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CONVEX POLYTOPES OF GENERALIZED DOUBLY STOCHASTIC MATRICES

SOOJIN CHO AND YUNSUN NAM

ABSTRACT. *Doubly stochastic matrices* are $n \times n$ nonnegative matrices whose row and column sums are all 1. Convex polytope Ω_n of doubly stochastic matrices and more generally $\mathfrak{A}(R,S)$, so called *transportation polytopes*, are important since they form the domains for the transportation problems. A theorem by Birkhoff classifies the extremal matrices of Ω_n, and extremal matrices of transportation polytopes $\mathfrak{A}(R,S)$ were all classified combinatorially.

In this article, we consider signed version of Ω_n and $\mathfrak{A}(R,S)$, obtain 'signed' Birkhoff theorem; we define a new class of convex polytopes $|\mathfrak{A}|(R,S)$, calculate their dimensions, and classify their extremal matrices. Moreover, we suggest an algorithm to express a matrix in $|\mathfrak{A}|(R,S)$ as a convex combination of extremal matrices. We also give an example that a polytope of signed matrices is used as a domain for a decision problem.

In the context of finite reflection(Coxeter) group theory, our generalization may also be considered as a generalization from type A_n to type B_n and D_n.

1. Introduction

For given positive vectors $R = (r_1, \ldots, r_m)$ and $S = (s_1, \ldots, s_n)$ with $\sum_i r_i = \sum_j s_j$, let $\mathfrak{A}(R,S)$ be the class of all $m \times n$ nonnegative matrices with row sum vector R and column sum vector S. The set $\mathfrak{A}(R,S)$ is a convex polytope, which is called a *transportation polytope.* A matrix in $\mathfrak{A}(R,S)$ is called a *transportation matrix.* Transportation polytopes have applications in many optimization problems and so have been extensively studied (see [4, 6, 8, 9]). Specially, their extremal matrices and

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facets were classified and their 1-skeleton graphs have been considered. We denote the set of extremal matrices of $\mathfrak{A}(R, S)$ by $\mathfrak{E}(R, S)$. A special case of the convex polytope $\mathfrak{A}(R, S)$ is the convex set Ω_n of $n \times n$ *doubly stochastic matrices* when $R = S = (1, 1, \ldots, 1)$. Birkhoff theorem tells that the set of extremal matrices of Ω_n is exactly the set of permutation matrices. A nonnegative $n \times n$ matrix whose row and column sums are all dominated by 1 is called a *doubly substochastic matrix.* The convex polytope Ω'_n of doubly substochastic matrices has been considered and the extremal matrices were classified in [12]. More generally, convex polytopes $\mathfrak{A}_{\leq}(R, S)$ are defined as the set of nonnegative matrices with row sum vector dominated componentwise by R and column sum vector by S respectively. The extremal matrices of $\mathfrak{A}_{\leq}(R, S)$ were classified in [3].

Convex polytopes Ω_n also appear in some different context. A. Barvinok and A. Vershik considered the polytopes of (image matrices of) representations of finite groups [5]. S. Onn [13] considered the *permutation polytopes* that is the convex polytopes of standard representation of subgroups of the symmetric group S_n. It is clear that Ω_n is the permutation polytope of standard representation of S_n.

Our natural question was on the 'signed' version of Birkhoff theorem. In other words, we wanted to know about the *signed permutation polytope* that is the polytope of the standard representation of *hyperoctahedral group.* Hyperoctahedral groups are reflection groups (Coxeter groups) of type B_n whereas the symmetric groups are of type A_n (see [7]). The group elements of hyperoctahedral group are the signed permutations, hence the order of hyperoctahedral group is $n!\,2^n$. Therefore, we may say that our question was on the Birkhoff Theorem of type B_n (or, of some other type than A_n).

In this article, we consider the convex polytope of signed permutations and its generalization, as Ω_n is understood as a special case of $\mathfrak{A}(R, S)$ for any positive vectors R, S rather than $R = S = (1, 1, \ldots, 1)$. The dimension of those polytopes are calculated and extremal matrices are determined, whence Birkhoff theorem of type B_n is obtained.

In the following section, we summarize known results on convex polytopes $\mathfrak{A}(R, S)$ and $\mathfrak{A}_{\leq}(R, S)$. In Section 3, we investigate the convex polytope $|\mathfrak{A}|(R, S)$, which is the polytope generated by those signed permutation matrices. We calculate the dimension and obtain the Birkhoff theorem of type B_n. In Section 4, we consider two polytopes generated by subsets of whole set of signed permutation matrices. One of them forms a domain for a decision problem and the other is a subpolytope of Ω_n generated by the signed permutations with even number of sign changes, which is a polytope of standard representation of reflection group of type D_n. Because of the Birkhoff theorem of type B_n, we can immediately know the extremal matrices of these polytopes.

For a finite set of vectors $\mathcal{S}$ in $\mathbb{R}^n$, Conv($\mathcal{S}$) is defined as the set of all convex combinations of elements in $\mathcal{S}$;

$$\mathrm{Conv}(\mathcal{S}) = \{\alpha_1 A_1 + \cdots + \alpha_k A_k \mid A_1, \ldots, A_k \in \mathcal{S}, \sum_{i=1}^{k} \alpha_i = 1, 0 \leq \alpha_i \leq 1\}.$$

In this case, we say that the convex polytope is *generated by* $\mathcal{S}$. The reader may refer to [14] for the basic definitions and theorems about convex polytopes.

2. Preliminaries

In this section, we give formal definitions of $\mathfrak{A}(R, S)$ and $\mathfrak{A}_{\leq}(R, S)$ with known results on those polytopes.

For given positive vectors $R = (r_1, \ldots, r_m)$ and $S = (s_1, \ldots, s_n)$ with $\sum_i r_i = \sum_j s_j$, $\mathfrak{A}(R, S)$ is the set of $m \times n$ nonnegative matrices $A = (a_{ij})$ satisfying

$$\sum_j a_{ij} = r_i \quad \text{for all } i\ ,$$

$$\sum_i a_{ij} = s_j \quad \text{for all } j\ .$$

$\mathfrak{A}_{\leq}(R, S)$ is the set of $m \times n$ nonnegative matrices $A = (a_{ij})$ satisfying

$$\sum_j a_{ij} \leq r_i \quad \text{for all } i\ ,$$

$$\sum_i a_{ij} \leq s_j \quad \text{for all } j\ .$$

Then it is easy to check that $\mathfrak{A}(R, S)$ and $\mathfrak{A}_{\leq}(R, S)$ are convex polytopes in $\mathbb{R}^{mn}$. We denote the set of extremal matrices of $\mathfrak{A}(R, S)$ and $\mathfrak{A}_{\leq}(R, S)$ by $\mathfrak{E}(R, S)$, $\mathfrak{E}_{\leq}(R, S)$, respectively.

Given an $m \times n$ matrix $A = (a_{ij})$, let $\mathcal{B}(A)$ denote the weighted bipartite graph with vertex set $\{R_1, R_2, \ldots, R_m\} \cup \{C_1, C_2, \ldots, C_n\}$, where there is an edge with weight a_{ij} between R_i and C_j if and only if $a_{ij} \neq 0$. We also let $P(A)$ be the $(0,1)$-matrix with 1's in the positions occupied by the non-zero entries of A and 0's elsewhere. A *line* of a matrix designates either a row or a column of the matrix.

In [4, 8, 9], the extremal matrices of $\mathfrak{A}(R,S)$, i.e. the elements of $\mathfrak{E}(R,S)$, were characterized.

PROPOSITION 1. *When A is a nonnegative $m \times n$ matrix in $\mathfrak{A}(R,S)$, the following conditions are equivalent:*

(i) $A \in \mathfrak{E}(R,S)$.
(ii) *Every submatrix of A contains a line with at most one positive entry.*
(iii) *Every submatrix A' of A of size $m' \times n'$ has at most $m' + n' - 1$ positive entries.*
(iv) *There is no matrix B in $\mathfrak{A}(R,S)$ such that $B \neq A$ and $P(B) = P(A)$.*
(v) *$\mathcal{B}(A)$ is a forest with no isolated vertex.*

Birkhoff theorem that the extremal matrices of Ω_n are the permutation matrices is a consequence of the Proposition 1 since the forest corresponding to an extremal matrix can only have single edges.

We say that a line sum of a matrix $A \in \mathfrak{A}_{\leq}(R,S)$ is *unattained* if the sum of the entries of the given line is strictly less than given r_i(or s_j).

The following characterizations of elements of $\mathfrak{E}_{\leq}(R,S)$ are given in [3].

PROPOSITION 2. *Let A be a matrix in $\mathfrak{A}_{\leq}(R,S)$. Then A is in $\mathfrak{E}_{\leq}(R,S)$ if and only if the connected components of $\mathcal{B}(A)$ are trees where at most one node of each tree corresponds to a line of A whose sum is unattained.*

PROPOSITION 3. *The elements of $\mathfrak{E}_{\leq}(R,S)$ are precisely those matrices obtained as follows: Take $A \in \mathfrak{A}(R,S)$ and in each of the trees of $\mathcal{B}(A)$ which are connected components, delete a set (possibly empty) of edges of a subtree. Replace by zero the positive entries of A which correspond to the edges of $\mathcal{B}(A)$ that were deleted.*

COROLLARY 4. *If a matrix $A = (a_{ij})$ is in $\mathfrak{E}_{\leq}(R,S)$, then there exists a matrix $B = (b_{ij})$ in $\mathfrak{E}(R,S)$ such that a_{ij} is either b_{ij} or 0.*

3. Convex polytope $|\mathfrak{A}|(R, S)$

We define $\Omega_n^{\pm}$ to be the convex polytope

$$\mathrm{Conv}(\{A \mid A \text{ is an } n \times n \text{ signed permutation matrix }\}) \subset \mathbb{R}^{n^2}.$$

In this section, we try to answer the questions that ask 'if signed permutations are extremal matrices of $\Omega_n^{\pm}$', and 'how the matrices in the convex polytope $\Omega_n^{\pm}$ are characterized'.

We define a class of convex polytopes which contains $\Omega_n^{\pm}$ as a special case; for given positive vectors $R = (r_1, \ldots, r_m)$ and $S = (s_1, \ldots, s_n)$ with $\sum_i r_i = \sum_j s_j$, let $|\mathfrak{A}|(R, S)$ be the set of matrices $A = (a_{ij})$ satisfying

$$\sum_j |a_{ij}| \leq r_i \quad \text{for all } i \ ,$$

$$\sum_i |a_{ij}| \leq s_j \quad \text{for all } j \ .$$

Note that $|\mathfrak{A}|(R, S)$ does not consist of only nonnegative matrices. We also can observe that the defining inequalities of $|\mathfrak{A}|(R, S)$ are obtained from those of $\mathfrak{A}_{\leq}(R, S)$ by substituting $|a_{ij}|$ for each a_{ij}. Hence $|\mathfrak{A}|(R, S)$ is the union of 2^{mn} copies of $\mathfrak{A}_{\leq}(R, S)$, and in this sense it is not clear that $|\mathfrak{A}|(R, S)$ is convex. However, it is easy to check that $|\mathfrak{A}|(R, S)$ is a convex polytope in $\mathbb{R}^{mn}$ by direct calculation.

We let $|\mathfrak{E}|(R, S)$ be the set of extremal matrices of $|\mathfrak{A}|(R, S)$. Given a matrix $A = (a_{ij})$, $|A|$ denotes the matrix $(|a_{ij}|)$, called the *absolute matrix* of A.

REMARK 1.
1. $\mathfrak{A}(R, S) \subset \mathfrak{A}_{\leq}(R, S) \subset |\mathfrak{A}|(R, S)$.
2. When A is an $m \times n$ real matrix,

$$A \in |\mathfrak{A}|(R, S) \ \text{ if and only if } \ |A| \in \mathfrak{A}_{\leq}(R, S).$$

A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ in $\mathbb{R}^n$ is *affinely independent* if the equation $\lambda_1 \mathbf{v}_1 + \cdots + \lambda_k \mathbf{v}_k = \mathbf{0}, \lambda_1 + \cdots + \lambda_k = 0$ has only trivial solution $\lambda_1 = \cdots = \lambda_k = 0$. The *dimension* of a convex polytope P is defined as the number one less than the maximum number of affinely independent vectors in P.

When R is an m-dimensional vector and S is an n-dimensional vector, it is well known that the dimension of $\mathfrak{A}(R, S)$ is $(m-1)(n-1)$. The following proposition gives the dimensions of $\mathfrak{A}_{\leq}(R, S)$ and $|\mathfrak{A}|(R, S)$.

PROPOSITION 5. *The dimension of the polytope* $\mathfrak{A}_{\le}(R,S)$ *is* mn. *Hence the dimension of* $|\mathfrak{A}|(R,S)$ *is also* mn.

Proof. Since $\mathfrak{A}_{\le}(R,S)$ and $|\mathfrak{A}|(R,S)$ are in $\mathbb{R}^{mn}$, the dimensions of $\mathfrak{A}_{\le}(R,S)$ and $|\mathfrak{A}|(R,S)$ are at most mn. For each $1 \le i \le m$ and $1 \le j \le n$, let $A(i,j) = (a_{kl})$ be the matrix defined by $a_{ij} = \min(r_i, s_j)$ and $a_{kl} = 0$ for $(k,l) \ne (i,j)$. Then the zero matrix and $A(i,j)$, $1 \le i \le m, 1 \le j \le n$, form a set of affinely independent matrices in $\mathfrak{A}_{\le}(R,S)$ and in $|\mathfrak{A}|(R,S)$ also. □

We classify the extremal matrices of $|\mathfrak{A}|(R,S)$ (elements of $|\mathfrak{E}|(R,S)$) in the following theorem. Since $|\mathfrak{A}|(R,S)$ is a union of copies of $\mathfrak{A}_{\le}(R,S)$, we may expect that the extremal matrices of $|\mathfrak{A}|(R,S)$ are obtained by changing signs of some entries of matrices in $\mathfrak{E}_{\le}(R,S)$. However, it is not so clear which subset of the set of those sign changed matrices in $\mathfrak{E}_{\le}(R,S)$ will form the extremal matrices of $|\mathfrak{A}|(R,S)$.

THEOREM 6. *A matrix* A *is in* $|\mathfrak{E}|(R,S)$ *if and only if* $|A|$ *is in* $\mathfrak{E}(R,S)$.

Proof. Suppose that A is in $|\mathfrak{E}|(R,S)$. Then $|A| \in \mathfrak{A}_{\le}(R,S)$, and so $|A|$ can be written as a convex combination of some matrices A_1, A_2, $\ldots$, A_ℓ in $\mathfrak{E}_{\le}(R,S)$. That is, $|A| = \sum_k \alpha_k A_k$ with $\sum_k \alpha_k = 1$ and $\alpha_k \ge 0$. For $k = 1, \ldots, \ell$, let $A_k = (a_{ij}^k)$. Since $A_k \in \mathfrak{E}_{\le}(R,S)$, by Corollary 4, there exists a matrix $B_k = (b_{ij}^k)$ in $\mathfrak{E}(R,S)$ such that a_{ij}^k is either b_{ij}^k or 0 for all (i,j). Define matrices $C_k = (c_{ij}^k)$ and $D_k = (d_{ij}^k)$ as follows: $c_{ij}^k = d_{ij}^k = \operatorname{sign}(a_{ij}) b_{ij}^k$ if $a_{ij} \ne 0$, and $c_{ij}^k = b_{ij}^k$ and $d_{ij}^k = -b_{ij}^k$ otherwise. Then $|C_k| = |D_k| = B_k$ and so $|C_k|, |D_k| \in \mathfrak{E}(R,S)$. Moreover, $A = \sum_k \alpha_k (\frac{1}{2} C_k + \frac{1}{2} D_k)$. Since A is an extremal matrix of $|\mathfrak{A}|(R,S)$ and $C_k, D_k \in |\mathfrak{A}|(R,S)$, A is one of $C_1, \ldots, C_\ell, D_1, \ldots, D_\ell$. This completes the proof of necessity.

Suppose that $|A|$ is in $\mathfrak{E}(R,S)$. Assume that $A = \alpha A' + (1-\alpha) A''$ with $0 < \alpha < 1$ and $A' = (a'_{ij})$, $A'' = (a''_{ij}) \in |\mathfrak{A}|(R,S)$. Then $|A'|, |A''| \in \mathfrak{A}_{\le}(R,S)$ and $|A| \le \alpha |A'| + (1-\alpha)|A''|$. We can claim that $|A| = \alpha|A'| + (1-\alpha)|A''|$. For otherwise, there exists (k,l) such that $|a_{kl}| < \alpha |a'_{kl}| + (1-\alpha)|a''_{kl}|$. Then $\sum_j |a_{kj}| < \alpha \sum_j |a'_{kj}| + (1-\alpha)\sum_j |a''_{kj}| \le r_k$, which contradicts the fact that $|A| \in \mathfrak{A}(R,S)$. By a similar argument, we can obtain that $\sum_j |a'_{ij}| = \sum_j |a''_{ij}| = r_i$ for all i and so $|A'|, |A''| \in \mathfrak{A}(R,S)$. We can conclude that $|A| = \alpha|A'| + (1-\alpha)|A''|$ and $|A'|, |A''| \in \mathfrak{A}(R,S)$. Since $|A|$ is in $\mathfrak{E}(R,S)$, $|A| = |A'| = |A''|$. Thus $a_{ij} = a'_{ij}$ or $-a'_{ij}$ for all i and j. If $a_{ij} = -a'_{ij} \ne 0$ for some (i,j), then $(1+\alpha)a_{ij} = (1-\alpha)a''_{ij}$ since

$A = \alpha A' + (1-\alpha)A''$. However $|A| = |A''|$, and so a_{ij} cannot be $-a'_{ij}$. Thus $A = A'$, and consequently $A = A''$. This completes the proof. □

The following is an immediate corollary of Theorem 6 and Birkhoff theorem, which gives the answer to the questions given at the beginning of this section.

COROLLARY 7. *The set of signed permutation matrices forms the set of extremal matrices of $\Omega_n^{\pm}$, and $\Omega_n^{\pm}$ is $|\mathfrak{A}|(R,S)$ with $R = S = (1,1,\ldots,1)$.*

4. Convex combination

The proof of Theorem 6 uses a result on the extremal matrices of $\mathfrak{A}_{\leq}(R,S)$, hence it does not show us how we can write a matrix in $|\mathfrak{A}|(R,S)$ as a convex combination of extremal matrices. In this section, we give a partial algorithm to write a matrix in $|\mathfrak{A}|(R,S)$ as a convex combination of its extremal matrices. We can have a complete algorithm if we know a way to write a matrix in $\mathfrak{A}(R,S)$ as a convex combination of its extremal matrices. We first prove two lemmas.

LEMMA 8. *Let $R = (r_1,\ldots,r_m)$ and $S = (s_1,\ldots,s_n)$ be positive vectors and c be a positive number. Let $R' = (r_1,\ldots,r_{i-1},r_i-c,r_{i+1},\ldots,r_m)$ and $S' = (s_1,\ldots,s_{j-1},s_j-c,s_{j+1},\ldots,s_n)$. If $A = (a_{ij})$ is a matrix such that $|A| \in \mathfrak{A}(R',S')$, then A can be written as*

$$A = \alpha A' + (1-\alpha)A'', \quad 0 < \alpha < 1$$

with $|A'|, |A''| \in \mathfrak{A}(R,S)$.

Proof. We define $A' = (a'_{kl})$ and $A'' = (a''_{kl})$ as follows:

$$a'_{kl} = \begin{cases} a_{kl} & \text{if } (k,l) \neq (i,j) \\ a_{ij} + \operatorname{sign}(a_{ij})\,c & \text{if } (k,l) = (i,j) \end{cases}$$

and

$$a''_{kl} = \begin{cases} a_{kl} & \text{if } (k,l) \neq (i,j) \\ -a_{ij} - \operatorname{sign}(a_{ij})\,c & \text{if } (k,l) = (i,j). \end{cases}$$

Then $|A'|, |A''| \in \mathfrak{A}(R,S)$. Now compare the (i,j) entry of A and $\alpha A' + (1-\alpha)A''$, then we have

$$a_{ij} = \alpha(a_{ij} + \operatorname{sign}(a_{ij})\,c) - (1-\alpha)(a_{ij} + \operatorname{sign}(a_{ij})\,c).$$

This means that if we define

$$\alpha = \frac{a_{ij} + (a_{ij} + \mathrm{sign}(a_{ij})\, c)}{2(a_{ij} + \mathrm{sign}(a_{ij})\, c)},$$

then $A = \alpha A' + (1-\alpha)A''$. Moreover, it is easy to check that $0 < \alpha < 1$. □

LEMMA 9. *Let $A = (a_{ij})$ be a matrix with $|A| \in \mathfrak{A}(R,S)$. Then A is a convex combination of matrices whose absolute matrices are in $\mathfrak{C}(R,S)$.*

Proof. Since $|A| \in \mathfrak{A}(R,S)$, $|A| = \sum_k \alpha_k A_k$ where $\alpha_k > 0$, $\sum_k \alpha_k = 1$ and $A_k \in \mathfrak{C}(R,S)$. For each k, let $A_k = (a^k_{ij})$ and let $B_k = (b^k_{ij})$ be the matrix defined by

$$b^k_{ij} = \begin{cases} -a^k_{ij} & \text{if } a_{ij} < 0 \\ a^k_{ij} & \text{otherwise} . \end{cases}$$

Then $A = \sum_k \alpha_k B_k$ and $|B_k| = A_k \in \mathfrak{C}(R,S)$. □

THEOREM 10. *Every matrix A in $|\mathfrak{A}|(R,S)$ is a convex combination of matrices A_k, where $|A_k| \in \mathfrak{C}(R,S)$.*

Proof. We use induction on the number of lines of the absolute matrix of a given matrix, which do not have full sum. If every line of $|A|$ has full sum, then $|A| \in \mathfrak{A}(R,S)$ so Lemma 9 finishes the proof. We assume that there is at least one line of $|A|$, which do not have full sum. Let i (respectively, j) be the least integer such that $\sum_j |a_{ij}| < r_i$ (respectively, $\sum_i |a_{ij}| < s_j$). Let $r'_i = \sum_j |a_{ij}|$, $s'_j = \sum_i |a_{ij}|$ and $c = \min(r_i - r'_i, s_j - s'_j)$. Then use Lemma 8 to write A as a convex sum of A', A'' where $|A'|, |A''|$ have the i^{th} row sum and the j^{th} column sum increased by c from those of A. It is clear that the number of lines of $|A'|$ and $|A''|$ which do not have full sum is strictly less than that of $|A|$. Hence by the induction hypothesis, we can write A' and A'' as convex combinations of matrices whose absolute matrices are in $\mathfrak{C}(R,S)$. Thus A is a convex combination of matrices whose absolute matrices are in $\mathfrak{C}(R,S)$. □

REMARK 2. Lemma 8, Lemma 9 and the proof of Theorem 10 give a way to write a matrix in $|\mathfrak{A}|(R,S)$ as a convex combination of matrices in $|\mathfrak{C}|(R,S)$. Note that the proof of Theorem 10 is done inductively and the proof of Lemma 8 gives an explicit way to write a matrix as a convex combination of other matrices. For the proof of Lemma 9, however, we need to borrow a way to write a matrix in $\mathfrak{A}(R,S)$ as a convex combination of matrices in $\mathfrak{E}(R,S)$. As we know, for the case $R = S = (1,1,\ldots,1)$ there are a few ways to write a matrix in $\mathfrak{A}(R,S)$ as a convex combination of matrices in $\mathfrak{E}(R,S)$ (see [11]). But for general R and S, there is no algorithm known. But at least for the case $R = S = (1,1,\ldots,1)$, Lemmas 8-9 and Theorem 10 give an algorithm to write a matrix in $|\mathfrak{A}|(R,S)$ as a convex combination of signed permutation matrices.

5. Examples

The following two examples deal with *sub*polytopes of $\Omega_n^{\pm}$. Even though the polytopes considered in this section are not exactly $|\mathfrak{A}|(R,S)$ that we considered in this article, they are subpolytopes of $|\mathfrak{A}|(R,S)$ generated by some subsets of $|\mathfrak{E}|(R,S)$. We, therefore, can immediately know the set of extremal matrices of those polytopes because of Corollary 7. The generating signed permutation matrices form the set of extremal matrices of given polytopes. Example 5 deals with a polytope generated by a subgroup of a hyperoctahedral group and it serves as a domain for a decision problem of isomorphism of two directed graphs. In Example 5, we consider the polytope of reflection group of another type D_n. Note that for a given optimization problem, (knowing) the set of extremal points of the base polytope of the problem plays an important role.

EXAMPLE 1. Any directed graph g on n vertices, labeled as $[n] = \{1,\ldots,n\}$, can be written as a $\{0,\pm 1\}$-valued vector $v = (v_1,\ldots,v_{\binom{n}{2}})$ in $\mathbb{R}^{\binom{n}{2}}$ in the following way;

1. Give a linear order to the set of 2-subsets of vertices.
2. If the k^{th} 2-subset $\{i,j\}$, $i<j$, of the vertices is
   (a) not an edge of g, then $v_k = 0$,
   (b) a directed edge of g from i to j, then $v_k = 1$,
   (c) a directed edge of g from j to i, then $v_k = -1$.

We let G be the set of directed graphs on n vertices. The group of permutations on n letters, S_n, acts on G by permuting the vertices. The S_n-action on G can be realized as the following matrix representation. We fix a basis of $\mathbb{R}^{\binom{n}{2}}$ as the set of 2-subsets of $\{1,2,\ldots,n\}$ and then, define a map ρ from S_n to the group of $\binom{n}{2} \times \binom{n}{2}$ nonsingular matrices as follows: For $\sigma \in S_n$ and the k^{th} 2-subset $\{i,j\}$, $i<j$, of $[n]$,

$$\rho(\sigma)(\{i,j\}) = (-1)^{\mathrm{inv}(\sigma,\{i,j\})}\{\sigma(i),\sigma(j)\}, \text{ where}$$

$$\mathrm{inv}(\sigma,\{i,j\}) = \begin{cases} 0 & \text{if } \sigma(i)<\sigma(j), \\ 1 & \text{otherwise.} \end{cases}$$

Then it is easy to check that ρ is a group homomorphism (actually it is a faithful representation of S_n), and the image of ρ is a subgroup of hyperoctahedral group. Hence, if we let $\mathcal{P}$ be the convex polytope generated by $\{\rho(\sigma)\,|\,\sigma \in S_n\}$, in $\mathbb{R}^{\binom{n}{2}\times\binom{n}{2}}$, then $\mathcal{P}$ is a subpolytope of the polytope $\Omega^{\pm}_{\binom{n}{2}}$. Now, by Theorem 6 or Corollary 7, the set of extremal matrices of $\mathcal{P}$ is exactly the set of images of ρ.

The following argument shows how $\mathcal{P}$ can be used in a decision problem:

Let $g, h \in \mathbb{R}^{\binom{n}{2}}$ be two directed graphs in G, having q edges. Letting $c = h \otimes g$, the decision problem

$$\max\{\langle c, x\rangle \;:\; x \in \mathcal{P}\} \geq q$$

on $\mathcal{P}$, where $\langle c, x\rangle$ is the usual inner product in the Euclidean space, is exactly the decision problem whether g and h are isomorphic as directed graphs.

EXAMPLE 2. A reflection group of type D_n, consisting of signed permutations with even number of sign changes, forms a subgroup of index 2 of a group of signed permutations(reflection group of type B_n). We investigate the convex polytope generated by those type D_n signed permutations, which will form a subpolytope of $\Omega^{\pm}_n$. This polytope may be thought in the context of [5]. One might expect $|\mathfrak{A}|(R,S)$ of type D_n for general R and S. It, however, is not so clear what should be a definition of $|\mathfrak{A}|(R,S)$ of type D_n for general R, S, and we only consider the case $R = S = (1,1,\ldots,1)$.

The polytope we consider is defined as follows;

$$\mathcal{P}_n = \mathrm{Conv}(\{A\,|\,A \text{ is an } n\times n \text{ signed permutation matrix with even number of sign changes}\}) \subset \mathbb{R}^{n^2}.$$

By Corollary 7, we know that the set of extremal matrices of $\mathcal{P}_n$ is exactly the generating set(set of signed permutations with even number of sign changes). Hence we have Birkhoff theorem of type D_n that the set of signed permutation matrices with even number of sign changes forms the set of extremal matrices of $\mathcal{P}_n$.

It is not so difficult to show that n^2 many standard basis matrices(the ones only one 1 and 0 elsewhere) and the zero matrix are all contained in $\mathcal{P}_n$ when $n > 2$, hence to show that the dimension of $\mathcal{P}_n$ is n^2 when $n > 2$. The dimension of $\mathcal{P}_2$ is 2.

6. Remarks

1. V. Klee and C. Witzgall [9] characterized and counted the facets as well as vertices of $\mathfrak{A}(R, S)$. We think that it might be an interesting problem to characterize the facets of $|\mathfrak{A}|(R, S)$.
2. On the polytope considered in Example 5, we do not have an answer to the question on the characterization of $\mathcal{P}_n$ that explains what kind of matrices are in $\mathcal{P}_n$.

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Soojin Cho
Department of Applied Mathematics
Sejong University
Seoul 143-747, Korea
E-mail: sjcho@sejong.ac.kr

Yunsun Nam
Computational Sci. & Eng. Lab.
Technical Consulting & Service Center
Samsung Advanced Institute of Technology
Yongin 449-900, Korea
E-mail: snam@hananet.net