the above three-dimensional geometries. Its *skeleton* P^1 is the weighted graph whose vertices and edges coincide with vertices and edges of the polyhedron P. A *weight* of the edge $e \in P^1$ is equal to n if the dihedral angle corresponding to this edge in P is equal to π/n.

Recall that a graph $H = (V(H), E(H))$ is said to be a *spanning subgraph* of a graph $G = (V(G), E(G))$ if sets of vertices of these graphs coincide: $V(H) = V(G)$, and the set of edges of H is a subset of the set of edges of G: $E(H) \subseteq E(G)$.

We will consider two type of graphs presented in Figure 1. First graph is the complete graph $\mathcal{K}(p)$ with four vertices with two (non-adjacent) edges labelled by $p > 1$. We will say that these two edges are *essential.* Let all other (*inessential*) edges be labelled by 2.



The graph $\mathcal{K}(p)$



The graph $\Theta^{\#}$

Figure 1.

Second graph is the double Θ-graph $\Theta^{\#}$ with four vertices and six edges labelled by 2. Two edges connecting pairs of vertices inside the circle are said to be *essential.* As before, all other edges are called *inessential.*

Two weighted graphs G and H with integer positive weights are said to be *homeomorphic* if these graphs can be obtained from some weighted graph by the subdivision of its inessential edges.

3. Main results

In this section we will describe the method for constructing of one-hyperelliptic manifolds in three-dimensional geometries. This approach is the generalization of the method for constructing of hyperelliptic manifolds given in [12, 13].

Consider the Hopf link 2_1^2 presented in Figure 2. The following fact is well-known and was firstly remarked by Wirtinger in 1904 and published by Tietze in 1908 (see [19], p. 270).

LEMMA 1. *For integers p and q such that $(p,q) = 1$ and $p > 1$ consider the spherical orbifold whose underlying space is S^3 and whose singular set is the Hopf link 2_1^2 labeled by p on both components. Let*

$\Gamma = \langle a, b \,|\, a^p = b^p = [a,b] = 1 \rangle$ *be its fundamental group, and* $\psi : \Gamma \to Z_p = \langle c \,|\, c^p = 1 \rangle$ *be the epimorphism, defined by* $\psi(a) = c$ *and* $\psi(b) = c^q$. *Denote* $G = Ker\ \psi$. *Then the quotient* S^3/G *is the lens space* $L(p,q)$.

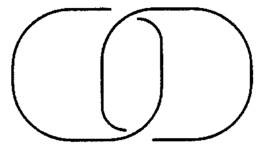


Figure 2. The Hopf link 2_1^2.

The following theorem gives a constructive method for obtaining $L(p,q)$-hyperelliptic manifolds admitting a geometrical structure.

THEOREM 1. *Let* P *be a Coxeter polyhedron in* $\mathbb{X}^3$, *where* $\mathbb{X}^3 = \mathbb{E}^3, \mathbb{H}^3, \mathbb{S}^2 \times \mathbb{E}^1, \mathbb{H}^2 \times \mathbb{E}^1$, *and* $\Delta(P)$ *be the group generated by reflections in faces of* P. *Suppose that the 1-skeleton* P^1 *of* P *contains a proper spanning subgraph* T *homeomorphic to* $\mathcal{K}(p)$ *and all edges of* $P^1 \setminus T$ *have weight two. Then there is a torsion-free subgroup* $\Gamma < \Delta(P)$ *of index* $8p$ *such that* $M^3 = \mathbb{X}^3/\Gamma$ *is a* $L(p,q)$*-hyperelliptic manifold.*

Proof. Remark that inessential edges of T form a hamiltonian cycle C in P^1. Denote by $\Delta^+(P)$ subgroup of $\Delta(P)$ consisting of all orientation-preserving isometries. The underlying space $|\mathcal{O}|$ of the orbifold $\mathcal{O} = \mathbb{X}^3/\Delta^+(P)$ is homeomorphic to S^3 and its singular set is the skeleton P^1. Denote by $\mathcal{H}$ the two-fold covering of $\mathcal{O}$ branched over C. Using that C is isotopic to an unknotted circle in $|\mathcal{O}| = S^3$, we get that the underlying space $|\mathcal{H}|$ of the orbifold $\mathcal{H}$ is homeomorphic to S^3. The singular set $\Sigma = \Sigma(\mathcal{H})$ of $\mathcal{H}$ consists of two parts $\Sigma = \Sigma_1 \cup \Sigma_2$ each of which is non-empty. The first part Σ_1 consists of two-component Hopf link 2_1^2 formed by preimages $\psi^{-1}(e_1)$ and $\psi^{-1}(e_2)$ of essential edges e_1 and e_2 of T under the projection $\psi : \mathcal{H} \to \mathcal{O}$.

The second part Σ_2 consists of a finite number of unknotted circles with weight two which are preimages of edges of $P^1 \setminus T$. We will call the parts Σ_1 and Σ_2 essential and inessential, respectively.

We will construct the regular $\mathbb{Z}_p \oplus \mathbb{Z}_2$-covering M^3 of the orbifold $\mathcal{H}$ and demonstrate that M^3 is a $L(p,q)$-hyperelliptic manifold.

Consider the fundamental group $\Gamma_\Sigma = \pi_1(S^3 \setminus \Sigma)$ of the complement of Σ, and suppose that Γ_Σ is generated by meridian loops around edges of Σ. Then relations can be obtained by the Wirtinger algorithm. Loops around essential (resp. inessential) components of Σ call essential (resp. inessential) loops.

Consider an epimorphism

$$\varphi : \Gamma_\Sigma \to \mathbb{Z}_p \oplus \mathbb{Z}_2 = \langle a : a^p = 1 \rangle \oplus \langle b : b^2 = 1 \rangle$$

defined by $\varphi(x) = a$ if x is an essential loop around the first component $\psi^{-1}(e_1)$ of the Hopf link; $\varphi(x) = a^q$, $(p,q) = 1$, if x is an essential loop around the second component $\psi^{-1}(e_2)$ of the Hopf link; and $\varphi(x) = b$ is x is an inessential loop.

As $\mathbb{Z}_p \oplus \mathbb{Z}_2$ is abelian, above described epimorphism φ is correctly defined and can be canonically extended to the epimorphism

$$\widehat{\varphi} : \Gamma_{\mathcal{H}} \to \mathbb{Z}_p \oplus \mathbb{Z}_2,$$

where $\Gamma_{\mathcal{H}}$ is the fundamental groups of the orbifold $\mathcal{H}$ such that $\mathcal{H} = \mathbb{X}^3/\Gamma_{\mathcal{H}}$. Such extension is possible in virtue of [4]: the presentation of $\Gamma_{\mathcal{H}}$ can be obtained from the presentation of Γ_Σ by adding the relation $x^p = 1$ if x is an essential loop and the relation $x^2 = 1$ if x in an inessential loop.

Put $\Gamma = \text{Ker } \widehat{\varphi}$ and consider the quotient space $M^3 = \mathbb{X}^3/\Gamma$. Remark that the group inclusion $\Gamma \triangleleft \Gamma_{\mathcal{H}}$ with $\Gamma_{\mathcal{H}}/\Gamma \cong \mathbb{Z}_p \oplus \mathbb{Z}_2$ induces the regular $\mathbb{Z}_p \oplus \mathbb{Z}_2$-covering

$$\pi : M^3 = \mathbb{X}^3/\Gamma \to \mathcal{H} = \mathbb{X}^3/\Gamma_{\mathcal{H}}$$

which is branched over all components of the singular set of $\mathcal{H}$.

Consider canonical coverings $\sigma : \mathbb{X}^3 \to M^3 = \mathbb{X}^3/\Gamma$ and $\omega : \mathbb{X}^3 \to \mathcal{H} = \mathbb{X}^3/\Gamma_{\mathcal{H}}$. Let us fix an arbitrary point $z \in M^3$ and its preimage $\tilde{z} \in \sigma^{-1}(z)$. Denote $z' = \omega(\tilde{z})$.

In a small enough ball $V_{\tilde{z}} = B(\tilde{z}, \epsilon)$ the covering $\pi : M^3 \to \mathcal{H}$ can be presented in the form $\pi : V_{\tilde{z}}/\Gamma_{\tilde{z}} \to V_{\tilde{z}}/(\Gamma_{\mathcal{H}})_{\tilde{z}}$, where $\Gamma_{\tilde{z}}$ and $(\Gamma_{\mathcal{H}})_{\tilde{z}}$ are stabilizers of the point $\tilde{z}$ in groups Γ and $\Gamma_{\mathcal{H}}$, respectively. In particular, the branching index of the covering π in the point z is equal to $|(\Gamma_{\mathcal{H}})_{\tilde{z}} : \Gamma_{\tilde{z}}|$.

For the point $z' \in \mathcal{H}$ one of the following cases holds: (i) $z' \notin \Sigma$; (ii) $z' \in \Sigma_1$; or (iii) $z' \in \Sigma_2$. Let us study the stabilizers $(\Gamma_{\mathcal{H}})_{\tilde{z}}$ and $\Gamma_{\tilde{z}}$ in each of the above cases. In the case (i) we have $(\Gamma_{\mathcal{H}})_{\tilde{z}} \cong \langle 1 \rangle$, whence $\Gamma_{\tilde{z}} \cong \langle 1 \rangle$. In the case (ii) z' belongs to an essential edge and $(\Gamma_{\mathcal{H}})_{\tilde{z}} \cong \mathbb{Z}_p$. By the definition of $\tilde{\varphi}$ we have $|(\Gamma_{\mathcal{H}})_{\tilde{z}} : \Gamma_{\tilde{z}}| = p$, then $\Gamma_{\tilde{z}} \cong \langle 1 \rangle$. In the case (iii) z' belongs to an inessential edge and $(\Gamma_{\mathcal{H}})_{\tilde{z}} \cong \mathbb{Z}_2$. By the definition of $\tilde{\varphi}$, $|(\Gamma_{\mathcal{H}})_{\tilde{z}} : \Gamma_{\tilde{z}}| = 2$, then $\Gamma_{\tilde{z}} \cong \langle 1 \rangle$.

Therefore, Γ acts on $\mathbb{X}^3$ without fixed points (equivalently, it is torsion-free) and M^3 is a manifold.

Consider an epimorphism

$$\chi : \mathbb{Z}_p \oplus \mathbb{Z}_2 = \langle a : a^p = 1 \rangle \oplus \langle b : b^2 = 1 \rangle \to \mathbb{Z}_p = \langle \alpha : \alpha^p = 1 \rangle$$

defined by $\chi(a) = \alpha$, $\chi(b) = 1$. Then

$$\chi \circ \widehat{\varphi} : \Gamma_{\mathcal{H}} \to \mathbb{Z}_p$$

send loops around the first component of the Hopf link to α, and loops around the second component of the Hopf link to α^q. By Lemma 1, the underlying space of the orbifold $\mathbb{X}^3/\text{Ker } (\chi \circ \widehat{\varphi})$ is homeomorphic to $L(p,q)$.

The group inclusion $\text{Ker } (\chi \circ \widehat{\varphi}) \triangleright \text{Ker } \widehat{\varphi} = \Gamma$ induces the two-fold covering of orbifolds:

$$M^3 = \mathbb{X}^3/\Gamma \to \mathbb{X}^3/\text{Ker } (\chi \circ \widehat{\varphi}).$$

Considering underlying spaces of above orbifolds, we conclude that M^3 is the two-fold branched covering of the lens space $L(p,q)$. The proof is complete. □

EXAMPLE. A hyperbolic $L(3,1)$-hyperelliptic manifold can be obtained from 24 copies of the polyhedron P presented in Figure 3. This polyhedron has 12 vertices, 18 edges, and 8 faces (four 5-gons and four 4-gons). Two of its dihedral angles are equal to $\pi/3$ and all other dihedral angles are equal to $\pi/2$.

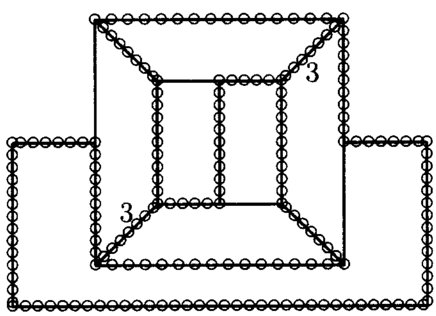


Figure 3. The example.

By Andreev theorem [20], this polyhedron can be realized in the hyperbolic space $\mathbb{H}^3$. Remark, that 1-skeleton of P contains a proper spanning subgraph T, shown by marks "o" in Figure 3, that is homeomorphic to $\mathcal{K}(3)$. Therefore, by Theorem 1, the group generated by reflections in faces of P has the subgroup of index 24 uniformizing a $L(3,1)$-hyperelliptic manifold.

Now we will consider $S^2 \times S^1$-hyperelliptic manifolds. The following fact is well-known and can be found in [17], p. 300.

LEMMA 2. *The two-fold branched covering of S^3 branched over the two-component trivial link O_1^2 is homeomorphic to the space $S^2 \times S^1$.*

The following theorem gives a constructive method for obtaining of $S^2 \times S^1$-hyperelliptic manifolds admitting geometrical structures.

THEOREM 2. *Let P be a Coxeter polyhedron in $\mathbb{X}^3$, where $\mathbb{X}^3 = \mathbb{E}^3, \mathbb{H}^3, \mathbb{S}^2 \times \mathbb{E}^1, \mathbb{H}^2 \times \mathbb{E}^1$, and $\Delta(P)$ be the group generated by reflections in faces of P. Suppose that the 1-skeleton P^1 of P contains a proper spanning subgraph T homeomorphic to the double theta graph $\Theta^{\#}$ and all edges of $P^1 \backslash T$ have weight two. Then there is a torsion-free subgroup $\Gamma < \Delta(P)$ of index 16 such that $M^3 = \mathbb{X}^3/\Gamma$ is a $S^2 \times S^1$-hyperelliptic manifold.*

Proof. The proof is similar to the proof of Theorem 1. Denote by $\Delta^+(P)$ subgroup of $\Delta(P)$ consisting of all orientation-preserving isometries. The underlying space $|\mathcal{O}|$ of the orbifold $\mathcal{O} = \mathbb{X}^3/\Delta^+(P)$ is homeomorphic to S^3 and its singular set is the skeleton P^1. Remark that inessential edges of T form a hamiltonian cycle C in P^1. Denote by $\mathcal{H}$ the two-fold covering of $\mathcal{O}$ branched over C. Since C is isotopic to an unknotted circle in $|\mathcal{O}| = S^3$, the underlying space $|\mathcal{H}|$ of the orbifold $\mathcal{H}$ is homeomorphic to S^3. The singular set $\Sigma = \Sigma(\mathcal{H})$ of $\mathcal{H}$ consists of two parts $\Sigma = \Sigma_1 \cup \Sigma_2$. The first part Σ_1 is the of two-component trivial link 0_1^2 formed by preimages $\psi^{-1}(e_1)$ and $\psi^{-1}(e_2)$ of essential edges e_1 and e_2 of T under the projection $\psi : \mathcal{H} \to \mathcal{O}$. The second part Σ_2 consists of a finite number of unknotted circles with weight two which are preimages of edges of $P^1 \setminus T$. We will call parts Σ_1 and Σ_2 essential and inessential, respectively.

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Consider the fundamental group $\Gamma_\Sigma = \pi_1(S^3 \setminus \Sigma)$ of the complement of Σ, and suppose that Γ_Σ is generated by meridian loops around edges of Σ. Then relations can be obtained by the Wirtinger algorithm. Loops around essential (resp. inessential) components of Σ call essential (resp. inessential) loops.

Consider an epimorphism

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defined by $\varphi(x) = a$ if x is an essential loop, and $\varphi(x) = b$ if x is an inessential loop.

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where $\Gamma_{\mathcal{H}}$ is the fundamental groups of the orbifold $\mathcal{H}$ such that $\mathcal{H} = \mathbb{X}^3/\Gamma_{\mathcal{H}}$. By [4] the presentation of $\Gamma_{\mathcal{H}}$ can be obtained from the presentation of Γ_Σ by adding the relation $x^2 = 1$ for generating loops around edges.

Put $\Gamma = \mathrm{Ker}\ \widehat{\varphi}$ and consider the quotient space $M^3 = \mathbb{X}^3/\Gamma$. Remark that the group inclusion $\Gamma \triangleleft \Gamma_{\mathcal{H}}$ with $\Gamma_{\mathcal{H}}/\Gamma \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ induces the regular $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-covering

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For the point $z' \in \mathcal{H}$ one of the following cases holds: (i) $z' \notin \Sigma$; or (ii) $z' \in \Sigma$. In the case (i) we have $(\Gamma_{\mathcal{H}})_{\tilde{z}} \cong \langle 1\rangle$, whence $\Gamma_{\tilde{z}} \cong \langle 1\rangle$. In the case (ii) we have $(\Gamma_{\mathcal{H}})_{\tilde{z}} \cong \mathbb{Z}_2$. By the definition of $\widehat{\varphi}$, $|(\Gamma_{\mathcal{H}})_{\tilde{z}} : \Gamma_{\tilde{z}}| = 2$, then $\Gamma_{\tilde{z}} \cong \langle 1\rangle$. Therefore, Γ acts on $\mathbb{X}^3$ without fixed points (equivalently, it is torsion-free) and M^3 is a manifold.

Consider an epimorphism

$$\chi : \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \langle a : a^2 = 1\rangle \oplus \langle b : b^2 = 1\rangle \to \mathbb{Z}_2 = \langle \alpha : \alpha^2 = 1\rangle$$

defined by $\chi(a) = \alpha$, $\chi(b) = 1$. Then

$$\chi \circ \widehat{\varphi} : \Gamma_{\mathcal{H}} \to \mathbb{Z}_2.$$

By Lemma 2, the underlying space of the orbifold $\mathbb{X}^3/\text{Ker } (\chi \circ \widehat{\varphi})$ is homeomorphic to $S^2 \times S^1$.

The group inclusion Ker $(\chi \circ \widehat{\varphi}) \triangleright$ Ker $\widehat{\varphi} = \Gamma$ induces the two-fold covering of orbifolds:

$$M^3 = \mathbb{X}^3/\Gamma \to \mathbb{X}^3/\text{Ker } (\chi \circ \widehat{\varphi}).$$

Considering underlying spaces of above orbifolds, we conclude that M^3 is the two-fold branched covering of the manifold $S^2 \times S^1$. The proof is complete. □

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