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A NOTE ON THE MODIFIED CONDITIONAL YEH-WIENER INTEGRAL

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ABSTRACT. In this paper, we first introduce the modified Yeh-Wiener integral and then consider the modified conditional Yeh-Wiener integral. Here we use the space of continuous functions on a different region which was discussed before. We also evaluate some modified conditional Yeh-Wiener integral with examples using the simple formula for the modified conditional Yeh-Wiener integral.

1. Introduction

Kitagawa ([5]) introduced the Wiener space of functions of two variables which is the collection of the continuous functions $x(s,t)$ on the unit square $[0,1] \times [0,1]$ satisfying $x(s,t) = 0$ for $st = 0$, and then he treated the integration on this space. Yeh([12]) treated the integration of this space for the more general function and made a firm logical foundation of this space. We call this space as a Yeh-Wiener space and the integral as a Yeh-Wiener integral.

In [13, 14], Yeh introduced the conditional expectation and evaluated the conditional Wiener integral for real-valued conditioning function using the inversion formulae. Chang and the first author ([3]) treated the conditional Wiener integral for vector-valued conditioning function. Chung and Ahn ([4]) considered the conditional Yeh-Wiener integral for real-valued conditioning function.

Park and Skoug ([7]) introduced the simple formula for the conditional Yeh-Wiener integral. Chang, Chung, Ahn, and Chang ([2, 4]) used the Yeh's inversion formula to evaluate the conditional Yeh-Wiener integral for real-valued conditioning function. But, Yeh's inversion formula is

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very complicated to evaluate the conditional Yeh-Wiener integral. Using the simple formula, Park and Skoug ([7–11]) treated the conditional Yeh-Wiener integral with vector-valued, sample path-valued, multiple path-valued, and boundary-valued conditioning function.

Recently, the first author ([1]) introduced the modified conditional Yeh-Wiener integral and obtain the simple formula for this integral. And he also treated the modified conditional Yeh-Wiener integral for the functional on various regions, for example, triangular, parabolic, and circular regions.

The purpose of this paper is to treat the modified conditional Yeh-Wiener integral for the more general region rather than the region in [1]. We can obtain the result in [1] as a special case and finally we discuss the examples of the modified conditional Yeh-Wiener integral involving the general region.

2. A note on the modified conditional Yeh-Wiener integral

Let $g(s)$ be a strictly decreasing and continuous function on $[a,b]$ with $g(b) \geq 0$ and let Ω be given by $\Omega = \{(s,t) \mid 0 \leq t \leq g(s) \text{ for } a \leq s \leq b\}$. And let $C(\Omega)$ denote the space of all real-valued continuous functions x on Ω satisfying $x(s,t) = 0$ for $(s-a)t = 0$.

Let $\{s_1, \cdots, s_m\}$ be a partition of $[a,b]$ with $a = s_0 < s_1 < \cdots < s_m = b$ and let $\{t_1, \cdots, t_n\}$ be a partition of $[0, g(a)]$ with $g(s_i) = t_{n-i}$ for $i = 0, 1, 2, \cdots, m$, and $0 = t_0 < t_1 < \cdots < t_{n-m} < \cdots < t_n = g(a)$ for $n \geq m$. Here, we note that $n = m$ if $g(b) = 0$. Let τ be a partition of Ω given by

$$\tau = \{(s_i, t_j) \mid j = 1, 2, \cdots, n-i, \text{ for } i = 1, 2, \cdots, m\}. \tag{2.1}$$

Let X_τ be a R^N-valued random vector on $C(\Omega)$ defined by

$$X_\tau(x) = (x(s_1,t_1), \cdots, x(s_1,t_{n-1}), \cdots, x(s_m,t_1), \cdots, x(s_m,t_{n-m})), \tag{2.2}$$

for $N = \frac{1}{2}m(2n-m-1)$. Let I be the interval or the cylinder set of the type

$$I = \{x \in C(\Omega) \mid X_\tau(x) \in B\} \tag{2.3}$$

for B in $\mathcal{B}^N$, Borel σ-algebra of R^N. Define the set function $\tilde{m}$ on the collection $\mathcal{I}$ of all the intervals I by

$$\tilde{m}(I) = \int_B K(\tau, \vec{u})\, d\vec{u}, \tag{2.4}$$

where

$$K(\tau,\vec{u}) = \Big\{(2\pi)^N (s_1-a)^{n-1}\cdots(s_m-s_{m-1})^{n-m} [t_1(t_2-t_1)\cdots(t_{n-m}-t_{n-m-1})]^m (t_{n-m+1}-t_{n-m})^{m-1}\cdots(t_{n-1}-t_{n-2})\Big\}^{-\frac{1}{2}} \tag{2.5}$$

$$\exp\Big\{-\sum_{j=1}^{n-1}\frac{(u_{1,j}-u_{1,j-1})^2}{2(s_1-a)(t_j-t_{j-1})}-\cdots -\sum_{j=1}^{n-m}\frac{(u_{m,j}-u_{m,j-1}-u_{m-1,j}+u_{m-1,j-1})^2}{2(s_m-s_{m-1})(t_j-t_{j-1})}\Big\}$$

with $\vec{u}=(u_{1,1},\cdots,u_{1,n-1},u_{2,1},\cdots,u_{m,n-m})$ in R^N and $u_{0,j}=u_{i,0}=0$ for every i and j. It can be shown that $\mathcal{I}$ is a semi-algebra of subsets of $C(\Omega)$ and the set function $\tilde{m}$ defined by (2.4) is a measure defined on $\mathcal{I}$ and the factor $K(\tau,\vec{u})$ in (2.5) is chosen to make $\tilde{m}(C(\Omega))=1$. The measure $\tilde{m}$ can be extended to a measure on the Caratheodory extension of interval class $\mathcal{I}$ in the usual way. With this Caratheodory extension, measurable functionals on $C(\Omega)$ may be defined and their integration on $C(\Omega)$ can be considered.

Let F be a real-valued integrable function on $C(\Omega)$ and let P_{X_τ} be the probability distribution of X_τ defined by $P_{X_\tau}(B)=\tilde{m}(X_\tau^{-1}(B))$ for B in $\mathcal{B}^N$. Then by the definition of conditional expectation ([13]), for each function F in $L_1(C(\Omega))$,

$$\int_{X_\tau^{-1}(B)} F(x)\,d\tilde{m}(x) = \int_B E(F(x)|X_\tau(x)=\vec{u})\,dP_{X_\tau}(\vec{u}) \tag{2.6}$$

for B in $\mathcal{B}^N$ and $E(F(x)|X_\tau(x)=\vec{u})$ is Borel measurable function of $\vec{u}$ which is unique up to Borel null sets in R^N. Here we call $E(F|X_\tau)(\vec{u})\equiv E(F(x)|X_\tau(x)=\vec{u})$ as a modified conditional Yeh-Wiener integral of F given by X_τ.

For each partition τ of Ω and x in $C(\Omega)$, define the quasi-polyhedric function $[x]$ on Ω by

$$[x](s,t) = x(s_{i-1},t_{j-1}) + \frac{(s-s_{i-1})(t-t_{j-1})}{\Delta_i s\Delta_j t}\Delta_{ij}x(s,t) + \frac{s-s_{i-1}}{\Delta_i s}(x(s_i,t_{j-1})-x(s_{i-1},t_{j-1})) + \frac{t-t_{j-1}}{\Delta_j t}(x(s_{i-1},t_j)-x(s_{i-1},t_{j-1})) \tag{2.7}$$

on each $\Omega_{ij} = (s_{i-1}, s_i] \times (t_{j-1}, t_j]$, $j = 1, 2, \cdots, n-i$ for $i = 1, 2, \cdots, m$, where $\Delta_i s = s_i - s_{i-1}, \Delta_j t = t_j - t_{j-1}$, and $\Delta_{ij} x(s,t) = x(s_i, t_j) - x(s_{i-1}, t_j) - x(s_i, t_{j-1}) + x(s_{i-1}, t_{j-1})$, and

$$\begin{aligned}[x](s,t) &= x(s_{i-1}, t_{n-i}) + \frac{s - s_{i-1}}{\Delta_i s}(x(s_i, t_{n-i}) - x(s_{i-1}, t_{n-i})) \\ &\quad + \frac{t - t_{n-i}}{\Delta_{n-i+1} t}(x(s_{i-1}, t_{n-i+1}) - x(s_{i-1}, t_{n-i}))\end{aligned} \tag{2.8}$$

on $\Omega_i = \{(s,t) \mid s_{i-1} < s \le s_i \ , \ t_{n-i} < t \le g(s)\}$, $i = 1, 2, \cdots, m$, and $[x](s,t) = 0$ if $(s-a)t = 0$.

Similarly, for $\vec{u} = (u_{1,1}, \cdots, u_{1,n-1}, u_{2,1}, \cdots, u_{m,n-m}) \in R^N$, we define the quasi-polyhedric function $[\vec{u}]$ of $\vec{u}$ on Ω by

$$\begin{aligned}[\vec{u}](s,t) &= u_{i-1,j-1} + \frac{(s - s_{i-1})(t - t_{j-1})}{\Delta_i s \Delta_j t}\Delta_{ij}\vec{u} \\ &\quad + \frac{s - s_{i-1}}{\Delta_i s}(u_{i,j-1} - u_{i-1,j-1}) \\ &\quad + \frac{t - t_{j-1}}{\Delta_j t}(u_{i-1,j} - u_{i-1,j-1}),\end{aligned} \tag{2.9}$$

on each Ω_{ij}, where $\Delta_{ij}\vec{u} = u_{i,j} - u_{i-1,j} - u_{i,j-1} + u_{i-1,j-1}$, and

$$\begin{aligned}[\vec{u}](s,t) &= u_{i-1,n-i} + \frac{s - s_{i-1}}{\Delta_i s}(u_{i,n-i} - u_{i-1,n-i}) \\ &\quad + \frac{t - t_{n-i}}{\Delta_{n-i+1} t}(u_{i-1,n-i+1} - u_{i-1,n-i})\end{aligned} \tag{2.10}$$

on each Ω_i, where $u_{0,j} = u_{i,0} = 0$ for all i, j and $[\vec{u}](s,t) = 0$ if $(s-a)t = 0$.

REMARK 2.1. (1) In [2, 4, 5, 7–12], they treated the space $C(\Omega)$ for the region Ω given by $\Omega = \{(s,t) \mid 0 \le t \le g(s) \text{ for } a \le s \le b\}$ only for a constant function g on $[a,b]$. But, in [1], the first author treated the space $C(\Omega)$ having a strictly decreasing function g on $[a,b]$ with $g(b) = 0$.

(2) In this paper we obtain a little bit generalized result rather than the result in [1]. That is, we treat $g(b) \ge 0$ and so the result (2.3) in [1] is a special case $m = n$ in (2.5). Here the measure $\tilde{m}$ given by (2.4) and (2.5) plays an important role. To obtain a modified conditional Yeh-Wiener integral for a monotone function g, our result of this paper is necessary.

In the rest of this section, we will state two important results without proof which can be obtained with a similar method as in [1].

THEOREM 2.A. *Let F be in $L_1(C(\Omega), \tilde{m})$, we have*

$$\int_{X_\tau^{-1}(B)} F(x)\, d\tilde{m}(x) = \int_B E(F(x - [x] + [\vec{u}]))\, dP_{X_\tau}(\vec{u}), \tag{2.11}$$

for $\mathcal{B}$ in $\mathcal{B}^N$.

THEOREM 2.B. *If $F \in L_1(C(\Omega), \tilde{m})$, then we have*

$$E(F(x)|X_\tau(x) = \vec{u}) = \hat{E}[F(x - [x] + [\vec{u}])], \tag{2.12}$$

where the right-hand side of (2.12) is a Borel measurable function of $\vec{u}$ which is equal to $E(F(x - [x] + [\vec{u}]))$ for a.e. $\vec{u}$ in R^N. In particular, if F is Borel measurable, then

$$E(F(x)|X_\tau(x) = \vec{u}) = E[F(x - [x] + [\vec{u}])]. \tag{2.13}$$

Theorem 2.B is a simple formula for the modified conditional Yeh-Wiener integral which comes from Theorem 2.A. And it is very convenient to apply in application rather than the Yeh's inversion formulae ([13, 14]). In fact, we will use this simple formula Theorem 2.B in the next section.

3. Examples of the modified conditional Yeh-Wiener integral

In [1], the first author treated the conditional Yeh-Wiener integral for the functional F on $C(\Omega)$ where the region Ω is given by the triangular, parabolic, and circular regions rather than the rectangular region in [7]. Here we will treat the slightly generalized and different region Ω rather than the region in [1].

EXAMPLE 3.1. Let Ω be a region in the first quadrant given by $\Omega = \{(s,t) \mid a \le s \le b,\ 0 \le t \le g(s)\}$ for $g(s) = \frac{k-T}{b-a}s + \frac{Tb-ka}{b-a}$ and $T > k \ge 0$. And let F on $C(\Omega)$ be given by $F(x) = \int_\Omega x(s,t)dsdt$. Then the modified conditional Yeh-Wiener integral of F given X_τ at $\vec{u}$ in R^N is

$$E(F|X_\tau)(\vec{u}) = \int_\Omega E\big(x(s,t) - [x](s,t) + [\vec{u}](s,t)\big)dsdt, \tag{3.1}$$

where the equality in (3.1) comes from Theorem 2.B and the Fubini theorem. Since $E(x(s,t)) = E([x](s,t)) = 0$ and $\tilde{m}(C(\Omega)) = 1$, we have

$$
\begin{aligned}
E(F|X_\tau)(\vec{u}) &= \int_\Omega [\vec{u}](s,t)dsdt \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n-i}\int_{\Omega_{ij}} [\vec{u}](s,t)dsdt \\
&\quad + \sum_{i=1}^{m}\int_{\Omega_i} [\vec{u}](s,t)dsdt,
\end{aligned} \tag{3.2}
$$

where $\Omega_i = \{\ (s,t) \mid s_{i-1} < s \le s_i,\ t_{n-i} < t \le g(s)\}$, $i = 1, 2, \cdots, m$, and $\Omega_{ij} = (s_{i-1}, s_i] \times (t_{j-1}, t_j]$, $j = 1, 2, \cdots, n-i$ for $i = 1, 2, \cdots, m$.

By (2.9) and (2.10), we have

$$
\int_{\Omega_{ij}} [u](s,t)dsdt = \frac{u_{i,j} + u_{i-1,j} + u_{i,j-1} + u_{i-1,j-1}}{4}\Delta_i s \Delta_j t, \tag{3.3}
$$

and

$$
\begin{aligned}
\int_{\Omega_i} [\vec{u}](s,t)dsdt &= u_{i-1,n-i}\int_{\Omega_i} dsdt \\
&\quad + \frac{u_{i,n-i} - u_{i-1,n-i}}{\Delta_i s}\int_{\Omega_i}(s - s_{i-1})dsdt \\
&\quad + \frac{u_{i-1,n-i+1} - u_{i-1,n-i}}{\Delta_{n-i+1} t}\int_{\Omega_i}(t - t_{n-i})dsdt,
\end{aligned} \tag{3.4}
$$

where $\Delta_i s = s_i - s_{i-1}$, $\Delta_j t = t_j - t_{j-1}$.

By the straight calculation, we have

$$
\int_{\Omega_i} dsdt = \frac{1}{2}[l_1(s_i + s_{i-1}) + 2(l_2 - t_{n-i})]\Delta_i s, \tag{3.5}
$$

$$
\begin{aligned}
\int_{\Omega_i}(s - s_{i-1})dsdt &= \frac{1}{6}\{2l_1(s_i^2 + s_i s_{i-1} + s_{i-1}^2) \\
&\quad + 3(l_2 - t_{n-i} - s_{i-1}l_1)(s_i + s_{i-1}) \\
&\quad - 6s_{i-1}(l_2 - t_{n-i})\}\Delta_i s,
\end{aligned} \tag{3.6}
$$

and

$$\begin{aligned}\int_{\Omega_i}(t-t_{n-i})dsdt = \frac{1}{6}\{&l_1^2(s_i^2+s_is_{i-1}+s_{i-1}^2)\\ &+3(l_2-t_{n-i})l_1(s_i+s_{i-1})\\ &+3(l_2-t_{n-i})^2\}\Delta_i s,\end{aligned} \tag{3.7}$$

where $l_1 = \frac{k-T}{b-a}$ and $l_2 = \frac{Tb-ka}{b-a}$. From (3.4), (3.5), (3.6), and (3.7), we have

$$\int_{\Omega_i}[\vec{u}](s,t)dsdt = \Delta_{n-i+1}t\ \Delta_i s\frac{u_{i-1,n-i+1}+u_{i,n-i}+u_{i-1,n-i}}{6}. \tag{3.8}$$

Equality in (3.8) comes from the fact that $t_{n-i} = g(s_i)$, $t_{n-i+1} = g(s_{i-1})$, and $\Delta_{n-i+1}t = -l_1\Delta_i s$, $i = 1,2,\cdots,m$. Combining (3.1), (3.2), (3.3), and (3.8), we have

$$\begin{aligned}&E(F|X_\tau)(\vec{u})\\ &=\sum_{i=1}^{m}\sum_{j=1}^{n-i}\frac{u_{i,j}+u_{i-1,j}+u_{i,j-1}+u_{i-1,j-1}}{4}\Delta_i s\Delta_j t\\ &\quad+\sum_{i=1}^{m}\frac{u_{i-1,n-i+1}+u_{i,n-i}+u_{i-1,n-i}}{3}\frac{\Delta_i s\Delta_{n-i+1}t}{2}\end{aligned} \tag{3.9}$$

for $\vec{u}$ in R^N.

The result (4.6) in [1] is a special case of (3.9) for $m = n$. Here we consider the space $C(\Omega)$ with the region Ω having a strictly decreasing function $g(s) = \frac{1}{s^2+1}$ on $[0, S]$ with $g(S) > 0$.

EXAMPLE 3.2. Let $\Omega = \{(s,t) \mid 0 \le s \le S,\ 0 \le t \le g(s)\}$ for $g(s) = \frac{1}{s^2+1}$ on $[0, S]$. And let $C(\Omega)$ denote the space of all real-valued continuous functions x on Ω satisfying $x(s,t) = 0$ for $st = 0$. Let F on $C(\Omega)$ be given by $F(x) = \int_\Omega x(s,t)dsdt$. Then we have

$$\begin{aligned}&E\Big(\int_\Omega x(s,t)dsdt \ \Big|\ X_\tau(x)=\vec{u}\Big)\\ &=\int_\Omega E\Big(x(s,t)-[x](s,t)+[\vec{u}](s,t)\Big)dsdt\\ &=\int_\Omega [u](s,t)dsdt\\ &=\sum_{i=1}^{m}\sum_{j=1}^{n-i}\int_{\Omega_{ij}}[\vec{u}](s,t)dsdt+\sum_{i=1}^{m}\int_{\Omega_i}[\vec{u}](s,t)dsdt,\end{aligned} \tag{3.10}$$

where Ω_i and Ω_{ij} are given as in Section 2. The first equality in (3.10) comes from Theorem 2.B and the Fubini theorem and the second equality comes from the fact that $\tilde{m}(C(\Omega)) = 1$ and $E(x(s,t)) = E([x](s,t)) = 0$. Using (2.10), we have

$$\begin{aligned}
&\int_{\Omega_i} [\vec{u}](s,t)dsdt \\
&= u_{i-1,n-i}(a_i - t_{n-i}\Delta_i s) \\
&\quad + \frac{u_{i,n-i} - u_{i-1,n-i}}{2\Delta_i s} \\
&\qquad \times \left\{ \ln \frac{1+s_i^2}{1+s_{i-1}^2} - t_{n-i}(s_i + s_{i-1})\Delta_i s - 2a_i s_{i-1} \right\} \\
&\quad + \frac{u_{i-1,n-i+1} - u_{i-1,n-i}}{4\Delta_{n-i+1} t} \\
&\qquad \times \left\{ a_i(1 - 4t_{n-i}) + \left(\frac{1 - s_i s_{i-1}}{(1+s_i^2)(1+s_{i-1}^2)} + 2t_{n-i}^2 \right) \Delta_i s \right\}
\end{aligned} \tag{3.11}$$

where $a_i = \tan^{-1} s_i - \tan^{-1} s_{i-1}$. From (3.3), (3.10), and (3.11) we can evaluate the modified conditional Yeh-Wiener integral $E(F|X_\tau)(\vec{u})$ for $\vec{u}$ in R^N.

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