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ON SUFFICIENT OPTIMALITY THEOREMS FOR NONSMOOTH MULTIOBJECTIVE OPTIMIZATION PROBLEMS

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ABSTRACT. We consider a nonsmooth multiobjective optimization problem (PE) involving locally Lipschitz functions and define generalized invexity for locally Lipschitz functions. Using Fritz John type optimality conditions, we establish Fritz John type sufficient optimality theorems for (PE) under generalized invexity.

1. Introduction and Preliminaries

Multiobjective optimization problems consist of conflicting objective functions and constraint sets, and the problems are to optimize the objective functions over the constraint sets under some concepts of solution, for example, properly efficient solutions, efficient solutions and weakly efficient solutions. There are two types of necessary optimality conditions. Those are Fritz John type necessary optimality conditions [13] and Kuhn-Tucker type necessary optimality conditions [3, 4, 8] for multiobjective optimization problems. The Kuhn-Tucker type optimality conditions are sufficient ones for feasible points to be (weakly) efficient under generalized convexity or invexity assumptions. Most of authors [5, 6, 7, 9, 11, 12, 14] have tried to obtain the Kuhn-Tucker type sufficient optimality theorems for multiobjective optimization problems.

In this paper, we consider a nonsmooth multiobjective optimization problem (PE) involving locally Lipschitz functions and define generalized invexity for locally Lipschitz functions. Using Fritz John type optimality

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conditions, we establish Fritz John type sufficient optimality theorems for (PE) under generalized invexity. Also, we extend Theorem 4 of Zhao [15] for a single optimization problem to (PE), and obtain a new Kuhn-Tucker type sufficient optimality theorem for (PE).

Let $g : X \to R$ be a function. Let X be a real Banach space and let X^* be the topological dual of X. Then g is said to be locally Lipschitz if $\forall x \in X$, $\exists$ a neighborhood $N(x)$ of x and $K_x > 0$ such that $\forall y, z \in N(x)$,

$$|g(y) - g(z)| \leq K_x \|y - z\|.$$

Consider the following nonsmooth multiobjective optimization problem with equality and inequality constraints:

(PE) Minimize $f(x)$
subject to $g(x) \leqq 0$, $h(x) = 0$,

where $f := (f_1, \cdots, f_p) : X \to R^p$, $g := (g_1, \cdots, g_m) : X \to R^m$ and $h := (h_1, \cdots, h_k) : X \to R^k$ are locally Lipschitz functions.

Optimization of (PE) is finding (weakly) efficient solutions defined as follows;

DEFINITION 1.1. (1) A point $\bar{x} \in X$ is said to be an efficient solution of (PE) if there exists no other feasible point $x \in X$ such that $f(x) \leqq f(\bar{x})$ and $f(x) \neq f(\bar{x})$.
(2) A point $\bar{x} \in X$ is said to be a weakly efficient solution of (PE) if there exists no other feasible point $x \in X$ such that $f(x) < f(\bar{x})$.

DEFINITION 1.2. ([1]) Let $g : X \to R$ be a locally Lipschitz function.

(1) The generalized directional derivative of the function g at x in the direction d is denoted by $g^o(x; d)$:

$$g^o(x; d) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{1}{t}[g(y + td) - g(y)].$$

(2) The Clarke generalized subgradient of the function g at x is denoted by

$$\partial^c g(x) = \{\xi \in X^* : \ g^o(x; d) \geq \langle \xi, \ d \rangle \ \forall d \in X\}.$$

DEFINITION 1.3. ([1]) A locally Lipschitz function $g : X \to R$ is said to be regular at x if

(i) for all $d \in X$, the usual one-sided directional derivative $g'(x; d)$ exists; and

(ii) for all $d \in X$, $g'(x; d) = g^0(x; d)$.

DEFINITION 1.4. ([2, 7]) Let $g : X \to R$ be a locally Lipschitz function and $\bar{x} \in X$.

(1) g is η-pseudo-invex function at $\bar{x}$ if and only if $\exists$ a function $\eta : X \times X \to X$ such that

$$\forall y \in X,\ g^o(\bar{x}; \eta(y, \bar{x})) \geqq 0 \text{ implies } g(y) \geqq g(\bar{x}),$$

equivalently,

$$\forall y \in X,\ \ \forall \xi \in \partial^c g(\bar{x}),\ \ g(y) < g(\bar{x}) \text{ implies } \langle \xi,\ \ \eta(y, \bar{x}) \rangle < 0.$$

(2) g is η-quasi-invex function at $\bar{x}$ if and only if $\exists$ a function $\eta : X \times X \to X$ such that

$$\forall y \in X,\ g(y) \leqq g(\bar{x}) \text{ implies } g^o(\bar{x}; \eta(y, \bar{x})) \leqq 0,$$

equivalently,

$$\forall y \in X,\ \ \forall \xi \in \partial^c g(\bar{x}),\ \ g(y) \leqq g(\bar{x}) \text{ implies } \langle \xi,\ \ \eta(y, \bar{x}) \rangle \leqq 0.$$

(3) g is strictly η-pseudo-invex function at $\bar{x}$ if and only if $\exists$ a function $\eta : X \times X \to X$ such that

$$\forall y \in X \text{ with } \bar{x} \neq y,\ g^o(\bar{x}; \eta(y, \bar{x})) \geqq 0 \text{ implies } g(y) > g(\bar{x}),$$

equivalently, $\forall \xi \in \partial^c g(\bar{x}),\ \ g(y) \leqq g(\bar{x})$ implies $\langle \xi,\ \ \eta(y, \bar{x}) \rangle < 0$.

By Theorem 6.1.1 in [1], we can obtain the following necessary Fritz John type optimality theorem of (PE);

THEOREM 1.1. *If $\bar{x}$ is a weakly efficient solution of (PE), then there exist $\bar{\mu} \in R^p$, $\bar{y} \in R^m$ and $\bar{z} \in R^k$ such that*

$$0 \in \sum_{i=1}^{p} \bar{\mu}_i \partial^c f_i(\bar{x}) + \sum_{j=1}^{m} \bar{y}_j \partial^c g_j(\bar{x}) + \sum_{l=1}^{k} \bar{z}_l \partial^c h_l(\bar{x}),$$

that is, there exist $a_i \in \partial^c f_i(\bar{x})$, $i = 1, \cdots, p$, $b_j \in \partial^c g_j(\bar{x})$, $j = 1, \cdots, m$ and $c_l \in \partial^c h_l(\bar{x})$, $l = 1, \cdots, k$ such that

$$0 = \sum_{i=1}^{p} \bar{\mu}_i a_i + \sum_{j=1}^{m} \bar{y}_j b_j + \sum_{l=1}^{k} \bar{z}_l c_l, \tag{1.1}$$

and

$$\bar{y}_j g_j(\bar{x}) = 0, \ j = 1, \cdots, m, \tag{1.2}$$

$$(\bar{\mu}_1, \cdots, \bar{\mu}_p, \bar{y}_1, \cdots, \bar{y}_m) \geqq 0, \tag{1.3}$$

$$(\bar{\mu}_1, \cdots, \bar{\mu}_p, \bar{y}_1, \cdots, \bar{y}_m, \bar{z}_1, \cdots, \bar{z}_k) \neq 0. \tag{1.4}$$

REMARK 1.1. In Theorem 1.1, if there exists $x^* \in X$ such that $\langle b_j, \ x^* \rangle < 0$, $j \in I(\bar{x}) := \{i : g_i(\bar{x}) = 0\}$, $\langle c_l, \ x^* \rangle = 0$, $l = 1, \cdots, k$ and $c_1, c_2, \cdots, c_k$ are linearly independent, then we have

$$(\bar{\mu}_1, \cdots, \bar{\mu}_p) \neq 0. \tag{1.5}$$

2. Sufficiency of the Fritz John type conditions

When $p = 1$ and all functions are continuously differentiable, the optimality conditions (1.1)-(1.4) reduce to the Fritz John ones for a single optimization problem found in [10]. So, we call the optimality conditions (1.1)-(1.4) the Fritz John type optimality ones for (PE). When $p = 1$ and all functions are continuously differentiable, the optimality conditions (1.1)-(1.3) and (1.5) become the Kuhn-Tucker ones for a single optimization problem found in [10]. So, we call the optimality conditions (1.1)-(1.3) and (1.5) the Kuhn-Tucker type optimality ones for (PE).

We now prove the following Fritz John type sufficient optimality theorem of (PE):

THEOREM 2.1. *Let $\bar{\mu} \in R^p$, $\bar{y} \in R^m$, $\bar{z} \in R^k$ and $\bar{x} \in X$, along with $\bar{\mu}$, $\bar{y}$ and $\bar{z}$ satisfy the following conditions ;*

$$0 \in \sum_{i=1}^{p} \bar{\mu}_i \partial^c f_i(\bar{x}) + \sum_{j=1}^{m} \bar{y}_j \partial^c g_j(\bar{x}) + \sum_{l=1}^{k} \bar{z}_l \partial^c h_l(\bar{x}), \tag{2.1}$$

$$\bar{y}^t g(\bar{x}) = 0, \tag{2.2}$$

$$g(\bar{x}) \leqq 0, \tag{2.3}$$

$$h(\bar{x}) = 0, \tag{2.4}$$

$$(\bar{\mu}, \bar{y}) \geqq 0, \quad (\bar{\mu}, \bar{y}, \bar{z}) \neq 0. \tag{2.5}$$

Assume that

(a) *f is η-quasi-invex at $\bar{x}$ and $\bar{y}^t g + \bar{z}^t h$ is strictly η-pseudo-invex at $\bar{x}$, and g and h are regular functions; or*

(b) *$\bar{\mu}^t f$ is η-quasi-invex at $\bar{x}$ and $\bar{y}^t g + \bar{z}^t h$ is strictly η-pseudo-invex at $\bar{x}$, and g and h are regular functions; or*

(c) *$\bar{\mu}^t f + \bar{y}^t g + \bar{z}^t h$ is strictly η-pseudo-invex at $\bar{x}$, and f, g and h are regular functions.*

Then $\bar{x}$ is an (weakly) efficient solution of (PE).

PROOF. (a) Suppose that $\bar{x}$ is not an efficient solution of (PE). Then there exists $x^* \in X$ such that $f(x^*) \leqq f(\bar{x})$, $f(x^*) \neq f(\bar{x})$, $g(x^*) \leqq 0$ and $h(x^*) = 0$. By the η-quasi-invexity of f at $\bar{x}$, we have $\langle \xi_i, \eta(x^*, \bar{x}) \rangle \leqq 0$ for any $\xi_i \in \partial^c f_i(\bar{x})$, $i = 1, \cdots, p$. Thus $\bar{\mu}_i \geqq 0$ implies that

$$\left\langle \sum_{i=1}^{p} \bar{\mu}_i \xi_i, \ \eta(x^*, \bar{x}) \right\rangle \leqq 0.$$

Therefore, from (2.1), we have

$$\left\langle \sum_{j=1}^{m} \bar{y}_j \tilde{\xi}_j + \sum_{l=1}^{k} \bar{z}_l \hat{\xi}_l, \ \eta(x^*, \bar{x}) \right\rangle \geqq 0$$

for some $\tilde{\xi}_j \in \partial^c g_j(\bar{x})$ and $\hat{\xi}_l \in \partial^c h_l(\bar{x})$. By the regularity of g_j and h_l at $\bar{x}$, $\langle \xi, \eta(x^*, \bar{x}) \rangle \geqq 0$ for some $\xi \in \partial^c(\bar{y}^t g + \bar{z}^t h)$ (see Corollary 3 at the page 40-th page in [1]) and hence $(\bar{y}^t g + \bar{z}^t h)^o(\bar{x}; \eta(x^*, \bar{x})) \geqq 0$. Thus, by the strict η-pseudo-invexity of $\bar{y}^t g + \bar{z}^t h$ at $\bar{x}$, we have

$$\bar{y}^t g(x^*) + \bar{z}^t h(x^*) > \bar{y}^t g(\bar{x}) + \bar{z}^t h(\bar{x}).$$

Since $h(x^*) = 0$, from (2.4), we have $\bar{y}^t g(x^*) > \bar{y}^t g(\bar{x})$. From (2.2), we have

$$\bar{y}^t g(x^*) > 0,$$

which contradicts the fact that $\bar{y}^t g(x^*) \leqq 0$. Hence the result holds.

(b) Suppose that $\bar{x}$ is not an efficient solution of (PE). Then there exists $x^* \in X$ such that $f(x^*) \leqq f(\bar{x})$, $f(x^*) \neq f(\bar{x})$, $g(x^*) \leqq 0$ and $h(x^*) = 0$. Since $\bar{\mu} \geqq 0$ and $f(x^*) \leqq f(\bar{x})$, we have $\bar{\mu}^t f(x^*) \leqq \bar{\mu}^t f(\bar{x})$. Thus, by the η-quasi-invexity of $\bar{\mu}^t f$ at $\bar{x}$, we have

$$\Big\langle \sum_{i=1}^{p} \bar{\mu}_i \xi_i,\ \ \eta(x^*, \bar{x}) \Big\rangle \leqq 0,$$

for any $\xi_i \in \partial^c f_i(\bar{x})$, $i = 1, \cdots, p$. By the same method as the proof of the part (a), we can prove the part (b).

(c) Suppose that $\bar{x}$ is not an efficient solution of (PE). Then there exists $x^* \in X$ such that $f(x^*) \leqq f(\bar{x})$, $f(x^*) \neq f(\bar{x})$, $g(x^*) \leqq 0$ and $h(x^*) = 0$. Let $I(\bar{x}) := \{i : g_i(\bar{x}) = 0\}$. Then it follows from (2.2), (2.3) and (2.5) that $\bar{y}_i = 0$ $\forall i \notin I(\bar{x})$. Since $g_j(x^*) \leqq g_j(\bar{x})$ $\forall j \in I(\bar{x})$ and $h_l(x^*) = h_l(\bar{x})$, we have

$$\sum_{i=1}^{p} \bar{\mu}_i f_i(x^*) + \sum_{j \in I(\bar{x})} \bar{y}_j g_j(x^*) + \sum_{l=1}^{k} \bar{z}_l h_l(x^*)$$
$$\leqq \sum_{i=1}^{p} \bar{\mu}_i f_i(\bar{x}) + \sum_{j \in I(\bar{x})} \bar{y}_j g_j(\bar{x}) + \sum_{l=1}^{k} \bar{z}_l h_l(\bar{x}).$$

By the regularity of $\sum_{i=1}^{p} \bar{\mu}_i f_i + \sum_{j \in I(\bar{x})} \bar{y}_j g_j + \sum_{l=1}^{k} \bar{z}_l h_l$ at $\bar{x}$,

$$\partial^c \Big(\sum_{i=1}^{p} \bar{\mu}_i f_i + \sum_{j \in I(\bar{x})} \bar{y}_j g_j + \sum_{l=1}^{k} \bar{z}_l h_l \Big)(\bar{x})$$
$$= \sum_{i=1}^{p} \bar{\mu}_i \partial^c f_i(\bar{x}) + \sum_{j \in I(\bar{x})} \bar{y}_j \partial^c g_j(\bar{x}) + \sum_{l=1}^{k} \bar{z}_l \partial^c h_l(\bar{x})$$

(see Corollary 3 at the 40-th page in [1]). So, by the strict η-pseudo-invexity of $\sum_{i=1}^{p} \bar{\mu}_i f_i + \sum_{j \in I(\bar{x})} \bar{y}_j g_j + \sum_{l=1}^{k} \bar{z}_l h_l$, we have $\forall \xi \in \partial^c (\sum_{i=1}^{p} \bar{\mu}_i f_i + \sum_{j \in I(\bar{x})} \bar{y}_j g_j + \sum_{l=1}^{k} \bar{z}_l h_l)(\bar{x})$,

$$\langle \xi, \eta(x^*, \bar{x}) \rangle < 0.$$

This contradicts (2.1). □

THEOREM 2.2. *Let $\bar{\mu} \in R^p$, $\bar{y} \in R^m$, $\bar{z} \in R^k$ and $\bar{x} \in X$, along with $\bar{\mu}$, $\bar{y}$ and $\bar{z}$ satisfy the (2.1)–(2.5) conditions ;*
Assume that

(a) *f is η-quasi-invex at $\bar{x}$, $\bar{y}^t g$ is strictly η-pseudo-invex at $\bar{x}$ and $\bar{z}^t h$ is η-quasi-invex at $\bar{x}$; or*

(b) *$\bar{\mu}^t f$ is η-quasi-invex at $\bar{x}$, $\bar{y}^t g$ is strictly η-pseudo-invex at $\bar{x}$ and $\bar{z}^t h$ is η-quasi-invex at $\bar{x}$.*

Then $\bar{x}$ is an (weakly) efficient solution of (PE).

PROOF. (a) Suppose that $\bar{x}$ is not an efficient solution of (PE). Then there exists $x^* \in X$ such that $f(x^*) \leqq f(\bar{x})$, $f(x^*) \neq f(\bar{x})$, $g(x^*) \leqq 0$ and $h(x^*) = 0$. Then by the η-quasi-invexity of f at $\bar{x}$, we have $\langle \xi_i, \eta(x^*, \bar{x}) \rangle \leqq 0$ for any $\xi \in \partial^c f_i(\bar{x})$, $i = 1, \cdots, p$. Since $\bar{\mu} \geqq 0$, we have

$$\left\langle \sum_{i=1}^{p} \bar{\mu}_i \xi_i,\ \eta(x^*, \bar{x}) \right\rangle \leqq 0, \tag{2.6}$$

for any $\xi \in \partial^c f_i(\bar{x})$, $i = 1, \cdots, p$. Since $\bar{z}^t h(x^*) = \bar{z}^t h(\bar{x})$, and $\bar{z}^t h$ is η-quasi-invex at $\bar{x}$, we have

$$\left\langle \sum_{l=1}^{k} \bar{z}_l \hat{\xi}_l,\ \eta(x^*, \bar{x}) \right\rangle \leqq 0, \tag{2.7}$$

for any $\hat{\xi}_l \in \partial^c h_l(\bar{x})$, $l = 1, \cdots, k$. From (2.1), we have

$$\left\langle \sum_{j=1}^{m} \bar{y}_j \tilde{\xi}_j,\ \eta(x^*, \bar{x}) \right\rangle \geqq 0,$$

for some $\tilde{\xi}_j \in \partial^c g_j(\bar{x})$, $j = 1, \cdots, m$. Thus, by the strict η-pseudo-invexity of $\bar{y}^t g$ at $\bar{x}$, we have $\bar{y}^t g(x^*) > \bar{y}^t g(\bar{x})$. From (2.2), we have

$$\bar{y}^t g(x^*) > 0,$$

which contradicts the fact that $\bar{y}^t g(x^*) \leqq 0$. Hence the result holds.

(b) Suppose that $\bar{x}$ is not an efficient solution of (PE). Then there exists $x^* \in X$ such that $f(x^*) \leqq f(\bar{x})$, $f(x^*) \neq f(\bar{x})$, $g(x^*) \leqq 0$ and

$h(x^*) = 0$. Since $\bar{\mu} \geqq 0$ and $f(x^*) \leqq f(\bar{x})$, we have $\bar{\mu}^t f(x^*) \leqq \bar{\mu}^t f(\bar{x})$. Thus, by the η-quasi-invexity of $\bar{\mu}^t f$ at $\bar{x}$, we have

$$\left\langle \sum_{i=1}^{p} \bar{\mu}_i \xi_i, \ \eta(x^*, \bar{x}) \right\rangle \leqq 0,$$

for any $\xi_i \in \partial^c f_i(\bar{x}), \ i = 1, \cdots, p$. By the same method as the proof of the part (a), we can prove the part (b). □

3. Sufficiency of the Kuhn-Tucker type conditions

The following are Kuhn-Tucker type necessary conditions for a weakly efficient solution $\bar{x}$ of (PE):
(KT) there exist $\bar{\mu} \in R^p, \ \bar{y} \in R^m, \ \bar{z} \in R^k, \ a_i \in \partial^c f_i(\bar{x}), \ i = 1, \cdots, p, \ b_j \in \partial^c g_j(\bar{x}), \ j = 1, \cdots, m, \ c_l \in \partial^c h_l(\bar{x}), \ l = 1, \cdots, k$, such that

$$\begin{aligned} & 0 = \sum_{i=1}^{p} \bar{\mu}_i a_i + \sum_{j=1}^{m} \bar{y}_j b_j + \sum_{l=1}^{k} \bar{z}_l c_l, \\ & y_j g_j(\bar{x}) = 0, \ j = 1, \cdots, m, \\ & (\bar{\mu}_1, \cdots, \bar{\mu}_p, \bar{y}_1, \cdots, \bar{y}_m) \geqq 0, \\ & (\bar{\mu}_1, \cdots, \bar{\mu}_p) \neq 0. \end{aligned}$$

Now we give a condition $(*)$ with $a_i, \ b_j, \ c_l$;

$(*)$ there exists a vector-valued function $\eta : X \to X$ such that for any feasible point of (PE),

$$\begin{aligned} & f_i(x) - f_i(\bar{x}) \geqq \langle a_i, \ \eta(x) \rangle, \ i = 1, \cdots, p \\ & - g_j(\bar{x}) \geqq \langle b_j, \ \eta(x) \rangle, \ j = 1, \cdots, m, \\ & 0 \ \geqq \left\langle \sum_{l=1}^{k} \bar{z}_l c_l, \eta(x) \right\rangle. \end{aligned}$$

Now we extend Theorem 4 of Zhao [15] for a single optimization problem to (PE), and obtain a new Kuhn-Tucker type sufficient optimality theorem of (PE) as follows;

THEOREM 3.1. *Let the conditions (KT) and* $(*)$ *hold at a feasible point* $\bar{x}$ *of (PE). Then* $\bar{x}$ *is a weakly efficient solution of (PE).*

PROOF. Suppose that $\bar{x}$ is not a weakly efficient solution of (PE). Then there exists $x^* \in X$ such that $f_i(x^*) < f_i(\bar{x}),\ i = 1, \cdots, p$. Then we have

$$\begin{aligned}
0 &> \sum_{i=1}^{p} \bar{\mu}_i (f_i(x^*) - f_i(\bar{x})) \\
&\geqq \Big\langle \sum_{i=1}^{p} \bar{\mu}_i a_i,\ \eta(x) \Big\rangle \qquad \text{(by condition } (*)) \\
&= -\Big\langle \sum_{j=1}^{p} \bar{y}_j b_j,\ \eta(x) \Big\rangle - \Big\langle \sum_{l=1}^{k} \bar{z}_l c_l,\ \eta(x) \Big\rangle \qquad \text{(by } (KT)) \\
&\geqq \sum_{j=1}^{m} \bar{y}_j g_j(\bar{x}) \qquad \text{(by condition } (*)) \\
&= 0 \qquad \text{(by } (KT)).
\end{aligned}$$

This is a contradiction. So, $\bar{x}$ is a weakly efficient solution of (PE). □

The following example shows that even though f_i and g_j are differentiable (and hence locally Lipschitzian), the converse of Theorem 3.1 does not hold in general.

EXAMPLE 3.1. Consider the following differentiable multiobjective problem:

$$(P) \qquad \text{Minimize} \quad (x, -x^2)$$
$$\text{subject to} \quad x \in Y := \{x \in R \mid -x + \frac{1}{2} \leqq 0\}.$$

Then for any $x \in Y$, x is a weakly efficient solution of (P). However, the condition $(*)$ does not hold for $\bar{x} = \frac{1}{2}$. Indeed, suppose that there exists a function $\eta : R \times R \to R$ such that for any $x \in Y$

$$f_1(x) - f_1(\bar{x}) \geqq f_1'(\bar{x})\eta(x)$$
$$f_2(x) - f_2(\bar{x}) \geqq f_2'(\bar{x})\eta(x).$$

Then for any $x \in Y$,

$$x - \frac{1}{2} \geqq \eta(x)$$
$$-x^2 + \frac{1}{4} \geqq -\eta(x).$$

Hence $x^2 - \frac{1}{4} \leqq \eta(x) \leqq x - \frac{1}{2}$ for any $x \in Y$. Thus $x^2 - \frac{1}{4} \leqq x - \frac{1}{2}$ for any $x \in Y$. This is impossible. Hence the condition $(*)$ does not hold at $\bar{x} = \frac{1}{2}$.

References

[1] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley-Interscience, New York, 1983.

[2] G. Giorgi and A. Guerraggio, *Various types of nonsmooth invexity*, J. Inform. Optim. Sci. **17** (1996), 137–150.

[3] V. Jeyakumar, *On subgradient duality with strong and weak convex functions*, J. Austal. Math. Soc. **40** (1986), 143–152.

[4] V. Jeyakumar and B. Mond, *On generalized convex mathematical programming*, J. Austal. Math. Soc. Ser. B **34** (1992), 43–53.

[5] D. S. Kim and G. M. Lee, *Optimality conditions and duality theorems for multiobjective invex programs*, J. Inform. Optim. Sci. **12** (1991), no. 2, 235–242.

[6] H. Kuk, G. M. Lee, and D. S. Kim, *Nonsmooth multiobjective programs with $V-\rho-$invexity*, Indian J. Pure Appl. Math. **29** (1998), no. 4, 405–412.

[7] G. M. Lee, *Optimality conditions in multiobjective optimization problems*, J. Inform. Optim. Sci. **13** (1993), no. 4, 39–41.

[8] ———, *Nonsmooth invexity in multiobjective programming*, J. Inform. Optim. Sci. **15** (1994), no. 1, 127–136.

[9] A. A. K. Majumdar, *Optimality conditions in differentiable multiobjective programming*, J. Optim. Th. Appl. **92** (1997), no. 2, 419–427.

[10] O. L. Mangasarian, *Nonlinear Programming*, McGraw-Hill, New York, 1969.

[11] C. Singh, *Optimality conditions in multiobjective differentiable programming*, J. Optim. Th. Appl. **53** (1987), 115–123.

[12] S. K. Suneja and M. K. Srivastava, *Optimality and duality in nondifferentiable multiobjective optimization involving d-type I and related functions*, J. Math. Anal. Appl. **206** (1997), 465–479.

[13] K. Tamura, *Multiobjective programming in Applications of Mathematical Programming: Theory,* Edited by M. Iri and M. Konno, Sankyo Pub. Co., 1982 (in Japanese).

[14] T. Weir and B. Mond, *Generalized convexity and duality in multiple objective programming*, Bull. Austral. Math. Soc. **39** (1989), 287–299.

[15] F. Zhao, *On sufficiency of the Kuhn-Tucker conditions in nondifferentiable programming*, Bull. Austral. Math. Soc. **46** (1992), 385–389.

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