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FINSLER SPACES WITH CERTAIN (α, β)-METRIC OF DOUGLAS TYPE

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ABSTRACT. We shall find the condition for a Finsler space with a special (α, β)-metric $L(\alpha, \beta)$ satisfying $L^2 = 2\alpha\beta$ to be a Douglas space. The special Randers change of the above Finsler metric by β is also studied.

1. Introduction

A Finsler space F^n with vanishing Douglas tensor D is called of Douglas type or Douglas space([2], [5]). It is known that if a Finsler space F^n is projective to a Berwald space, then F^n is a Douglas space [1]. Recently S. Bácsó and M. Matsumoto [2] introduced the new notion of Douglas space as a generalization of Berwald space from the view point of geodesic equation in the Finsler space. A Finsler metric $L(x, y)$ is called an (α, β)-metric, when L is a positively homogeneous function $L(\alpha, \beta)$ of the first degree in two variables: $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and 1-form $\beta = b_i(x)y^i$. The (α, β)-metric L satisfying $L^2 = 2\alpha\beta$ is one of the generalized Randers metric $L^2 = c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2$, where c_i are constants ([1], [9]). Some properties of the Finsler metric L satisfying $L^2 = 2\alpha\beta$ have been investigated by S. Hōjō ([4]).

The present paper is devoted to studying the condition for a Finsler space with (α, β)-metric satisfying $L^2 = 2\alpha\beta$ to be of Douglas type. The Randers metric $L = \alpha + \beta$ is considered as the modification of a Riemannian metric α by 1-form β. We consider generally the change of Finsler metric $L \longrightarrow \bar{L} = L + \rho$, where ρ is a 1-form. This change is called the Randers change by ρ. In the last section, we consider a

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special Randers change of Finsler space with (α, β)-metric L satisfying $L^2 = 2\alpha\beta$ by β which coincides with 1-form β of the metric L, and devoted to studying the Douglas space obtained by Randers change by β.

2. Preliminaries

Geodesics of an n-dimensional Finsler space $F^n = (M^n, L)$ are given by the system of differential equations ([1]):

$$\frac{d^2x^i}{dt^2}y^j - \frac{d^2x^j}{dt^2}y^i + 2\{G^i(x,y)y^j - G^j(x,y)y^i\} = 0, \quad y^i = \frac{dx^i}{dt},$$

in parameter t. The functions $G^i(x,y)$ are given by

$$2G^i(x,y) = g^{ij}(x,y)(y^r \dot{\partial}_j \partial_r F - \partial_j F),$$

where $\dot{\partial}_i = \partial/\partial y^i$, $\partial_i = \partial/\partial x^i$, $F = L^2/2$ and $g^{ij}(x,y)$ is the inverse of Finsler metric $g_{ij}(x,y)$. The Finsler space F^n is said to be of *Douglas type* or called a *Douglas space* if the *Douglas tensor* D with components

$$D_h{}^i{}_{jk} = G_h{}^i{}_{jk} - \frac{1}{n+1}(G_{hjk}y^i + G_{hj}\delta^i_k + G_{jk}\delta^i_h + G_{kh}\delta^i_j)$$

vanishes identically, where $G_h{}^i{}_{jk} = \dot{\partial}_h G_j{}^i{}_k$ is the hv-curvature tensor of the Berwald connection $B\Gamma = (G_j{}^i{}_k, G^i{}_j, 0)$, $G_{ij} = G_i{}^r{}_{jr}$ and $G_{ijk} = \dot{\partial}_k G_{ij}$ ([1], [5]). The Douglas tensor D is invariant under projective change in F^n. If F^n is projective to a Berwald space, then F^n is a Douglas space, that is, $D = 0$. It is shown that $D = 0$ is one of half of the necessary and sufficient conditions for a generalized affine space to be a projectively flat. We put

$$D^{ij} = G^i(x,y)y^j - G^j(x,y)y^i. \tag{2.1}$$

It is known that F^n is of Douglas type if and only if D^{ij} defined by (2.1) are homogeneous polynomials in (y^i) of degree three [2]. We shall denote the homogeneous polynomials in (y^i) of degree r by $hp(r)$ for brevity. The space $R^n = (M^n, \alpha)$ is called the Riemannian space associated with F^n. In R^n, we have the covariant differentiation (;) with respect to the Levi-Civita connection $\{_j{}^i{}_k\}(x)$. We shall use the symbols as follows:

$$r_{ij} = \frac{1}{2}(b_{i;j} + b_{j;i}), \quad s_{ij} = \frac{1}{2}(b_{i;j} - b_{j;i}), \quad s^i{}_j = a^{ir}s_{rj}, \quad s_j = b_r s^r{}_j.$$

According to [7] the functions $G^i(x,y)$ of F^n with (α,β)-metric $L(\alpha,\beta)$ are written in the form

$$(2.2)\qquad \begin{aligned} 2G^i &= \{{}_j{}^i{}_k\}(x)y^jy^k + 2B^i(x,y), \\ B^i &= \left\{\frac{\beta L_\beta}{\alpha L}y^i - \frac{\alpha L_{\alpha\alpha}}{L_\alpha}(\frac{1}{\alpha}y^i - \frac{\alpha}{\beta}b^i)\right\}C^* + \frac{\alpha L_\beta}{L_\alpha}s^i{}_0, \end{aligned}$$

where $L_\alpha = \partial L/\partial\alpha, L_\beta = \partial L/\partial\beta, L_{\alpha\alpha} = \partial^2 L/\partial\alpha\partial\alpha$, the subscript 0 means the contraction by y^i and put

$$\begin{aligned} C^* =& \frac{\alpha\beta(r_{00}L_\alpha - 2\alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})}, \\ b^i =& a^{ij}b_j, \qquad b^2 = a^{ij}b_ib_j, \quad \gamma^2 = b^2\alpha^2 - \beta^2. \end{aligned}$$

Since $\{{}_0{}^i{}_0\}(x)$ are $hp(2)$, F^n with (α,β)-metric is of Douglas type, if and only if $B^{ij} \equiv B^iy^j - B^jy^i$ are $hp(3)$. From (2.1) and (2.2), we have

$$(2.3)\qquad B^{ij} = \frac{\alpha L_\beta}{L_\alpha}(s^i{}_0y^j - s^j{}_0y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha}C^*(b^iy^j - b^jy^i).$$

The following Lemma ([3]) will be useful for our purposes.

LEMMA M. *If $\alpha^2 \equiv 0(mod.\,\beta)$, that is, $a_{ij}(x)y^iy^j$ contains $b_i(x)y^i$ as a factor, then the dimension is two and $b^2 = 0$. In this case, we have $\delta = d_i(x)y^i$ satisfying $\alpha^2 = \beta\delta$ and $d_ib^i = 2$.*

3. A Finsler space with $L^2 = 2\alpha\beta$

We consider the condition for a Finsler space F^n with (α,β)-metric $L(\alpha,\beta)$ satisfying

$$(3.1)\qquad L^2 = 2\alpha\beta$$

to be of Douglas type. Then (2.3) gives

$$(3.2)\qquad \begin{aligned} &2(3\beta^2 - b^2\alpha^2)\{\beta B^{ij} - \alpha^2(s^i{}_0y^j - s^j{}_0y^i)\} \\ &+ \alpha^2(r_{00}\beta - 2s_0\alpha^2)(b^iy^j - b^jy^i) = 0. \end{aligned}$$

Suppose that F^n is of Douglas type, that is, B^{ij} are $hp(3)$. Since the terms $6\beta^3B^{ij}$ of (3.2) seemingly do not contain α^2, we must have hp (4) : u_4^{ij} satisfying

$$(3.3)\qquad 6\beta^3B^{ij} = \alpha^2u_4^{ij}.$$

First, we are concerned with the case of $\alpha^2 \equiv 0(\mathrm{mod.}\,\beta)$. In this case, $n = 2$, $b^2 = 0$ and $\alpha^2 = \beta\delta$ by Lemma M.

According to [1], the main scalar I of F^2 with (α, β)-metric is defined by

$$\epsilon I^2 = \frac{1}{\alpha^2 T^3}(T_\beta F)^2(b^2 - \frac{\beta^2}{\alpha^2}),$$

where $\epsilon = \pm 1$ and $T = 2FF_\alpha/\alpha^3 + (2FF_{\beta\beta} - F_\beta{}^2)(b^2 - \beta^2/\alpha^2)/\alpha^2$. Hence in two dimensional Finsler space with $L(\alpha, \beta)$ satisfying (3.1), we obtain $\epsilon I^2 = -4/3$, that is, the main scalar I is a constant. Thus F^2 with (3.1) is a Berwald space ([1]).

Second, we are concerned with the case of $\alpha^2 \not\equiv 0(\mathrm{mod.}\,\beta)$. From (3.3), there exists $hp(1)$ such that

$$6B^{ij} = \alpha^2 u^{ij}. \tag{3.4}$$

Therefore, we get $u_4^{ij} = \beta^3 u^{ij}$. Hence (3.2) can be rewritten in the form

$$\begin{aligned}&(3\beta^2 - b^2\alpha^2)\{\beta u^{ij} - 6(s^i{}_0 y^j - s^j{}_0 y^i)\}\\ &+ 3(r_{00}\beta - 2s_0\alpha^2)(b^i y^j - b^j y^i) = 0.\end{aligned} \tag{3.5}$$

The terms which seemingly do not contain β in (3.5) are

$$6b^2\alpha^2(s^i{}_0 y^j - s^j{}_0 y^i) - 6s_0\alpha^2(b^i y^j - b^j y^i).$$

Therefore we must have $hp(3) : v_3^{ij}$ such that

$$6\alpha^2\{b^2(s^i{}_0 y^j - s^j{}_0 y^i) - s_0(b^i y^j - b^j y^i)\} = \beta v_3^{ij},$$

which implies

$$b^2(s^i{}_0 y^j - s^j{}_0 y^i) - s_0(b^i y^j - b^j y^i) = \beta v^{ij}, \tag{3.6}$$

where v^{ij} are $hp(1)$ satisfying $v_3^{ij} = 6\alpha^2 v^{ij}$. Thus (3.5) is rewritten as

$$(3\beta^2 - b^2\alpha^2)(b^2 u^{ij} - 6v^{ij}) + 3(b^2 r_{00} - 6s_0\beta)(b^i y^j - b^j y^i) = 0. \tag{3.7}$$

On the other hand, we put $v^{ij} = v_k^{ij}(x)y^k$. Then (3.7) is written as

$$b^2\{(s^i{}_h\delta_k^j + s^i{}_k\delta_h^j) - [i, j]\} - \{b^i(s_h\delta_k^j + s_k\delta_h^j) - [i, j]\} = b_h v_k^{ij} + b_k v_h^{ij}, \tag{3.8}$$

where $[i,j]$ denotes the interchange of indices i,j of previous terms. Contracting j and k, from (3.8) we obtain the following form :

$$n(b^2 s^i{}_h - b^i s_h) = b_h v^{ir}_r + b_r v^{ir}_h. \tag{3.9}$$

Transvection of (3.8) by $b_j b^h$ is reduced to

$$b^2(b^2 s^i{}_k - s^i b_k - b^i s_k) = b^2 b_r v^{ir}_k + b_k b_r v^{ir}_s b^s. \tag{3.10}$$

Furthermore, transvecting (3.10) by b^k, we have

$$b_r v^{ir}_s b^s = -b^2 s^i, \tag{3.11}$$

provided that $b^2 \neq 0$. Substituting (3.11) in (3.10), we get $b_r v^{ir}_k = b^2 s^i{}_k - b^i s_k$ and (3.9) is rewritten as

$$(n-1)(b^2 s^i{}_h - b^i s_h) = b_h v^{ir}_r.$$

Consequently, if we put $v_j = (a_{ij} v^{ir}_r)/(n-1)$, then we have

$$b^2 s_{ih} = v_i b_h + b_i s_h. \tag{3.12}$$

Since s_{ih} is skew symmetric, we have $(v_i b_h + b_i s_h) y^i y^h = 0$, which implies $\beta(v_i + s_i) y^i = 0$, that is, $v_i = -s_i$. Hence, from (3.12) we have

$$s_{ih} = \frac{1}{b^2}(b_i s_h - b_h s_i). \tag{3.13}$$

Thus (3.6) is reduced to $v^{ij} = y^i s^j - y^j s^i$ and (3.7) is rewritten as

$$\begin{aligned}(3\beta^2 - b^2\alpha^2)\{b^2 u^{ij} - 6(y^i s^j - y^j s^i)\}\\ + 3(b^2 r_{00} - 6\beta s_0)(b^i y^j - b^j y^i) = 0.\end{aligned} \tag{3.14}$$

Transvecting (3.14) by $b_i s_j$, we have

$$(3\beta^2 - b^2\alpha^2)(b^2 u^{ij} b_i s_j - 6 s^j s_j \beta) + 3(b^2 r_{00} - 6 s_0 \beta) b^2 s_0 = 0. \tag{3.15}$$

Suppose that there exists $u = u(x) y^i$ such that $3\beta^2 - b^2\alpha^2 = b^2 s_0 u$. Then this is written in the form $2(3b_i b_j - b^2 a_{ij}) = b^2(s_i u_j + s_j u_i)$. Transvection of this equation by $b^i b^j$ gives $b^2 = 0$, which is a contradiction. Consequently (3.15) shows that we have a function $k = k(x)$ satisfying

$$b^2 u^{ij} b_i s_j - 6 s^j s_j \beta = k b^2 s_0, \quad 3(b^2 r_{00} - 6 s_0 \beta) = k(b^2\alpha^2 - 3\beta^2). \tag{3.16}$$

From the latter of (3.16), we have

$$b^2 r_{00} = \frac{k}{3}(b^2\alpha^2 - 3\beta^2) + 6s_0\beta,$$

which implies

$$r_{ij} = \frac{1}{b^2}\left\{\frac{k}{3}(b^2 a_{ij} - 3b_i b_j) + 6(s_i b_j + s_j b_i)\right\}. \tag{3.17}$$

From (3.13) and (3.17), we have

$$b_{i;j} = \frac{1}{b^2}\left\{\frac{k}{3}(3b_i b_j - b^2 a_{ij}) + (7b_i s_j + 5b_j s_i)\right\}. \tag{3.18}$$

Conversely, if (3.18) holds, then we see that

$$B^{ij} = \frac{\alpha^2}{b^2}\left\{(y^i s^j - y^j s^i) - \frac{k}{3}(b^i y^j - b^j y^i)\right\},$$

are $hp(3)$, that is, F^n is a Douglas space. Thus we have

THEOREM 3.1. *Let F^n be a Finsler space with (α,β)-metric L satisfying* (3.1).

(1) $\alpha^2 \equiv 0(mod.\,\beta)$: *$n = 2$, and F^2 is a Berwald space with constant main scalar.*

(2) $\alpha^2 \not\equiv 0(mod.\,\beta)$: *$F^n$ with non-zero b^2 is of Douglas type, if and only if $b_{i;j}$ are given by* (3.18), *where $k = k(x)$.*

4. A special Randers change by β

We consider the condition for a Finsler space which is obtained by a special Randers change by β of the Finsler metric L satisfying (3.1) to be of Douglas type, where the modified 1-form ρ coincides with β of (3.1).

Let $\overline{F}^n = (M^n, \overline{L})$ be a Finsler space which is obtained by Randers change of L satisfying $L^2 = 2\alpha\beta$ by β. Since $\overline{L} = 2\alpha\beta + \beta$ is also an (α,β)-metric, (2.3) in $\overline{F}^n$ gives $\overline{B}^{ij} = B^{ij} + W^{ij}$, where

$$\begin{aligned} W^{ij} &= \frac{\alpha E}{\beta}(s^i{}_0 y^j - s^j{}_0 y^i) \\ &\quad + \frac{s_0\alpha^3 E}{\beta(3\beta^2 - b^2\alpha^2)}(b^i y^j - b^j y^i), \end{aligned} \tag{4.1}$$

and $E = (2\alpha\beta)^{1/2}$. Thus we have

PROPOSITION 4.1. *Let a Finsler space F^n with an (α, β)-metric L satisfying* (3.1) *be a Douglas space and $\overline{F}^n$ a Finsler space which is obtained by the special Randers change F^n by β. Then $\overline{F}^n$ is also a Douglas space, if and only if W^{ij} are $hp(3)$.*

Equation (4.1) is rewritten as

$$\beta(3\beta^2 - b^2\alpha^2)W^{ij} - \alpha E\{(3\beta^2 - b^2\alpha^2)(s^i{}_0 y^j - s^j{}_0 y^i) + s_0\alpha^2(b^i y^j - b^j y^i)\} = 0.$$

Suppose W^{ij} are $hp(3)$. Since αE is irrational, we have

$$\beta(3\beta^2 - b^2\alpha^2)W^{ij} = 0, \tag{4.2}$$

$$(3\beta^2 - b^2\alpha^2)(s^i{}_0 y^j - s^j{}_0 y^i) + s_0\alpha^2(b^i y^j - b^j y^i) = 0. \tag{4.3}$$

Transvecting (4.3) by $b_i y_j$, we get $2s_0\alpha^2\beta^2 = 0$, that is, $s_0 = 0$. From this and (4.3), we get $s_{ij} = 0$. Therefore we get $W^{ij} = 0$ from (4.1), that is, $\overline{B}^{ij} = B^{ij}$. Thus we have

PROPOSITION 4.2. *Let a Finsler space F^n with an (α, β)-metric L satisfying* (3.1) *be a Douglas space and $\overline{F}^n$ a Finsler space which is obtained by the special Randers change of F^n by β. Then $\overline{F}^n$ is also a Douglas space.*

Now we shall find the condition that $\overline{F}$ is of Douglas type. Since $\overline{L}$ is also an (α, β)-metric, (2.2) is reduced to

$$\begin{aligned} &2(3\beta^2 - b^2\alpha^2)\{\beta\overline{B}^{ij} - \alpha(\alpha + E)(s^i{}_0 y^j - s^j{}_0 y^i)\} \\ &+ \alpha^2\{r_{00}\beta - 2s_0\alpha(\alpha + E)\}(b^i y^j - b^j y^i) = 0. \end{aligned} \tag{4.4}$$

Transvecting (4.4) by $b_i y_j$, we get

$$\begin{aligned} &2(3\beta^2 - b^2\alpha^2)\beta\overline{B}^{ij} b_i y_j - 2s_0(3\beta^2 - b^2\alpha^2)\alpha^4 \\ &+ \alpha^2(r_{00}\beta - 2s_0\alpha^2)(b^2\alpha^2 - \beta^2) - 4s_0\alpha^3\beta^2 E = 0. \end{aligned} \tag{4.5}$$

Suppose that $\overline{F}^n$ is of Douglas type, that is, $\overline{B}^{ij}$ are $hp(3)$. Since $\alpha^3 E$ is an irrational in (y^i), we have $s_0 = 0$, that is, $s_i = 0$. Substituting this in (4.5), we have

$$2(3\beta^2 - b^2\alpha^2)\overline{B}^{ij} b_i y_j = \alpha^2 r_{00}(b^2\alpha^2 - \beta^2). \tag{4.6}$$

For $b^2 \neq 0$, if $(b^2\alpha^2 - \beta^2)$ is a divisor of $3\beta^2 - b^2\alpha^2$, then we must have a function $f = f(x)$ such that $3\beta^2 - b^2\alpha^2 = f(b^2\alpha^2 - \beta^2)$. This is written in the form

$$3b_i b_j - b^2 a_{ij} = f(b_i b_j - b^2 a_{ij}).$$

Transvection of the above by $b^i b^j$ gives $b^2 = 0$, which is a contradiction. Therefore, for $b^2 \neq 0$, there exists a function $g = g(x)$ such that $3\beta^2 - b^2\alpha^2 = r_{00} g$, from which

$$r_{ij} = h(3b_i b_j - b^2 a_{ij}), \tag{4.7}$$

where $h = 1/g$.

On the other hand, from $s_0 = 0$, (4.4) is reduced to

$$\begin{aligned} &2(3\beta^2 - b^2\alpha^2)\{\beta\overline{B}^{ij} - \alpha(\alpha + E)({s^i}_0 y^j - {s^j}_0 y^i)\} \\ &+ \alpha^2 r_{00}\beta(b^i y^j - b^j y^i) = 0. \end{aligned} \tag{4.8}$$

The terms which seemingly do not contain β in (4.8) are

$$2b^2\alpha^3(\alpha + E)({s^i}_0 y^j - {s^j}_0 y^i).$$

First, we suppose $\alpha^2 \not\equiv 0 (\mathrm{mod.}\, \beta)$. Since β is not a factor of $\alpha + E$, we have $hp(1) : w^{ij} = w^{ij}_k(x) y^k$ such that

$$2b^2({s^i}_0 y^j - {s^j}_0 y^i) = \beta w^{ij}.$$

The above equation is written as

$$2b^2({s^i}_h \delta^j_k - {s^j}_h \delta^i_k + {s^i}_k \delta^j_h - {s^j}_k \delta^i_h) = b_h w^{ij}_k + b_k w^{ij}_h. \tag{4.9}$$

Contraction of (4.9) by $j = k$ yields

$$2nb^2 {s^i}_h = b_h w^{ir}_r + b_r w^{ir}_h. \tag{4.10}$$

On the other hand, transvection of (4.9) by $b_j b^k$ yields

$$2b^4 s^i{}_h = b_h b_r b^s w_s^{ir} + b^2 b_r w_h^{ir}. \tag{4.11}$$

Furthermore, transvecting (4.11) by b^h, we have $b_r b^s w_s^{ir} = 0$ because of $s_i = 0$ and $b^2 \neq 0$. Hence (4.11) gives $2b^2 s^i{}_h = b_r w_h^{ir}$ and (4.10) is rewritten as $2(n-1)b^2 s^i{}_h = b_h w_r^{ir}$. Consequently, if we put $w_j = (a_{ij} w_r^{ir})/2(n-1)$, then we have $b^2 s_{ih} = w_i b_h$, from which $w_i = -s_i$ $(= 0)$, and hence $s_{ij} = 0$. Therefore, from (4.7) we have

$$b_{i;j} = h(3b_i b_j - b^2 a_{ij}). \tag{4.12}$$

Conversely, if (4.12) holds, then $r_{00} = h(3\beta^2 - b^2\alpha^2)/2$ and $s_{ij} = 0$, which implies $s_0 = 0$. Hence we see that $\overline{B}^{ij}$ is $hp(3)$ from (4.4), that is, $\overline{F}^n$ is a Douglas space.

Second, in the case of $\alpha^2 \equiv 0(\mathrm{mod}.\,\beta)$, $n = 2$, $b^2 = 0$ and $\alpha^2 = \beta\delta$ by Lemma M. Then from (4.5)

$$6\overline{B}^{ij} b_i y_j - 6s_0\beta\delta^2 - \beta\delta(r_{00}\beta - s_0\delta) - 4s_0\alpha\delta E = 0.$$

Since αE is an irrational in (y^i), we have $s_0 = 0$. Substituting this and $b^2 = 0$ in (4.4), we get

$$\begin{aligned} &6\beta^2\{\beta\overline{B}^{ij} - \beta\delta(s^i{}_0 y^j - s^j{}_0 y^i)\} + r_{00}\beta^2\delta(b^i y^j - b^j y^i) \\ &- 6\beta^2\alpha E(s^i{}_0 y^j - s^j{}_0 y^i) = 0. \end{aligned}$$

The above equation is divided into two following equations

$$\begin{aligned} 6\beta\{\overline{B}^{ij} - \delta(s^i{}_0 y^j - s^j{}_0 y^i)\} + r_{00}\delta(b^i y^j - b^j y^i) &= 0, \\ 6\beta^2(s^i{}_0 y^j - s^j{}_0 y^i) &= 0. \end{aligned} \tag{4.13}$$

From the latter of (4.13), we get $s_{ij} = 0$. Substitution of this in the former of (4.13) leads to

$$6\beta\overline{B}^{ij} + r_{00}\delta(b^i y^j - b^j y^i) = 0. \tag{4.14}$$

Since $\beta \not\equiv 0(\mathrm{mod}.\,\delta)$, there exists a function $\mu = \mu(x)$ such that

$$6\overline{B}^{ij} = -r_{00}(b^i y^j - b^j y^i)\mu. \tag{4.15}$$

Substituting (4.15) in (4.14), we have

$$r_{00}(\delta - \mu\beta)(b^i y^j - b^j y^i) = 0. \tag{4.16}$$

Transvecting (4.16) by $b_i y_j$, we get $r_{00} = 0$. From this and $s_{ij} = 0$, we have $b_{i;j} = 0$. Consequently we have

THEOREM 4.3. *Let F^n be a Finsler space with an (α, β)-metric $L(\alpha, \beta)$ satisfying (3.1) and $\overline{F}^n$ be a Finsler space which is obtained by a special Randers change by β of F^n. $\overline{F}^n$ is of Douglas type, if and only if*

(1) $\alpha^2 \not\equiv 0(mod.\beta)$: $b^2 \neq 0$ and $b_{i;j}$ is satisfied (4.12)
(2) $\alpha^2 \equiv 0(mod.\beta)$: $n = 2$ and $b_{i;j} = 0$.

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