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AN INTERARRIVAL HYPEREXPONENTIAL MACHINE INTERFERENCE MODEL: $H_r/M/c/k/N$ WITH BALKING AND RENEGING

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ABSTRACT. The aim of this paper is to derive the analytical solution of the queue: $H_r/M/c/k/N$ for machine interference with balking and reneging, and FIFO (first in, first out) service discipline.

1. Introduction

Kleinrock [5] studied the queue: $M/M/c/k/N$ for machine interference without balking and reneging, Gross and Harris [3] discussed the system: $M/M/c/m/m$ with spares only and Medhi [6] treated the system: $M/M/c/m/m$ without balking and reneging. Gupta [4] treated numerically the interarrival hyperexponential queue: $H_r/M/1/m$ with state dependent arrival and service rates, and Shawky [7] studied the system: $M/M/c/k/N$ with balking, reneging and spares. This paper aims to derive the analytical solution of the queue: $H_r/M/c/k/N$ for machine interference model with balking and reneging.

2. Description of the system

As in Goyal [2], the arrival channel consists of r independent branches. A unit arriving for service joins the ith branch with a fraction σ_i of the time on the average, so that $\sum_{i=1}^{r} \sigma_i = 1$. Only one unit can occupy any one of the branches at a time and if a unit is present in any one of the branches, the arrival channel is busy and no other unit can enter any other branch. The unit in the ith branch joins the system (queue

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or service) with rate λ_i per unit time. We assume that we have a finite source (population) of N customers, c servers (repairmen) are available, that the service times are identical exponential random variables with rate μ and also the system has finite storage room such that the total number of customers (machines) in the system is no more than k. The queue discipline is assumed to be first come, first served.

Let β be the probability that a unit joins the queue when the system size is greater than or equal c.

It is also assumed that the units in the queue may renege according to an exponential distribution, $f(t) = \alpha e^{-\alpha t}$, $t \geq 0$, with parameter α. The probability of reneging in a short period of time Δt is given by $r_n = (n-c)\alpha\Delta\, t$ for $c < n \leq k$ and $r_n = 0$ for $0 \leq n \leq c$.

3. The steady–state analytical solution

Let $P_{n,i}$ denote the equilibrium probability that there are n units in the system and the unit in the arrival channel is in the ith branch, where $n = 0, 1, 2, ..., k; i = 1, 2, ..., r$.

The steady-state probability difference equations are

$$N\lambda_i P_{0,i} = \mu P_{1,i}, \qquad i = 1, 2, ..., r, \tag{1}$$

$$\begin{aligned}&[(N-n)\lambda_i + n\mu]P_{n,i}\\ =\ & \sigma_i(N-n+1)\sum_{s=1}^{r}\lambda_s P_{n-1,s} + (n+1)\mu P_{n+1,i},\ n = 1, 2, ..., c-1,\end{aligned} \tag{2}$$

$$\begin{aligned}&[(N-c)\beta\lambda_i + c\mu]P_{c,i}\\ =\ & \sigma_i(N-c+1)\sum_{s=1}^{r}\lambda_s P_{c-1,s} + (c\mu+\alpha)P_{c+1,i},\ \ n = c,\end{aligned} \tag{3}$$

$$\begin{aligned}&[(N-n)\beta\lambda_i + c\mu + (n-c)\alpha]P_{n,i}\\ =\ & \sigma_i(N-n+1)\sum_{s=1}^{r}\beta\lambda_s P_{n-1,s}\\ &+\{c\mu + (n-c+1)\alpha\}P_{n+1,i},\ \ n = c+1, ..., k-1,\end{aligned} \tag{4}$$

$$
\begin{aligned}
& [(N-k)\beta\lambda_i + c\mu + (k-c)\alpha]P_{k,i} \\
(5) \qquad = \quad & \sigma_i(N-k+1)\sum_{s=1}^{r}\beta\lambda_s P_{k-1,s} \\
& +\sigma_i(N-k)\sum_{s=1}^{r}\beta\lambda_s P_{k,s}, \quad n=k.
\end{aligned}
$$

Summing up equations (1)–(4) over i, and adding the results obtained for $n = 1, 2, ..., k-1$, we get

$$(6) \qquad (N-n)\sum_{i=1}^{r}\rho_i P_{n,i} = (n+1)\sum_{i=1}^{r}P_{n+1,i}, \quad n = 0,1,2,...,c-1,$$

$$
\begin{aligned}
(7) \qquad & (N-n)\sum_{i=1}^{r}\beta\rho_i P_{n,i} \\
= \quad & [c+(n+1-c)\delta]\sum_{i=1}^{r}P_{n+1,i}, \quad n = c, c+1, ..., k-1,
\end{aligned}
$$

where $\rho_i = \frac{\lambda_i}{\mu}, \delta = \frac{\alpha}{\mu}$.
From (2) and (6), we have

$$B(n,i)P_{n,i} - n\sigma_i\sum_{s=1}^{r}P_{n,s} = (n+1)P_{n+1,i}, \quad n = 1,2,...,c-1,$$

where

$$B(n,i) = (N-n)\rho_i + n,$$

which can be written in the matrix form as

$$(8) \qquad \mathbf{BP} = (n+1)\mathbf{Q}, \ n = 1,2,...,c-1,$$

where

$$\mathbf{B} = [b_{ij}],$$

such that

$$
\begin{aligned}
& b_{ij} = -n\sigma_i, \ i \neq j \\
& b_{ii} = \mathbf{B}(n,i) - n\sigma_i, \\
& \mathbf{P}^T = [P_{n,1}, P_{n,2}, ..., P_{n,r}],
\end{aligned}
$$

and

$$\mathbf{Q}^T = [P_{n+1,1}, P_{n+1,2}, ..., P_{n+1,r}],$$

where T denotes the transpose of a matrix. Now, $\mathbf{B}^{-1}$ is given by

$$\mathbf{B}^{-1} = [b^{\backslash}_{ij}],$$

where

$$b^{\backslash}_{ij} = \frac{n\sigma_i}{B(n,i)B(n,j)D_n}, \; i \neq j$$

$$b^{\backslash}_{ii} = \frac{1}{B(n,i)} + \frac{n\sigma_i}{B^2(n,i)D_n},$$

$$D_n = 1 - n\sum_{i=1}^{r} \frac{\sigma_i}{B(n,i)}, \; n = 1, 2, ..., c-1, \; D_0 = 1.$$

Using this value of $\mathbf{B}^{-1}$ in (8), we get

$$\text{(9)} \qquad P_{n,i} = \frac{n+1}{B(n,i)} \left\{ P_{n+1,i} + \sum_{j=1}^{r} \frac{n\sigma_i P_{n+1,j}}{B(n,j)D_n} \right\}, \; n = 0, 1, ..., c-1.$$

From (4) and (7) we obtain

$$[(N-n)\beta\rho_i + c + (n-c)\delta]P_{n,i} - \sigma_i\,[c + (n-c)\delta]\sum_{s=1}^{r} P_{n,s}$$
$$= \; [c + (n-c+1)\delta]\,P_{n+1,i}, \quad n = c, c+1, ..., k-1,$$

which can be written in the form

$$\text{(10)} \qquad \mathbf{DP} = \{c + (n-c+1)\delta\}\mathbf{Q}, \; n = c, c+1, ..., k-1,$$

where

$$\mathbf{D} = [d_{ij}],$$

such that

$$d_{ij} = -\sigma_i\{c + (n-c)\delta\}, \; i \neq j,$$
$$d_{ii} = A(n,i) - \sigma_i\{c + (n-c)\delta\},$$
$$A(n,i) = (N-n)\beta\rho_i + c + (n-c)\delta.$$

Now, $\mathbf{D}^{-1}$ is given by

$$\mathbf{D}^{-1} = [d^{\backslash}_{ij}],$$

where

$$d^{\backslash}_{ij} = \frac{\sigma_i\{c + (n-c)\delta\}}{A(n,i)A(n,j)D^{\backslash}_n}, \; i \neq j,$$

$$d^{\backslash}_{ii} = \frac{1}{A(n,i)} + \frac{\sigma_i\{c + (n-c)\delta\}}{A^2(n,i)D^{\backslash}_n},$$

and

$$D^{\backslash}_n = 1 - \sum_{i=1}^{r} \frac{\sigma_i\{c + (n-c)\delta\}}{A(n,i)}, \; n = c, c+1, ..., k-1.$$

Using the value of $\mathbf{D}^{-1}$ in (10) we get

$$(11)\quad P_{n,i} = \frac{c+(n-c+1)\delta}{A(n,i)}\left\{P_{n+1,i} + \sum_{j=1}^{r}\frac{\sigma_i\{c+(n-c)\delta\}P_{n+1,j}}{A(n,j)D_n^{\backslash}}\right\},$$

$n = c, c+1, ..., k-1.$

Similarly, from (5) and (7) we have

$$(12)\qquad A(k,i)P_{k,i} = \sigma_i \sum_{s=1}^{r} A(k,s)P_{k,s},\ i = 1,2,...,r.$$

It is easy to verify that the determinant formed by the coefficients of $P_{k,i}, i = 1,2,...,r$, is zero and therefore we can solve equation (12) for any $r-1$ probabilities involved in terms of $P_{k,r}$. Leaving out the r^{th} equation, we have the matrix representation of (12) as

$$(13)\qquad \mathbf{ER} = -A(k,r)\mathbf{G}P_{k,r},$$

where $\mathbf{E}$ is the $(r-1)\times(r-1)$ matrix

$$\mathbf{E} = [e_{ij}]$$

such that

$$\begin{aligned} e_{ij} &= \sigma_i A(k,j),\ i \neq j, \\ e_{ii} &= (\sigma_i - 1)A(k,i), \end{aligned}$$

where

$$\begin{aligned} \mathbf{R}^T &= [P_{k,1}, P_{k,2}, ..., P_{k,r-1}], \\ \mathbf{G}^T &= [\sigma_1, \sigma_2, ..., \sigma_{r-1}]. \end{aligned}$$

Now, $\mathbf{E}^{-1}$ is given by

$$\mathbf{E}^{-1} = [e_{ij}^{\backslash}]$$

such that

$$\begin{aligned} e_{ij}^{\backslash} &= \frac{-\sigma_i}{\sigma_r A(k,i)},\ i \neq j, \\ e_{ii}^{\backslash} &= \frac{-(\sigma_i + \sigma_r)}{\sigma_r A(k,i)}. \end{aligned}$$

As before,

$$(14)\qquad P_{k,i} = \frac{\sigma_i A(k,r)}{\sigma_r A(k,i)}P_{k,r},\ i = 1,2,...,r-1.$$

Thus, we have expressed all the probabilities $P_{n,i}$ for $n = 0,1,2,...,k$; $i = 1,2,...,r$ in terms of one unknown probability, namely $P_{k,r}$. This unknown probability may now be evaluated by using the normalizing condition:

$$\sum_{n=0}^{k}\sum_{i=1}^{r} P_{n,i} = 1, \tag{15}$$

and hence all the probabilities are completely known.

The following example illustrates the method discussed above.

EXAMPLE. In the above system: $H_r/M/c/k/N$ with balking and reneging, letting $r = 2, c = 3, k = 5$ and $N = 8$, i.e., the system: $H_2/M/3/5/8$ with balking and reneging, the results are

$$\begin{aligned}
P_{5,1} &= \eta P_{5,2},\ P_{4,1} = aP_{5,2},\ P_{4,2} = bP_{5,2},\\
P_{3,1} &= dP_{5,2},\ P_{3,2} = eP_{5,2},\ P_{2,1} = fP_{5,2},\\
P_{2,2} &= gP_{5,2},\ P_{1,1} = hP_{5,2},\ P_{1,2} = \ell P_{5,2},\\
P_{0,1} &= \frac{h}{8\rho_1}P_{5,2},\ P_{0,2} = \frac{\ell}{8\rho_2}P_{5,2},
\end{aligned}$$

where

$$\eta = \frac{(3\beta\rho_2 + 3 + 2\delta)\sigma_1}{(3\beta\rho_1 + 3 + 2\delta)\sigma_2},$$

$$\rho_i = \frac{\lambda_i}{\mu},\ i = 1, 2,$$

$$\delta = \frac{\alpha}{\mu},\ \sigma_1 + \sigma_2 = 1,$$

$$a = \frac{3+2\delta}{4\beta\rho_1 + 3 + \delta}\left\{\eta + \frac{\sigma_1(3+\delta)}{D_4^{\backslash}}\left[\frac{\eta}{4\beta\rho_1 + 3 + \delta} + \frac{1}{4\beta\rho_2 + 3 + \delta}\right]\right\},$$

$$b = \frac{3+2\delta}{4\beta\rho_2 + 3 + \delta}\left\{1 + \frac{\sigma_2(3+\delta)}{D_4^{\backslash}}\left[\frac{\eta}{4\beta\rho_1 + 3 + \delta} + \frac{1}{4\beta\rho_2 + 3 + \delta}\right]\right\},$$

$$d = \frac{3+\delta}{5\beta\rho_1 + 3}\left\{a + \frac{3\sigma_1}{D_3^{\backslash}}\left[\frac{a}{5\beta\rho_1 + 3} + \frac{b}{5\beta\rho_2 + 3}\right]\right\},$$

$$e = \frac{3+\delta}{5\beta\rho_2 + 3}\left\{b + \frac{3\sigma_2}{D_3^{\backslash}}\left[\frac{a}{5\beta\rho_1 + 3} + \frac{b}{5\beta\rho_2 + 3}\right]\right\},$$

$$f = \frac{3}{6\rho_1 + 2}\left\{d + \frac{2\sigma_1}{D_2}\left[\frac{d}{6\rho_1 + 2} + \frac{e}{6\rho_2 + 2}\right]\right\},$$

$$g = \frac{3}{6\rho_2 + 2}\left\{e + \frac{2\sigma_2}{D_2}\left[\frac{d}{6\rho_1 + 2} + \frac{e}{6\rho_2 + 2}\right]\right\},$$
$$h = \frac{2}{7\rho_1 + 1}\left\{f + \frac{\sigma_1}{D_1}\left[\frac{f}{7\rho_1 + 1} + \frac{g}{7\rho_2 + 1}\right]\right\},$$
$$\ell = \frac{2}{7\rho_2 + 1}\left\{g + \frac{\sigma_2}{D_1}\left[\frac{f}{7\rho_1 + 1} + \frac{g}{7\rho_2 + 1}\right]\right\},$$
$$D_1 = \frac{7\rho_1\sigma_1}{7\rho_1 + 1} + \frac{7\rho_2\sigma_2}{7\rho_2 + 1},$$
$$D_2 = \frac{6\rho_1\sigma_1}{6\rho_1 + 2} + \frac{6\rho_2\sigma_2}{6\rho_2 + 2},$$
$$D_3^{\backslash} = \frac{5\beta\rho_1\sigma_1}{5\beta\rho_1 + 3} + \frac{5\beta\rho_2\sigma_2}{5\beta\rho_2 + 3},$$
$$D_4^{\backslash} = \frac{4\beta\rho_1\sigma_1}{4\beta\rho_1 + 3 + \delta} + \frac{4\beta\rho_2\sigma_2}{4\beta\rho_2 + 3 + \delta}.$$

From the normalizing condition: $\sum_{n=0}^{5}\sum_{s=1}^{2} P_{n,s} = 1$, we have

$$P_{5,2}^{-1} = \left[h(1 + \frac{1}{8\rho_1}) + \ell(1 + \frac{1}{8\rho_2}) + f + g + e + d + a + b + \eta + 1\right].$$

The expected number of units in the system and in the queue are, respectively,

$$\begin{aligned} L &= \sum_{n=1}^{5}\sum_{i=1}^{2} nP_{n,i} \\ &= \{h + \ell + 2(g + f) + 3(d + e) + 4(a + b) + 5(1 + \eta)\}P_{5,2}, \\ L_q &= \sum_{n=4}^{5}\sum_{i=1}^{2}(n - 3)P_{n,i} = [a + b + 2(1 + \eta)]P_{5,2}. \end{aligned}$$

The machine availability (rate of production per machine is

$$M.A. = 1 - \frac{L}{5}.$$

The operative efficiency (utilization) is

$$\begin{aligned} O.E. &= 1 - \sum_{n=0}^{2}\sum_{i=1}^{2}(1 - \frac{n}{3})P_{n,i} \\ &= 1 - \left[h(\frac{1}{8\rho_1} + \frac{2}{3}) + \ell(\frac{1}{8\rho_2} + \frac{2}{3}) + \frac{1}{3}(f + g)\right]P_{5,2}. \end{aligned}$$

4. Special cases

i) Let $\sigma_i = \delta_{is}$, where δ_{is} is the Kronecker delta function, in the above system, we can get the Markovian machine interference system: $M/M/c/k/N$ with balking and reneging which had been discussed by Shawky [7].

ii) Moreover, let $\alpha = 0$ and $\beta = 1$ one has the system: $M/M/c/k/N$ without any concept, which studied by Kleinrock [5]. If $N = k$, the system becomes $M/M/c/k/k$ without any concept which had been studied by White et al. [8], Medhi [6], Gross and Harris [3] and Bunday [1].

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