Comm. Korean Math. Soc. **16** (2001), No. 4, pp. 621–626

CERTAIN MAXIMAL OPERATOR AND ITS WEAK TYPE $L^1(\mathbb{R}^n)$-ESTIMATE

YONG-CHEOL KIM

ABSTRACT. Let $\{A_t = \exp(M \log t)\}_{t>0}$ be a dilation group where M is a real $n \times n$ matrix whose eigenvalues has strictly positive real part, and let ϱ be an A_t-homogeneous distance function defined on $\mathbb{R}^n$. Suppose that $\mathcal{K}$ is a function defined on $\mathbb{R}^n$ such that $|\mathcal{K}(x)| \leq \mathfrak{K} \circ \varrho(x)$ for a decreasing function $\mathfrak{K}(t)$ on $\mathbb{R}_+$ satisfying $\mathfrak{K} \circ \varrho \in L^1(\omega_0)$ where $\omega_0(x) = |\log|\log \varrho(x)||$. For $f \in L^1(\mathbb{R}^n)$, define $\mathfrak{M}f(x) = \sup_{t>0} |\mathcal{K}_t * f(x)|$ where $\mathcal{K}_t(x) = t^{-\nu}\mathcal{K}(A_{1/t}x)$ and ν is the trace of M. Then we show that $\mathfrak{M}$ is a bounded operator of $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$.

1. Introduction

Let M be a real $n \times n$ matrix whose eigenvalues λ_i satisfy Re $(\lambda_i) > 0$; set $\lambda_0 = \min_{1\leq i\leq n} \text{Re } (\lambda_i)$ and $\lambda_m = \max_{1\leq i\leq n} \text{Re } (\lambda_i)$, and let ν be the trace of M. Then the linear transformations $A_t = \exp(M \log t), t > 0$, form a dilation group generated by the infinitesimal generator M. We now introduce A_t-homogeneous distance functions ϱ defined on $\mathbb{R}^n$; that is, ϱ is a smooth function on $\mathbb{R}^n_0 \fallingdotseq \mathbb{R}^n \setminus \{0\}$ with strictly positive values satisfying the generalized homogeneity condition $\varrho(A_t x) = t\varrho(x)$ for all $x \in \mathbb{R}^n$ and $t > 0$. Then such ϱ's satisfy the generalized triangle inequality, i.e. there is a constant $\mu \geq 1$ such that

$$\varrho(x + y) \leq \mu[\varrho(x) + \varrho(y)]$$

for any $x, y \in \mathbb{R}^n$.

Received April 11, 2001.
2000 Mathematics Subject Classification: 42B10, 42B20, 42B25.
Key words and phrases: maximal operator, weak type $L^1(\mathbb{R}^n)$-estimate.
The author was supported in part by Dong-A University Research Grant.

Next we define generalized polar coordinates with respect to the quasi-distance function ϱ, which are given by the diffeomorphism

$$\mathbb{R}_+ \times \Sigma_\varrho \to \mathbb{R}^n_0,\ (\varrho, \xi) \mapsto A_\varrho \zeta = x,$$

for $\varrho > 0$ and $\zeta \in \Sigma_\varrho \risingdotseq \{\zeta \in \mathbb{R}^n \,|\, \varrho(\zeta) = 1\}$. Then the Lebesgue measure dx transforms by way of

$$dx = \varrho^{\nu-1} \langle M\zeta, n(\zeta)\rangle \, d\varrho \, d\sigma(\zeta) \tag{1.1}$$

where $d\sigma$ denotes the surface measure on the unit sphere Σ_ϱ and $n(\zeta)$ is the outer unit normal vector to Σ_ϱ at $\zeta \in \Sigma_\varrho$.

We now introduce the quasi-Banach space $L^{1,\infty}(\mathbb{R}^n)$, which is called weak-$L^1$ space, with the norm given by

$$\|f\|_{L^{1,\infty}} \risingdotseq \sup_{s>0} s|\{x \in \mathbb{R}^n \,|\, |f(x)| > s\}| < \infty.$$

That is, $L^{1,\infty}(\mathbb{R}^n)$ is the space of all measurable functions f defined on $\mathbb{R}^n$ such that $\|f\|_{L^{1,\infty}} < \infty$. As in [2], we introduce weighted integrable functions with a weight $\omega(x)$. We denote by $L^1(\omega)$ the space of all measurable functions f defined on $\mathbb{R}^n$ for which

$$\int_{\mathbb{R}^n} |f(x)| \, \omega(x) \, dx < \infty.$$

Let us introduce weighted integrable functions with a weight $\omega_0(x) = |\log|\log \varrho(x)||$. Then it is easy to see that the space $L^1(\omega_0)$ is a proper subspace of $L^1(\mathbb{R}^n)$. Our main result is to obtain weak type $L^1(\mathbb{R}^n)$-estimate for certain maximal operator to be defined in the following. In what follows, we always suppose that $\mathcal{K}$ is a function defined on $\mathbb{R}^n$ such that

$$|\mathcal{K}(x)| \le \mathfrak{K} \circ \varrho(x)$$

where $\mathfrak{K}(t)$ is a function defined on $\mathbb{R}_+$. For $f \in L^1(\mathbb{R}^n)$, we now define

$$\mathfrak{M} f(x) = \sup_{t>0} |\mathcal{K}_t * f(x)|$$

where $\mathcal{K}_t(x) = t^{-\nu} \mathcal{K}(A_{1/t}\, x)$ for $t > 0$.

THEOREM 1.1. *If $\mathfrak{K}(t)$ is decreasing on $\mathbb{R}_+$ and $\mathfrak{K}\circ\varrho \in L^1(\omega_0)$ where $\omega_0(x) = |\log|\log \varrho(x)||$, then $\mathfrak{M}$ is a bounded operator of $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$; that is, there is a constant $C = C(n)$ such that for any $f \in L^1(\mathbb{R}^n)$,*

$$|\{x \in \mathbb{R}^n |\, \mathfrak{M}f(x) > s\}| \leq \frac{C}{s}\, \|f\|_{L^1}, \;\; s > 0.$$

REMARK. (i) It is well-known that the maximal operator associated with isotropic dilation on the kernel with decreasing radial $L^1(\mathbb{R}^n)$-majorant is dominated by the classical Hardy-Littlewood maximal operator. However, this no longer works for anisotropic cases.

(ii) The weak type $L^1(\mathbb{R}^n)$-estimate for $\mathfrak{M}$ under the stronger assumption is found in [2]; in fact, they assumed that the kernel has a certain ellipsoidal majorant that is bounded, decreasing, and in the maximal weighted L^1-space related with a weight $\omega_\varepsilon(x) = (1 + \|x\|_Q)^\varepsilon$, $\varepsilon > 0$, where Q is a certain positive definite and real symmetric matrix so that $\|A_t\, x\|_Q$ is increasing in t for the norm $\|\cdot\|_Q$ which is defined by $\|x\|_Q = \langle Qx, x\rangle^{1/2}$ for $x \in \mathbb{R}^n$. If $1 < p \leq \infty$ and the kernel $\mathcal{K}$ has a quasiradial majorant which is decreasing and integrable, then it is well-known (see [4]) that the maximal operator $\mathfrak{M}$ is bounded on $L^p(\mathbb{R}^n)$. When p is near below 1, if the kernel $\mathcal{K}$ has a quasiradial majorant which is bounded, decreasing, and in the maximal weighted L^1-space, then it is known (see [3]) that the maximal operator $\mathfrak{M}$ is a bounded operator of certain anisotropic Hardy space $H^p(\mathbb{R}^n; \varrho)$ into the weak L^p-space $L^{p,\infty}(\mathbb{R}^n)$ where ϱ is an A_t-homogeneous distance function on $\mathbb{R}^n$.

2. Weak type $L^1(\mathbb{R}^n)$-estimate

We first of all introduce a Vitali family [5] and the result of Stein and Weiss [7] on summing up weak type functions.

Suppose that $\{\mathcal{U}_s |\, s > 0\}$ is a family of open subsets of $\mathbb{R}^n$ whose closure is compact. Then we say that $\{\mathcal{U}_s |\, s > 0\}$ is a Vitali family with constant $A > 0$, if the followings are satisfied; (i) $\mathcal{U}_s \subset \mathcal{U}_{s'}$ for $s \leq s'$ and $\cap_{s>0}\mathcal{U}_s = \{0\}$, (ii) $|\mathcal{U}_s - \mathcal{U}_s| \leq A\,|\mathcal{U}_s|$ for all $s > 0$, and (iii) $\lim_{k\to\infty} |\mathcal{U}_{s_k}| = |\mathcal{U}_s|$ when $\lim_{k\to\infty} s_k = s$.

LEMMA 2.1. *Suppose that $\{g_j\}$ is a sequence of nonnegative functions on a measure space for which $\|g_j\|_{L^{1,\infty}} \leq A$ where $A > 0$ is a* *constant. Let* $\{\alpha_j\}$ *be a sequence of positive numbers with* $\sum_j \alpha_j = 1$. *Then we have that*

$$\Big\|\sum_j \alpha_j\, g_j\Big\|_{L^{1,\infty}} \le 2A(N+2)$$

where $N = \sum_j \alpha_j \log(1/\alpha_j)$.

PROOF OF THEOREM 1.1. For $k \in \mathbb{Z}$, set $\mathcal{U}_k = \{y \in \mathbb{R}^n \,|\, \varrho(y) \le 2^k\}$. Then it follows from simple computation that

$$\begin{aligned}
|\mathcal{K}(x)| &\le \sum_{k\in\mathbb{Z}} (\mathfrak{K}\circ\varrho(x))\chi_{\mathcal{U}_k}(x) \\
&\le \sum_{k\in\mathbb{Z}} \mathfrak{K}(2^{k-1})\chi_{\mathcal{U}_k}(x) \\
&= \sum_{k\in\mathbb{Z}} 2^{k\nu}\mathfrak{K}(2^{k-1})\frac{1}{2^{k\nu}}\chi_{\mathcal{U}_k}(x) \\
&\le C\sum_{k\in\mathbb{Z}} 2^{k\nu}\mathfrak{K}(2^{k-1})\frac{1}{|\mathcal{U}_k|}\chi_{\mathcal{U}_k}(x).
\end{aligned} \tag{2.1}$$

For $k \in \mathbb{Z}$, set $d_k = 2^{k\nu}\mathfrak{K}(2^{k-1})$ and $d = \sum_{k\in\mathbb{Z}} 2^{k\nu}\mathfrak{K}(2^{k-1}) < \infty$. If $c_k = d_k/d$, then we first show that

$$\mathcal{J} \fallingdotseq \sum_{k\in\mathbb{Z}} c_k[1 + \log^+(1/c_k)] < \infty. \tag{2.2}$$

Let $\mathbb{E} = \{k \in \mathbb{Z}\setminus\{0\} \,|\, d_k \le 1/k^2\}$. Since $t(1+\log(1/t))$ is increasing on $(0,1]$, we then have that

$$\begin{aligned}
&\frac{1}{d}\sum_{k\in\mathbb{E}} d_k[1 + \log^+ d + \log^+(1/d_k)] \\
\le &\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{k^2}[1 + \log^+ d + \log(k^2)] < \infty.
\end{aligned}$$

We observe that the assumption $\mathfrak{K}\circ\varrho \in L^1(\omega_0)$ is equivalent to the condition

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} \mathfrak{K}(2^k)2^{k\nu}(1 + \log|k|) < \infty.$$

Hence this makes it possible to get

$$\frac{1}{d}\sum_{k\notin\mathbb{E}} d_k(1+\log^+ d+\log^+(1/d_k))$$
$$\leq \sum_{k\in\mathbb{Z}\setminus\{0\}} d_k(1+\log^+ d+\log(k^2))<\infty.$$

Thus we obtain that $\mathcal{J}<\infty$.

Next we show that if we let $\mathcal{U}_t^k=\{y\in\mathbb{R}^n \,|\, \varrho(A_{1/t}\,y)\leq 2^k\}$ for $k\in\mathbb{Z}$ and $t>0$, then for each $k\in\mathbb{Z}$, $\{\mathcal{U}_t^k \,|\, t>0\}$ is a Vitali family with constant $(2\mu)^\nu$. It is clear to show that given $k\in\mathbb{Z}$, $\mathcal{U}_t^k\subset\mathcal{U}_{t'}^k$ for $t\leq t'$ and $\cap_{t>0}\mathcal{U}_t^k=\{0\}$. By the generalized triangle inequality, we have that $\mathcal{U}_t^k-\mathcal{U}_t^k\subset\mathcal{U}_{2\mu t}^k$, and so

$$\begin{aligned}|\mathcal{U}_t^k-\mathcal{U}_t^k|\leq|\mathcal{U}_{2\mu t}^k|&=\int_{\varrho(y)\leq 2\mu t2^k} dy\\ &=\int_{\Sigma_\varrho}\langle M\zeta,n(\zeta)\rangle\int_0^{2\mu t2^k}\varrho^{\nu-1}d\varrho\, d\sigma(\zeta)\\ &=(2\mu)^\nu|\mathcal{U}_t^k|.\end{aligned}$$

Since $|\mathcal{U}_t^k|=|\mathcal{B}(0;1)|\,2^\nu t^\nu$, we finally get that $\lim_{k\to\infty}|\mathcal{U}_{t_k}|=|\mathcal{U}_t|$ when $\lim_{k\to\infty}t_k=t$. This implies that $\{\mathcal{U}_t^k \,|\, t>0\}$ is a Vitali family with constant $(2\mu)^\nu$. It easily follows from (2.1) that

$$\begin{aligned}|\mathcal{K}_t*f(x)|&\leq C\sum_{k\in\mathbb{Z}}2^{k\nu}\mathfrak{K}(2^{k-1})\frac{1}{2^{k\nu}t^\nu}\chi_{\mathcal{U}_t^k}*|f|(x)\\ &\leq C\sum_{k\in\mathbb{Z}}2^{k\nu}\mathfrak{K}(2^{k-1})\mathcal{M}_k f(x),\end{aligned}$$

where

$$\mathcal{M}_k f(x)=\sup_{t>0}\frac{1}{2^{k\nu}t^\nu}\chi_{\mathcal{U}_t^k}*|f|(x).$$

Then by the maximal theorem [5] on a Vitali family, we have that for each $k\in\mathbb{Z}$,

$$|\{x\in\mathbb{R}^n \,|\, \mathcal{M}_k f(x)>s\}|\leq\frac{(2\mu)^\nu}{s}\,\|f\|_{L^1},\ s>0.$$

Thus by Lemma 2.1 and (2.2) we have that

$$|\{x\in\mathbb{R}^n \,|\, \mathfrak{M}f(x)>s\}|\leq\frac{2C(2\mu)^\nu\,(\mathcal{J}+2)}{s}\,\|f\|_{L^1},\ s>0.$$

Therefore, we complete the proof. □

References

[1] R. Fefferman, *A theory of entropy in Fourier analysis*, Adv. Math. **30** (1978), 171–201.

[2] Y.-C. Kim, *Weak type $L^1(\mathbb{R}^n)$-estimate for certain maximal operators*, J. Korean Math. Soc. **34** (1997), no. 4, 1029–1036.

[3] ———, *Weak type estimates for maximal operators on certain anisotropic Hardy space*, Proc. of Real Analysis Symp. in Japan (Sendai) (1997), 35–38.

[4] W. Madych, *On Littlewood-Paley functions*, Studia Math. **50** (1974), 43–63.

[5] N. M. Riviére, *Singular integrals and multiplier operators*, Ark. Mat. **9** (1971), 243–278.

[6] E. M. Stein and S. Wainger, *Problems in harmonic analysis related to curvature*, Bull. Amer. Math. Soc. **84** (1978), 1239–1295.

[7] E. M. Stein and N. J. Weiss, *On the convergence of Poisson integrals*, Trans. Amer. Math. Soc. **140** (1969), 34–54.

Department of Mathematics
Dong-A University
Pusan 604–714, Korea
E-mail: yckim@daunet.donga.ac.kr

CURRENT ADDRESS :
Department of Mathematics Education
Korea University
Seoul 136–701, Korea
E-mail: ychkim@korea.ac.kr