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ON THE TRANSFINITE POWERS OF THE JACOBSON RADICAL OF A DICC RING

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ABSTRACT. A ring is a DICC ring if every chain of right ideals indexed by the integers stabilizes to the left or to the right or to both sides. A counterexample is given to an assertion of Karamzadeh and Motamedi that a transfinite power of the Jacobson radical of a right DICC ring is zero. We determine the behavior of the transfinite powers of the Jacobson radical relative to a torsion theory and consequently can obtain their correct behavior in the classical setting.

Introduction

The Double Infinite Chain Condition (DICC for short) was introduced by M. Contessa [5], [6], [7] for modules over a commutative ring, extended by B. Osofsky [10] to objects in a Grothendieck category, and studied by O.A.S. Karamzadeh and M. Motamedi [9] for modules over an arbitrary ring. Recently, T. Albu and M. Teply [3] extended some results from these papers to a more general setting. One of the results of Karamzadeh and Modamedi that was not generalized in [3] was [9, Proposition 1.2]: if J is the Jacobson radical of a DICC ring R, then $J^{\alpha} = 0$ for some ordinal α. Unfortunately, that proposition is not always true. In this paper, we provide a counterexample to [9, Proposition 1.2]. Then we determine the behavior of the transfinite powers of the Jacobson radical relative to a torsion theory in the more general setting of [3] and consequently the correct result can be obtained for the classical setting of [9].

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Terminology

Let R denote a ring with identity element. All modules will be unitary right R-modules.

We will need concepts from torsion theory that can be found in [1], [2], [4], and [8]. Throughout, $\tau = (\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory on the category Mod-R of all unitary right R-modules, and $\tau(M)$ denotes the *τ-torsion submodule* of a right R-module M. Note that $\tau(R) = \tau(R_R)$ is a two-sided ideal of R and that $M/\tau(M)$ is τ-torsionfree for each right module M. A submodule N of M is *τ-dense* (in M) if $\tau(M/N) = M/N$, i.e., if M/N is τ-torsion; it is *τ-closed* if $\tau(M/N) = 0$, i.e., if M/N is τ-torsionfree. The *τ-closure* of N (in M) is the submodule $\overline{N} = \bigcap\{C | N \subseteq C \subseteq M, \tau(M/C) = 0\}$. It is the smallest τ-closed submodule of M that contains N and also the largest submodule of M in which N is τ-dense. Also, M is said to be *τ-finitely generated* if there exists a finitely generated submodule F of M such that $M/F \in \mathcal{T}$.

A module M is called *τ-noetherian* (*τ-artinian*) if it satisfies the ascending (descending) chain condition on τ-closed submodules. We use $\tau = 0$ to denote the torsion theory whose only torsion module is the zero module; i.e., every module is τ-torsionfree. A module M is said to be τ-DICC if any infinite chain of τ-closed submodules indexed by the integers $\mathbb{Z}$ stabilizes to the right or to the left or to both sides; i.e., for any chain

$$\ldots \subseteq M_{-2} \subseteq M_{-1} \subseteq M_0 \subseteq M_1 \subseteq M_2 \subseteq \ldots$$

of τ-closed submodules of M, there exists $m \in \mathbb{Z}$ such that $M_{i+1} = M_i$ for all $i \geq m$ or for all $i \leq m$. A module is said to be DICC if M is τ-DICC for the trivial torsion theory $\tau = 0$. A ring R is said to be τ-DICC (DICC) if R is a τ-DICC (DICC) right R-module.

The *τ-Krull dimension* $k_\tau(M)$ of a right R-module M is the deviation of the poset of τ-closed submodules of M. Thus, $k_\tau(M) = -1$ if M is τ-torsion, and $k_\tau(M) = \alpha$ for an ordinal $\alpha > -1$ if $k_\tau(M) \not< \alpha$ and, given any descending chain

$$M \supset C_1 \supset C_2 \supset \cdots \supset C_i \supset C_{i+1} \supset \cdots$$

of (τ-closed) submodules C_i of $M, k_\tau(C_i/C_{i+1}) < \alpha$ for all but finitely many i. Note that for $\tau = 0, k_\tau(M)$ is the usual Krull dimension $K(M)$ of M. A τ-torsionfree right R-module $M \neq 0$ is *τ-critical* if it has τ-Krull dimension and $k_\tau(M/N) < k_\tau(M)$ for any submodule $0 \neq N \subseteq M$. A τ-critical module M with $k_\tau(M) = \alpha$ is called *α-τ-critical*. We also call a 0-τ-critical module *τ-simple*. The right ideals K such that R/K

are τ-simple are called the *τ-maximal ideals*. We define the *τ-Jacobson radical* as the intersection of the τ-maximal right ideals (or R if no τ-maximal right ideals exist.) As in [9], we define the transfinite powers of J by $J^{\beta+1} = J^{\beta}J$ and, for a limit ordinal γ, $J^{\gamma} = \cap_{\beta<\gamma} J^{\beta}$.

For a τ-torsionfree module M, define $Soc_{\tau}(M)$ to be the sum of the τ-simple submodules of M (or 0 if there are no τ-simple submodules.) Then we define the τ-Loewy series of a module M by transfinite recursion:

$$S_{\tau}^{0}(M) = \tau(M)\,,\ S_{\tau}^{\beta+1}(M)/S_{\tau}^{\beta}(M) = \overline{Soc_{\tau}(M/S_{\tau}^{\beta}(M))},$$

and

$$S_{\tau}^{\gamma} = \overline{\bigcup_{\beta<\gamma} S_{\tau}^{\beta}(M)}$$

when γ is a limit ordinal.

As in [1] a torsion theory τ of R is called *I-invariant* for an ideal I if I/DI is τ-torsion for each τ-dense right ideal D. From [1, Lemma 6.1] we have the equivalent conditions:
(a) τ is I-invariant;
(b) $\overline{A}\,\overline{I} \subseteq \overline{AI}$ for every right ideal A of R;
(c) $\overline{A}I \subseteq \overline{AI}$ for every right ideal A of R;
(d) if $KI = 0$ for a τ-dense submodule K of a τ-torsionfree module M, then $MI = 0$.

Results

Now that we have given the basic terminology and notation that we will use, we are ready to present our example and our main result.

EXAMPLE 1. (Counterexample to [9, Proposition 1.2].) Let L be the localization of the integers $\mathbb{Z}$ at a nonzero prime ideal (p), and let R be the ring of all 2×2 matrices of the form:

$$\begin{pmatrix} x & y \\ 0 & x \end{pmatrix},$$

where $x \in L$ and $y \in \mathbb{Z}(p^{\infty})$, the smallest divisible abelian p-group. We will show the three conditions of [9, Theorem 1.2] (cf. [3, Corollary 4.6]) for a ring to be DICC hold for R. Let

$$A = \begin{pmatrix} 0 & \mathbb{Z}(p^{\infty}) \\ 0 & 0 \end{pmatrix}.$$

Since A is artinian as an abelian group, it is also artinian as a right R-module. Also, $R/A \cong L$ is (right) noetherian. Finally, suppose that $P_R \subseteq R$ such that $P \not\subseteq A$. We claim that $A \subseteq P$. Since $P \not\subseteq A$, then P contains an element of the form

$$\begin{pmatrix} p^m & y \\ 0 & p^m \end{pmatrix}.$$

Given any $x \in \mathbb{Z}(p^\infty)$, the divisibility of $\mathbb{Z}(p^\infty)$ gives $w \in \mathbb{Z}(p^\infty)$ such that $wp^m = x$. Hence

$$(*) \qquad \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} p^m & y \\ 0 & p^m \end{pmatrix} \begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix} \in P$$

for each $x \in \mathbb{Z}(p^\infty)$. Thus $R/(A \cap P) = R/A \cong L$ is noetherian, and hence R is a DICC ring.

Now we compute directly that

$$J^n = \left\{ \begin{pmatrix} p^n x & y \\ 0 & p^n x \end{pmatrix} \mid x \in L,\ y \in \mathbb{Z}(p^\infty) \right\}$$

for each $n < \omega$, $J^\omega = \cap J^n = A$, and $J^{\omega+1} = AJ = A$ by an equation symmetric to $(*)$. Hence $J^\beta = A$ for all $\beta > \omega$. In particular, $J^\beta \neq 0$ for any β, which contradicts the claim in [9, Proposition 1.2].

In Example 1, $A^2 = 0$. In [9] the proof of Proposition 1.2 seems to use the formula $(J^\alpha)^n = J^{\alpha n}$. Although the notation is suggestive of this formula, it is not necessarily true for limit ordinals, as our Example 1 demonstrates.

The true facts about the transfinite powers of the Jacobson radical can be obtained from the following theorem by setting $\tau = 0$.

THEOREM 2. *Let R be a τ-DICC ring for a torsion theory τ and let J be the τ-Jacobson radical of R. Let J^α be the first stationary transfinite power of J (i.e., $J^\alpha = J^{\alpha+1}$.)*

(1) *If J^α is τ-finitely generated, then $J^\alpha \subseteq \tau(R)$.*
(2) *If J^α is not τ-finitely generated, $\alpha < \omega$, and τ is J^α-invariant, then $J^\alpha \subseteq \tau(R)$.*
(3) *If J^α is not τ-finitely generated and τ is I-invariant for each $I \subseteq \overline{J^\alpha}$, then $J^\alpha \cdot J^\alpha \subseteq \tau(R)$.*

We prove Theorem 2 by establishing a sequence of five lemmas.

As in [4], a ring R with torsion theory τ is called τ-max if every τ-finitely generated R-module with a proper τ-closed submodule has a maximal τ-closed submodule.

LEMMA 3. *If R is τ-DICC, then R is τ-max.*

Proof. Let M be τ-finitely generated; say $\sum_{i=1}^{n} x_i R$ is τ-dense in M. Let N be a proper τ-closed submodule of M. Then at least one $x_i \notin N$; say $x_1 \notin N$. If $\overline{x_1 R + N} \neq M$, there exists at least one $x_j \notin \overline{x_1 R + N}$. Inductively, we eventually find a τ-closed submodule N' and an element $x_p \notin N'$ such that $\overline{x_p R + N'} = M$. Since R is τ-DICC, so is its homomorphic image $(x_p R + N')/N'$.

If $(x_p R + N')/N'$ is τ-artinian, then this cyclic module has a τ-Loewy series whose length is not a limit ordinal. So $(x_p R + N')/N'$ must have a maximal τ-closed submodule K/N'; whence K is a maximal τ-closed submodule of M.

If $(x_p R + N')/N'$ is not τ-artinian, [3, Corollary 4.6] implies that $(x_p R + N')/N'$ has a maximal artinian submodule A/N' such that $(x_p R + N')/A$ is τ-noetherian. Thus $(x_p R + N')/A$ must have a proper maximal τ-closed submodule K'/A; whence K' is a maximal τ-closed submodule of M. □

LEMMA 4. *Conclusion (1) of Theorem 2 is true.*

Proof. Since R has τ-DICC, R has τ-max by Lemma 3. Hence the Generalized Nakayama's Lemma (see [11] or [4, Proposition 2.3.15]) implies that $J^{\alpha} \subseteq \tau(R)$. □

LEMMA 5. *If M is a τ-DICC module and N is τ-closed in M, then $(S^1_\tau(M/N))J = 0$, where J is the τ-Jacobson radical of R.*

Proof. Since M is a τ-DICC module, then M/N has finite uniform dimension via [3, Proposition 4.3]. Hence there exist τ-simple modules $S_1, S_2, \ldots, S_n$ such that $\bigoplus_{i=1}^{n} S_i$ is essential in $S^1_\tau(M)$. Choose K_j maximal with respect to $\bigoplus_{i \neq j} S_i \subseteq K_j$ and $K_j \cap S_j = 0$. Then each S_j is essential in $S^1_\tau(M/N)/K_j$ and $S^1_\tau(M/N)/K_j$ is τ-simple. Therefore,

$$(S^1_\tau(M/N)/K_j)J = 0$$

for each $j \leq n$. But $S^1_\tau(M/N)$ embeds in $\bigoplus_{j=1}^{n}(S^1_\tau(M/N))/K_j$; so $S^1_\tau(M/N)\ J = 0$, as desired. □

LEMMA 6. *Conclusion (2) of Theorem 2 is true.*

Proof. Since $\alpha < \omega$, then J^α is an idempotent ideal of R. Since J^α is not τ-finitely generated, then J^α is τ-artinian by [3, Proposition 4.7]. Note that $S^1_\tau(R)J^\alpha \subseteq \tau(R)$ by Lemma 5. Inductively, suppose that $S^\beta_\tau(R)J^\alpha \subseteq \tau(R)$ for each $\beta < \gamma$. If γ is not a limit ordinal, then Lemma 5, the idempotence of J^α, and the induction hypothesis imply that $S^\gamma_\tau(R)J^\alpha \subseteq \tau(R)$. If γ is a limit ordinal, then $(\cup_{\beta<\gamma} S^\beta_\tau(R))J^\alpha \subseteq \tau(R)$ by induction; so $(S^\gamma_\tau(R))J^\alpha \subseteq \tau(R)$ by the J^α-invariance of τ. Thus $S^\gamma(R)J^\alpha \subseteq \tau(R)$ for all γ.

Since R is a τ-DICC ring, there is a τ-artinian ideal A of R such that R/A is τ-noetherian. Since $A \subseteq S^\lambda_\tau(R)$ for some λ, then $AJ^\alpha \subseteq \tau(R)$. Let

$$A = C_0 \subset C_1 \subset C_2 \subset \cdots \subset C_k = R$$

be a chain of right ideals C_i such that each C_{i+1}/C_i is β_i-τ-critical for some ordinal β_i, where $0 \le \beta_1 \le \beta_2 \le \ldots \le \beta_k$. If we show that $DJ^\alpha = 0$ for any β-τ-critical module D, then $J^\alpha = R(J^\alpha)^{k+1} \subseteq AJ^\alpha \subseteq \tau(R)$, as desired.

Let D be β-τ-critical. We have already seen that $DJ^\alpha = 0$ when $\beta = 0$; so we may assume $\beta \ge 1$. If $DJ^\alpha \ne 0$, then DJ^α contains a τ-simple submodule S, as J^α is τ-artinian. Since submodules of β-τ-critical modules are β-τ-critical, then S is β-τ-critical; whence $\beta = 0$, which is a contradiction. □

LEMMA 7. *Conclusion (3) of Theorem 2 is true.*

Proof. Since J^α is not τ-finitely generated, then J^α is τ-artinian by [3, Proposition 4.7]. Consider the descending chain of ideals:

$$J^\alpha \supseteq J \cdot J^\alpha \supseteq J^2 \cdot J^\alpha \supseteq \cdots .$$

Since J^α is τ-artinian, there exists $k < \omega$ such that $\overline{J^k J^\alpha} = \overline{J^{k+1} J^\alpha}$.

We now show inductively that $S^\gamma_\tau(J^\alpha) \cdot \overline{J^k J^\alpha} \subseteq \tau(R)$ for each ordinal γ. The case $\gamma = 1$ is immediate from Lemma 5. Inductively, assume that $S^\beta_\tau(J^\alpha)\overline{J^k J^\alpha}$ for all $\beta < \gamma$. If $\gamma = \beta+1$, then by the $J^{k+1}J^\alpha$-invariance of τ, Lemma 5, and the induction hypothesis, we have $(S^{\beta+1}_\tau(J^\alpha))(\overline{J^k J^\alpha}) = (S^{\beta+1}_\tau(J^\alpha))(\overline{J^{k+1} J^\alpha}) \subseteq \overline{S^{\beta+1}(J^\alpha)(J^{k+1}J^\alpha)} \subseteq \overline{S^\beta(J^\alpha)J^k J^\alpha} \subseteq \overline{\tau(R)} = \tau(R)$. Finally, if γ is a limit ordinal, then $(\bigcup_{\beta<\gamma} S^\beta_\tau(J^\alpha))\overline{J^k J^\alpha} \subseteq \tau(R)$ by induction; so $S^\gamma_\tau(J^\alpha) \cdot \overline{J^k J^\alpha} \subseteq \tau(R)$ by the $\overline{J^k J^\alpha}$-invariance of τ.

Since J^α is τ-artinian, then $J^\alpha = S^\lambda_\tau(J^\alpha)$ for some λ. As a consequence of the previous paragraph, associativity and the definition of J^α, we now have $\tau(R) \supseteq J^\alpha(J^k J^\alpha) = (J^{\alpha+k})J^\alpha = J^\alpha \cdot J^\alpha$, as desired. □

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