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ERROR BOUNDS FOR SIMPSON'S QUADRATURE THROUGH ZERO MEAN GAUSSIAN WITH COVARIANCE

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Abstract. We computed zero mean Gaussian of average error bounds of Simpson's quadrature with covariances in [2]. In this paper, we compute zero mean Gaussian of average error bounds between Simpson's quadrature and composite Simpson's quadrature on four consecutive subintervals. The reason why we compute these on subintervals is because these results enable us to compute *a posteriori* error bounds on the whole interval in the later paper.

1. Introduction

Many numerical computations in science and engineering can only be solved approximately since the available information is partial. For instance, for problems defined on a space of functions, information about f is typically provided by a few function values, $N(f) = [f(x_1), f(x_2), \dots, f(x_n)]$. Knowing $N(f)$, the solution is approximated by a numerical method. The error between the true solution and the approximation depends on a problem setting. In the worst case setting, the error of a numerical scheme is defined by its worst performance with respect to the given class of functions. Many results are known in this setting; see [4] and [6] for hundreds of references. In this paper, we concentrate on another setting, the average case setting. In this setting, we assume that the class F of input functions is equipped with a probability measure. Then the average case error of an algorithm is defined by its expectation, rather than by its worst case performance.

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It is well known that the average case setting requires the space of functions to be equipped with a probability measure. The average case error of an algorithm is defined by its expectation, rather than by its worst case performance. The average case analysis is important and significant number of results have already been obtained (see, e.g., [6] and the references cited therein). In this paper, we choose a probability measure μ_r which is a variant of an r-fold Wiener measure ω_r. The probability measure ω_r is a Gaussian measure with zero mean and correlation function given by

$$M_{\omega_r}(f(x)\, f(y)) \;=\; \int_F f(x) f(y)\, \omega_r(df) \;=\; \int_0^1 \frac{(x-t)_+^r}{r!} \, \frac{(y-t)_+^r}{r!}\, dt,$$

where $(z-t)_+^r = [\max\{0,(z-t)\}]^r$. Equivalently, f distributed according to ω_r can be viewed as a Gaussian stochastic process with zero mean and autocorrelation given above. However, since ω_r is concentrated on functions with boundary conditions $f(0) = f'(0) = \cdots = f^{(r)}(0) = 0$, we choose to study a slightly modified measure μ_r that preserves basic properties of ω_r, yet does not require any boundary conditions. More precisely, we assume that a function f, as a stochastic process, is given by

$$f(x) \;=\; f_1(x) + f_2(1-x), \quad x \in [0,1],$$

where f_1 and f_2 are independent and distributed according to ω_r. Then the corresponding probability measure μ_r is a zero mean Gaussian with the correlation function given by

$$M_{\mu_r}(f(x)\, f(y)) \;=\; \int_0^1 \frac{(x-t)_+^r (y-t)_+^r \;+\; (t-x)_+^r (t-y)_+^r}{r!\; r!}\, dt.$$

We study the problem of approximating an integral $I(f) = \int_0^1 f(x)\, dx$ for $f \in F = C^r[0,1]$, assuming that the class of integrands is equipped with the probability measure μ_r.

2. Definitions

Assume that we have m subintervals (not necessarily of equal length) partitioning $[0,1]$ and choose five equally spaced points from each subintervals. For simplicity of presentation, we let x_i and x_{i+4} be the left end and right end points, and $x_{i+k} = x_i + kh_i$, for $k = 0, \ldots, 4$. With this indexing, we get

$$I_i(f) = \int_{x_i}^{x_{i+4}} f(x)\,dx \quad \text{and} \quad S_i(f) = \frac{2h_i}{3}\{f(x_i) + 4f(x_{i+2}) + f(x_{i+4})\},$$

while S_i is the basic *Simpson's quadrature* that uses $f(x_i)$, $f(x_{i+2})$, and $f(x_{i+4})$. Let $\overline{S}_i$ be the composite Simpson's quadrature that uses $f(x_i)$, $f(x_{i+1})$, $f(x_{i+2})$, $f(x_{i+3})$, and $f(x_{i+4})$, i.e.,

$$\overline{S}_i(f) = \frac{h_i}{3}\{f(x_i) + 4f(x_{i+1}) + 2f(x_{i+2}) + 4f(x_{i+3}) + f(x_{i+4})\}.$$

Let

$$X_i(f) := I_i(f) - \overline{S}_i(f) \qquad \text{and} \qquad Y_i(f) := \frac{1}{15}(\overline{S}_i - S_i).$$

3. Error bounds on subintervals

In this section, we compute error bounds of the distributions of X_i, Y_i and $X_i - Y_i$ on four consecutive subintervals. We will explore error bounds on the whole interval in the later paper. Recall that the space $F = C^r[0,1]$ is equipped with the probability measure μ_r defined in chapter 2. Since f is a zero-mean Gaussian process, X_i's, Y_i's and $X_i - Y_i$'s are Gaussian with zero-mean and covariances given in the following theorems.

The general references for this paper are [1, 3, 4, 5, 6].

THEOREM 1. *For $i \le j$,*

$$M_{\mu_r}(X_iX_j) = \begin{cases} \delta_{ij} \cdot c_r \cdot h_i^{2r+3} & \text{if } r \le 3, \\ c_{ijr} \cdot h_i^5 h_j^5 & \text{if } r \ge 4, \end{cases}$$

where c_r is independent of h_i and equals respectively;

$$c_0 = \frac{8}{9}, \quad c_1 = \frac{4}{135}, \quad c_2 = \frac{2}{945}, \quad \text{and} \quad c_3 = \frac{1}{2268}.$$

For $r = 4$,

$$c_{ii4} = \frac{1}{45^2}\left(1 - \frac{931}{792}h_i\right) \quad \text{and} \quad c_{ij4} = \frac{1}{45^2}(x_i + 1 - x_{j+4} + 2h_i + 2h_j).$$

For $r \geq 5$, $c_{ijr} = c_{ijr}(h_i, h_j)$ is bounded from below by

$$a_r[x_i^{r-3}(x_j - x_i)^{r-4} + x_i^{r-3}x_j^{r-4} + h_i^{r-3}(x_j - x_{i+4})^{r-4} + (1 - x_{j+4})^{r-3}(x_{j+4} - x_{i+4})^{r-4} + (1 - x_{j+4})^{r-3}(1 - x_{i+4})^{r-4} + h_j^{r-3}(x_j - x_{i+4})^{r-4}]$$

and from above by

$$a'_r[x_{i+4}^{r-3}x_{j+4}^{r-4} + h_i^{r-3}(x_{j+4} - x_i)^{r-4} + (1 - x_j)^{r-3}(1 - x_i)^{r-4} + h_j^{r-3}(x_{j+4} - x_i)^{r-4}],$$

where a_r and a'_r are positive constants depend on r, but not on h_i's.

PROOF. See Choi and Hong [2]. □

The following is the main theorem of this paper.

THEOREM 2. *For $i \leq j$,*

$$(1) \qquad M_{\mu_r}(Y_iY_j) = \begin{cases} \delta_{ij} \cdot c'_r \cdot h_i^{2r+3} & \text{if } r \leq 3, \\ c'_{ijr} \cdot h_i^5 h_j^5 & \text{if } r \geq 4, \end{cases}$$

where

$$c'_0 = \frac{8}{405}, \quad c'_1 = \frac{16}{6075}, \quad c'_2 = \frac{4}{6075}, \quad \text{and} \quad c'_3 = \frac{302}{637875}.$$

For $r = 4$,

$$c'_{ii4} = \frac{1}{45^2}\left(1 - \frac{1487}{2268}h_i\right) \quad \text{and} \quad c'_{ij4} = \frac{1}{45^2}(x_i + 1 - x_{j+4} + 2h_i + 2h_j).$$

For $r \geq 5$, $c'_{ijr} = c'_{ijr}(h_i, h_j)$ is bounded in the same way as c_{ijr} in Theorem 1.

$$(2) \qquad M_{\mu_r}([X_i - Y_i][X_j - Y_j]) = \begin{cases} \delta_{ij} \cdot c''_r \cdot h_i^{2r+3} & \text{if } r \leq 5, \\ c''_{ijr} \cdot h_i^5 h_j^5 & \text{if } r \geq 6, \end{cases}$$

where

$$c''_0 = \frac{416}{405}, \quad c''_1 = \frac{256}{6075}, \quad c''_2 = \frac{941}{212625}, \quad c''_3 = \frac{6443}{1134000},$$

$$c_4'' = 7.5895 \times 10^{-2}, \quad \textit{and} \quad c_5'' = 8.4999 \times 10^{-3}.$$

For $r \geq 6$, $c''_{ijr} = c''_{ijr}(h_i, h_j)$ *is bounded from below by*

$$\begin{aligned} b_r[x_i^{r-5}(x_j - x_i)^{r-6} + x_i^{r-5}x_j^{r-6} \\ + h_i^{r-5}(x_j - x_{i+4})^{r-6} + (1 - x_{j+4})^{r-5}(x_{j+4} - x_{i+4})^{r-6} \\ + (1 - x_{j+4})^{r-5}(1 - x_{i+4})^{r-6} + h_j^{r-5}(x_j - x_{i+4})^{r-6}] \end{aligned}$$

and from above by

$$\begin{aligned} b'_r[x_{i+4}^{r-5}x_{j+4}^{r-6} + h_i^{r-5}(x_{j+4} - x_i)^{r-6} \\ + (1 - x_j)^{r-5}(1 - x_i)^{r-6} + h_j^{r-5}(x_{j+4} - x_i)^{r-6}], \end{aligned}$$

where b_r *and* b'_r *are positive constants that depend on* r*, but not on* h_i*'s.*

PROOF. We first prove (1). Since f_1 and f_2 are independent,

$$M_{\mu_r}(Y_iY_j) = M_{\omega_r}(Y_{i1}Y_{j1}) + M_{\omega_r}(Y_{i2}Y_{j2}).$$

It is easy to verify that $Y_i(f) = -\frac{h_i}{45}\nabla_i^4 f = -\frac{h_i}{45}\nabla_i^4 f_1 - \frac{h_i}{45}\nabla_i^4 f_2$, where $h_i = (x_{i+4} - x_i)/4$ and $\nabla_i^4 f$ is the backward difference of degree 4 of f at x_{i+4}, i.e., $\nabla_i^4 f = f(x_i) - 4f(x_{i+1}) + 6f(x_{i+2}) - 4f(x_{i+3}) + f(x_{i+4})$. Now, if L_{i1} is the first term and L_{j1} is the second in the next integral,

$$\begin{aligned} M_{\omega_r}(Y_{i1}Y_{j1}) &= \int_0^1 \left[-\frac{h_i}{45}\nabla_i^4\left(\frac{(\cdot - t)_+^r}{r!}\right)\right]\left[-\frac{h_j}{45}\nabla_j^4\left(\frac{(\cdot - t)_+^r}{r!}\right)\right] dt \\ &= \int_0^1 L_{i1}(t)\cdot L_{j1}(t)dt = \int_0^{x_{i+4}} L_{i1}(t)\cdot L_{j1}(t)\,dt, \end{aligned}$$

since $L_{i1}(t) = 0$ for $t \in [x_{i+4}, 1]$. Similarly,

$$\begin{aligned} M_{\omega_r}(Y_{i2}Y_{j2}) &= \int_{x_j}^1 \left[-\frac{h_i}{45}\nabla_i^4\left(\frac{(t - \cdot)_+^r}{r!}\right)\right]\left[-\frac{h_j}{45}\nabla_j^4\left(\frac{(t - \cdot)_+^r}{r!}\right)\right] dt \\ &= \int_{x_j}^1 L_{i2}(t)\cdot L_{j2}(t)dt. \end{aligned}$$

Consider first $r \leq 3$. Since ∇_i^4 applied to polynomials of degree ≤ 3 is zero, $L_{j1}(t) = 0$ for $t \leq x_{i+4}$ and $L_{i2}(t) = 0$ for $t \geq x_j$. Thus, $M_{\mu_r}(Y_iY_j) = 0$ when $i < j$. For $i = j$,

$$M_{\omega_r}(Y_{i1}^2) = \int_{x_i}^{x_{i+4}} \left[-\frac{h_i}{45}\nabla_i^4\left(\frac{(\cdot - t)_+^r}{r!}\right)\right]^2 dt = c'_{r1}h_i^{2r+3},$$

where

$$c'_{r1} = \frac{4^{2r+3}}{180^2} \int_0^1 \left[\nabla_1^4 \left(\frac{(\cdot - u)_+^r}{r!} \right) \right]^2 du,$$

and

$$\nabla_1^4 \left(\frac{(\cdot - u)_+^r}{r!} \right) = \frac{(0-u)_+^r}{r!} - \frac{4(\frac{1}{4}-u)_+^r}{r!} + \frac{6(\frac{1}{2}-u)_+^r}{r!} - \frac{4(\frac{3}{4}-u)_+^r}{r!} + \frac{(1-u)_+^r}{r!}. \tag{3}$$

Similarly,

$$M_{\omega_r}(Y_{i2}^2) = \int_{x_i}^{x_{i+4}} \left[-\frac{h_i}{45} \nabla_i^4 \left(\frac{(t - \cdot)_+^r}{r!} \right) \right]^2 dt = c'_{r2} h_i^{2r+3},$$

where

$$c'_{r2} = \frac{4^{2r+3}}{180^2} \int_0^1 \left[\nabla_1^4 \left(\frac{(u - \cdot)_+^r}{r!} \right) \right]^2 du = c'_{r1}.$$

Thus, $c'_r = c'_{r1} + c'_{r2} = 2c'_{r1}$. Since it is straightforward to get the corresponding values of c'_r, we omit this part. This completes the proof of (1) for $r \le 3$.

Next consider $r \ge 4$. Let

$$\begin{aligned} M_{\omega_r}(Y_{i1}Y_{j1}) &= \int_0^{x_{i+4}} L_{i1}(t) \cdot L_{j1}(t)\, dt \\ &= \left(\int_0^{x_i} + \int_{x_i}^{x_{i+4}} \right) L_{i1}(t) \cdot L_{j1}(t)\, dt. \end{aligned}$$

Then

$$\begin{aligned} \int_0^{x_i} L_{i1}(t) \cdot L_{j1}(t)\, dt &= \frac{h_i^5 h_j^5}{45^2} \int_0^{x_i} \frac{(\xi_t - t)^{r-4} (\eta_t - t)^{r-4}}{(r-4)!(r-4)!}\, dt \\ &= A_{ijr} \cdot h_i^5 h_j^5, \end{aligned}$$

$$\begin{aligned} \int_{x_i}^{x_{i+4}} L_{i1}(t) \cdot L_{j1}(t)\, dt &= \int_{x_i}^{x_{i+4}} L_{i1}(t) \left[-\frac{h_j^5}{45} \frac{(\eta_t - t)^{r-4}}{(r-4)!} \right] dt \\ &= B_{ijr}(h_i) \cdot h_i^5 h_j^5, \end{aligned}$$

where $\xi_t \in (x_i, x_{i+4})$, $\eta_t \in (x_j, x_{j+4})$. Here A_{ijr} is bounded from below by

$$\begin{aligned} A_{ijr} &\geq a_1 \sum_{p=0}^{r-4} \frac{(x_j - x_i)^p}{p!(2r-7-p)} \frac{x_i^{2r-7-p}}{(r-4-p)!} \\ &\geq a_2 \left[x_i^{r-3}(x_j - x_i)^{r-4} + x_i^{r-3} x_j^{r-4} \right], \end{aligned}$$

and from above by

$$\begin{aligned} A_{ijr} &\leq a_3 \frac{x_{i+4}^{r-3}}{(r-4)!} \sum_{p=0}^{r-4} \binom{r-4}{p} (x_{j+4} - x_{i+4})^p \frac{x_{i+4}^{r-4-p}}{(2r-7-p)} \\ &\leq a_4 x_{i+4}^{r-3} x_{j+4}^{r-4}. \end{aligned}$$

Obviously, $B_{ijr}(h_i)$ is bounded by

$$a_5 h_i^{r-3}(x_j - x_{i+4})^{r-4} \;\leq\; B_{ijr}(h_i) \;\leq\; a_5 h_i^{r-3}(x_{j+4} - x_i)^{r-4},$$

and a_1, a_2, a_3, a_4, and a_5 are positive constants that depend only on r. Similarly,

$$M_{\omega_r}(Y_{i2}Y_{j2}) \;=\; A'_{ijr} \cdot h_i^5 h_j^5 + B'_{ijr}(h_j) \cdot h_i^5 h_j^5.$$

Therefore, for $r \geq 4$,

$$M_{\mu_r}(Y_i Y_j) \;=\; \left\{ A_{ijr} + A'_{ijr} + B_{ijr}(h_i) + B'_{ijr}(h_j) \right\} \cdot h_i^5 h_j^5 \;=\; c'_{ijr} \cdot h_i^5 h_j^5.$$

It is straightforward to get the bounds on c'_{ijr}, so we skip this part. This completes the proof of (1).

To show (2),

$$M_{\omega_r}\left([X_{i1} - Y_{i1}][X_{j1} - Y_{j1}] \right) \;=\; \int_0^{x_{i+4}} L_{i1}(t) \cdot L_{j1}(t)\, dt,$$

where

$$L_{i1}(t) \;=\; \int_{x_i}^{x_{i+4}} \frac{(x-t)_+^r}{r!} dx - \overline{S}_i \left(\frac{(\cdot - t)_+^r}{r!} \right) + \frac{h_i}{45} \nabla_i^4 \left(\frac{(\cdot - t)_+^r}{r!} \right)$$

and

$$L_{j1}(t) \;=\; \int_{x_j}^{x_{j+4}} \frac{(y-t)_+^r}{r!}\, dy - \overline{S}_j \left(\frac{(\cdot - t)_+^r}{r!} \right) + \frac{h_j}{45} \nabla_j^4 \left(\frac{(\cdot - t)_+^r}{r!} \right).$$

Similarly,

$$M_{\omega_r}\left([X_{i2}-Y_{i2}][X_{j2}-Y_{j2}]\right) = \int_{x_j}^{1} L_{i2}(t)\cdot L_{j2}(t)\,dt.$$

Consider $r \leq 3$. Since $L_{j1}(t) = 0$ for $t \leq x_{i+4}$ and $L_{i2}(t) = 0$ for $t \geq x_j$, $M_{\mu_r}([X_i - Y_i][X_j - Y_j]) = 0$ when $i < j$. For $i = j$, by the change of variables, $z = (x - x_i)/4h_i$, $u = (t - x_i)/4h_i$, we have
(4)

$$\begin{aligned} M_{\omega_r}([X_{i1}-Y_{i1}]^2) = & \int_{x_i}^{x_{i+4}} \Big[\int_{x_i}^{x_{i+4}} \frac{(x-t)_+^r}{r!}dx - \overline{S}_i\left(\frac{(\cdot - t)_+^r}{r!}\right) \\ & \qquad + \frac{h_i}{45}\nabla_i^4\left(\frac{(\cdot - t)_+^r}{r!}\right)\Big]^2 dt \\ = & (4h_i)^{2r+3}\int_0^1 \Big[\int_0^1 \frac{(z-u)_+^r}{r!}dz - \overline{S}\left(\frac{(\cdot - u)_+^r}{r!}\right) \\ & \qquad + \frac{1}{180}\nabla_1^4\left(\frac{(\cdot - u)_+^r}{r!}\right)\Big]^2 du, \end{aligned}$$

where $\overline{S}$ denotes composite *Simpson's quadrature* on $[0,1]$ based on the points 0, 1/4, 1/2, 3/4, and 1, and ∇_1^4 is from (3) $M_{\omega_r}\left([X_{i2}-Y_{i2}]^2\right)$ can be found in a similar way. Since it is straightforward to get the values of c_r'', we omit this part. This completes the proof for $r \leq 3$.

Secondly, consider $r = 4$. Then $L_{j1}(t) = 0$ for $t \leq x_{i+4}$, since $X_{j1} = -h_j^5/45$ and $Y_{j1} = -h_j^5/45$. Similarly, $L_{i2}(t) = 0$ for $t \geq x_j$. Hence, for $i < j$, $M_{\mu_r}([X_i - Y_i][X_j - Y_j]) = 0$. For $i = j$, by (4), $M_{\mu_r}\left([X_i - Y_i]^2\right) = c_4'' \cdot h_i^{11}$.

Thirdly, let $r = 5$. By the Binomial theorem, we have

$$\frac{(y-t)^r}{r!} = \sum_{\ell=0}^{r} \frac{(x_j - t)^{r-\ell}}{(r-\ell)!}\frac{(y-x_j)^\ell}{\ell!}.$$

Then, for $t \in [0, x_j]$,

$$\begin{aligned} L_{j1}(t) &= \int_{x_j}^{x_{j+4}} \frac{(y-t)^r}{r!}dy - \overline{S}_j\left(\frac{(\cdot - t)^r}{r!}\right) + \frac{h_j}{45}\nabla_j^4\left(\frac{(\cdot - t)^r}{r!}\right) \\ &= \sum_{\ell=0}^{r}\frac{(x_j - t)^{r-\ell}}{(r-\ell)!}W_\ell, \end{aligned}$$

where

$$W_\ell = \int_{x_j}^{x_{j+4}} \frac{(y-x_j)^\ell}{\ell!}\,dy - \overline{S}_j\left(\frac{(\cdot - x_j)^\ell}{\ell!}\right) + \frac{h_j}{45}\nabla_j^4\left(\frac{(\cdot - x_j)^\ell}{\ell!}\right).$$

Note that $W_\ell = 0$ for $\ell = 0,1,2,3,4$. For $\ell = 5$,

$$\begin{aligned} W_5 &= \int_{x_j}^{x_{j+4}} \frac{(y-x_j)^5}{5!}\,dy - \overline{S}_j\left(\frac{(\cdot - x_j)^5}{5!}\right) + \frac{h_j}{45}\nabla_j^4\left(\frac{(\cdot - x_j)^5}{5!}\right) \\ &= \frac{(4h_j)^6}{6!} - \frac{h_j}{3}\left(\frac{4h_j^5}{5!} + \frac{2(2h_j)^5}{5!} + \frac{4(3h_j)^5}{5!} + \frac{(4h_j)^5}{5!}\right) \\ &\quad + \frac{h_j}{45}\left(-\frac{4h_j^5}{5!} + \frac{6(2h_j)^5}{5!} - \frac{4(3h_j)^5}{5!} + \frac{(4h_j)^5}{5!}\right) \quad = \quad 0. \end{aligned}$$

Therefore, $L_{j1}(t) = 0$ for $t \le x_{i+4}$ and similarly $L_{i2}(t) = 0$ for $t \ge x_j$, and hence, for $i < j$, $M_{\mu_r}([X_i - Y_i][X_j - Y_j]) = 0$. For $i = j$, by (4), $M_{\mu_r}\left([X_i - Y_i]^2\right) = c_5'' h_i^{13}$.

Finally, consider $r \ge 6$. Then, for $\ell = 6$, we have

$$\begin{aligned} W_6 &= \int_{x_i}^{x_{i+4}} \frac{(x-x_i)^6}{6!}\,dx - \overline{S}_i\left(\frac{(\cdot - x_i)^6}{6!}\right) + \frac{h_i}{45}\nabla^4\left(\frac{(\cdot - x_i)^6}{6!}\right) \\ &= \frac{8}{945}h_i^7, \end{aligned}$$

and for $\ell \ge 7$, we can find that $W_\ell = \Theta(h_i^{\ell+1})$. Thus, for $t \in [0, x_i]$,

$$\begin{aligned} &M_{\omega_r}([X_{i1} - Y_{i1}][X_{j1} - Y_{j1}]) \\ &= \int_0^{x_i}\left[\sum_{\ell=0}^{r}\frac{(x_i - t)^{r-\ell}}{(r-\ell)!}W_\ell\right]\left[\sum_{k=0}^{r}\frac{(x_j - t)^{r-k}}{(r-k)!}W_k\right]dt \\ &= \left(\frac{8}{945}\right)^2 h_i^7 h_j^7 \int_0^{x_i}\frac{(x_i - t)^{r-6}}{(r-6)!}\frac{(x_j - t)^{r-6}}{(r-6)!}\,dt \\ &\quad + \text{(higher order terms)} \\ &= A_{ijr}\cdot h_i^7 h_j^7, \end{aligned}$$

where A_{ijr} is independent of h_i, bounded from below by

$$\begin{aligned} A_{ijr} &\ge a_1 \sum_{p=0}^{r-6}\frac{(x_j - x_i)^p}{p!(2r-11-p)}\frac{x_i^{2r-11-p}}{(r-6-p)!} \\ &\ge a_2\left[x_i^{r-5}(x_j - x_i)^{r-6} + x_i^{r-5}x_j^{r-6}\right], \end{aligned}$$

and bounded from above by

$$\begin{aligned} A_{ijr} &\le a_3 \frac{x_{i+4}^{r-5}}{(r-6)!} \sum_{p=0}^{r-6} \binom{r-6}{p} (x_{j+4} - x_{i+4})^p \frac{x_{i+4}^{r-6-p}}{(2r-11-p)} \\ &\le a_4 x_{i+4}^{r-5} x_{j+4}^{r-6}, \end{aligned}$$

and a_1, a_2, a_3, and a_4 are positive constants that depend only on r. For $t \in [x_i, x_{i+4}]$,

$$\begin{aligned} M_{\omega_r}([X_{i1} - Y_{i1}][X_{j1} - Y_{j1}]) &= \int_{x_i}^{x_{i+4}} L_{i1}(t) L_{j1}(t) dt \\ = \int_{x_i}^{x_{i+4}} L_{i1}(t) &\left[\sum_{\ell=0}^{r} \frac{(x_j - t)^{r-\ell}}{(r-\ell)!} W_\ell \right] dt \\ = \frac{8}{945} h_j^7 \int_{x_i}^{x_{i+4}} &L_{i1}(t)\, dt + \quad \text{(higher order terms)} \\ = \frac{8}{945} h_j^7 (4h_i)^{r+2} &\int_0^1 \left[\int_0^1 \frac{(z-u)_+^r}{r!} dz - \overline{S} \left(\frac{(\cdot - u)_+^r}{r!} \right) \right. \\ &\left. + \frac{1}{180} \nabla_1^4 \left(\frac{(\cdot - u)_+^r}{r!} \right) \right] du \\ + \quad &\text{(higher order terms)} \\ = B_{ijr}(h_i) \cdot h_i^7 h_j^7, & \end{aligned}$$

where $B_{ijr}(h_i)$ is bounded by

$$a_5 h_i^{r-5} (x_j - x_{i+4})^{r-6} \le B_{ijr}(h_i) \le a_5 h_i^{r-5} (x_{j+4} - x_i)^{r-6},$$

and a_5 is a positive constant that depends only on r. Similarly,

$$M_{\omega_r}([X_{i2} - Y_{i2}][X_{j2} - Y_{j2}]) = A'_{ijr} \cdot h_i^7 h_j^7 + B'_{ijr}(h_j) \cdot h_i^7 h_j^7.$$

Therefore, for $r \ge 6$, $M_{\mu_r}([X_i - Y_i][X_j - Y_j]) = c''_{ijr} h_i^7 h_j^7$, where $c''_{ijr} = A_{ijr} + A'_{ijr} + B_{ijr}(h_i) + B'_{ijr}(h_j)$. It is straightforward to get the bounds on c''_{ijr}. This completes the proof. □

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