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A GENERALIZED STABILITY OF THE GENERAL EULER-LAGRANGE FUNCTIONAL EQUATION

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ABSTRACT. Rassias obtained the Hyers-Ulam stability of the general Euler-Lagrange functional equation. In this paper we generalized this stability in the spirit of Hyers, Ulam, Rassias and Găvruta.

1. Introduction

It is well known that the stability of functional equations had been first posed by Ulam [See 12] in 1940. In 1941 Hyers [3] showed that if for fixed $\delta > 0$, $f : X \longrightarrow Y$ with X, Y Banach spaces, satisfies that

$$\|f(x+y) - f(x) - f(y)\| \leq \delta \quad \text{for all } x, y \in X,$$

then there exists a unique additive mapping $g : X \longrightarrow Y$ such that

$$\|f(x) - g(x)\| \leq \delta,$$

for all $x, y \in X$. Furthermore if $f(tx)$ is continuous in t for each x, then g is linear. In 1978, a generalized solution to the Ulam problem for approximately additive mapping was given by Rassias [11]. In 1994, Găvruta [2] verified the following theorem : Let X be an abelian group and Y a Banach space. Let $\varphi : X \times X \longrightarrow [0, \infty)$ be a mapping such that

$$\Phi(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \varphi(2^{i-1}x, 2^{i-1}y) < \infty$$

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for all $x, y \in X$. If function $f : X \longrightarrow Y$ satisfies

$$\|f(x+y) - f(x) - f(y)\| \leq \varphi(x, y)$$

for all $x, y \in X$, then there is a unique mapping $g : X \longrightarrow Y$ such that

$$g(x+y) = g(x) + g(y) \quad \text{for all } x, y \in G,$$

and

$$\|f(x) - g(x)\| \leq \Phi(x, x) \quad \text{for all } x \in G.$$

Later, many Rassias and Găvruta type theorems concerning the stability of different functional equations were obtained by numerous authors (see for instance, [1, 4-9]).

In this paper we deal with general Euler-Lagrange functional equation

$$(1) \qquad f(a_1x + a_2y) + f(a_2x - a_1y) = (a_1^2 + a_2^2)(f(x) + f(y)).$$

Rassias [10] investigate Hyers-Ulam stability of this equation. We prove the stability of equation (1) in the spirit of Hyers, Ulam, Rassias and Găvruta.

2. A generalized stability

Throughout this section X and Y will be a normed linear space and Banach space, respectively.

Let $\varphi : X \times X \longrightarrow [0, \infty)$ be a mapping such that

$$\Phi(x, y) = \sum_{i=1}^{\infty} \frac{1}{m^{2i-1}} \varphi(m^{i-1}x, m^{i-1}y) < \infty$$

for all $x, y \in X$. Note that

$$\sum_{i=1}^{\infty} \frac{1}{m^{2i}} \varphi(a_1 m^{i-1}x, a_2 m^{i-1}y) = \frac{1}{m}\Phi(a_1x, a_2y) < \infty$$

for all $x, y \in X$.

THEOREM 1. *If the function $f : X \longrightarrow Y$ satisfies*

$$(2) \qquad \begin{aligned} &\|f(a_1x + a_2y) + f(a_2x - a_1y) - (a_1^2 + a_2^2)(f(x) + f(y))\| \\ &\leq \varphi(x, y) \end{aligned}$$

for every $x, y \in X$ and fixed $a_1, a_2 \in R$ with $a_1^2 + a_2^2 > 1$, then there exists a unique function $g : X \longrightarrow Y$ such that

$$g(a_1x + a_2y) + g(a_2x - a_1y) = (a_1^2 + a_2^2)(g(x) + g(y))$$

for all $x \in X$ and

$$\|f(x) - g(x)\| \leq \frac{1}{m}\Phi(a_1x, a_2x) + \Phi(x, 0) + \frac{m^2+1}{m^2-1}\|f(0)\|$$

for all $x \in X$, where $m = a_1^2 + a_2^2$. The function g is given by

$$g(x) = \lim_{n\to\infty} \frac{f((a_1^2 + a_2^2)^n x)}{(a_1^2 + a_2^2)^{2n}}$$

for all $x \in X$.

PROOF. Putting $y = 0$ and dividing by m in (2), we get

$$\left\|\frac{f(a_1x) + f(a_2x)}{m} - f(x)\right\| \leq \frac{\varphi(x, 0)}{m} + \|f(0)\| \tag{3}$$

for all $x \in X$. Replacing x by a_1x and y by a_2x, we obtain

$$\|f(mx) + f(0) - m(f(a_1x) + f(a_2x))\| \leq \varphi(a_1x, a_2x) \tag{4}$$

for all $x \in X$. By dividing by m^2 in (4), we have

$$\left\|\frac{f(mx)}{m^2} - \frac{f(a_1x) + f(a_2x)}{m}\right\| \leq \frac{1}{m^2}\varphi(a_1x, a_2x) + \frac{\|f(0)\|}{m^2}. \tag{5}$$

By (3) and (5), we have

$$\left\|f(x) - \frac{f(mx)}{m^2}\right\| \leq \frac{\varphi(x, 0)}{m} + \frac{1}{m^2}\varphi(a_1x, a_2x) + \|f(0)\| + \frac{\|f(0)\|}{m^2} \tag{6}$$

for all $x \in X$. Replacing x by mx and dividing by m^2 in (6), we obtain

$$\begin{aligned} &\left\|\frac{f(mx)}{m^2} - \frac{f(m^2x)}{m^4}\right\| \\ &\leq \frac{\varphi(mx, 0)}{m^3} + \frac{\varphi(a_1mx, a_2mx)}{m^4} + \frac{\|f(0)\|}{m^2} + \frac{\|f(0)\|}{m^4} \end{aligned} \tag{7}$$

for all $x \in X$. Replacing x by m^{n-1} and dividing by $m^{2(n-1)}$ in (6), we have

$$\begin{aligned}\left\|\frac{f(m^{n-1}x)}{m^{2(n-1)}} - \frac{f(m^n x)}{m^{2n}}\right\| \le\ & \frac{1}{m^{2n}}\varphi(a_1 m^{n-1}x, a_2 m^{n-1}x) \\ & + \frac{1}{m^{2n-1}}\varphi(m^{n-1}x, 0) \\ & + \|f(0)\|\left(\frac{1}{m^{2n}} + \frac{1}{m^{2(n-1)}}\right)\end{aligned}$$

for all $x \in X$. Induction argument implies

$$\begin{aligned}\left\|f(x) - \frac{f(m^n x)}{m^{2n}}\right\| \le\ & \sum_{i=1}^{n} \frac{1}{m^{2i}}\varphi(a_1 m^{i-1}x, a_2 m^{i-1}x) \\ & + \sum_{i=1}^{n} \frac{1}{m^{2i-1}}\varphi(m^{i-1}x, 0) \\ & + \sum_{i=1}^{n}\left(\frac{1}{m^{2i}} + \frac{1}{m^{2(i-1)}}\right)\|f(0)\| \\ \le\ & \frac{1}{m}\Phi(a_1x, a_2x) + \Phi(x, 0) + \frac{m^2+1}{m^2-1}\|f(0)\| \qquad (8)\end{aligned}$$

for all $x \in X$ and $n \in N$. Hence

$$\begin{aligned}\left\|\frac{f(m^n x)}{m^{2n}} - \frac{f(m^k x)}{m^{2k}}\right\| \le\ & \sum_{i=k+1}^{n} \frac{1}{m^{2i}}\varphi(a_1 m^{i-1}x, a_2 m^{i-1}x) \\ & + \sum_{i=k+1}^{n} \frac{1}{m^{2i-1}}\varphi(m^{i-1}x, 0) \\ & + \sum_{i=k+1}^{n}\left(\frac{1}{m^{2i}} + \frac{1}{m^{2(i-1)}}\right)\|f(0)\|\end{aligned}$$

for all $x \in X$ and $n, k \in N$ with $n > k$. This shows that $\left\{\frac{f(m^n x)}{m^{2n}}\right\}$ is a Cauchy sequence for all $x \in X$ and thus converges. Therefore we can define a function $g : X \longrightarrow Y$ by

$$g(x) := \lim_{n\to\infty} \frac{f(m^n x)}{m^{2n}} \quad \text{for all } x \in X.$$

By (2) we have

$$\begin{aligned}
&\left\|g(a_1x+a_2y)+g(a_2x-a_1y)-(a_1^2+a_2^2)(g(x)+g(y))\right\| \\
=&\lim_{n\to\infty}\frac{1}{m^{2n}}\|f(a_1m^nx+a_2m^ny)+f(a_2m^nx-a_1m^ny) \\
&\qquad -(a_1^2+a_2^2)(f(m^nx)+f(m^ny))\| \\
=&\lim_{n\to\infty}\frac{1}{m^{2n}}\varphi(m^nx,m^ny) \\
=&\,0
\end{aligned}$$

for all $x, y \in X$ and for all $n \in N$. Hence

$$g(a_1x+a_2y)+g(a_2x-a_1y)=(a_1^2+a_2^2)(g(x)+g(y))$$

for all $x, y \in X$. Taking the limit as $n \to \infty$ in (8), we have

$$\|f(x)-g(x)\| \le \frac{1}{m}\Phi(a_1x,a_2x)+\Phi(x,0)+\frac{m^2+1}{m^2-1}\|f(0)\| \tag{9}$$

for all $x \in X$. Now we prove the uniqueness. Let h be an another function satisfying;

$$h(a_1x+a_2y)+h(a_2x-a_1y)=(a_1^2+a_2^2)(h(x)+h(y)) \tag{10}$$

for all $x, y \in X$ and

$$\|f(x)-h(x)\| \le \frac{1}{m}\Phi(a_1x,a_2x)+\Phi(x,0)+\frac{m^2+1}{m^2-1}\|f(0)\| \tag{11}$$

for all $x \in X$. Then $h(0) = 0$. Replacing y by 0 in (10), we have

$$h(a_1x)+h(a_2x)=mh(x) \tag{12}$$

for all $x \in X$. Replacing x by a_1x and y by a_2x in (10), we get

$$h(mx)=m(h(a_1x)+h(a_2x)) \tag{13}$$

for all $x \in X$ and $n \in N$. By (12) and (13)

$$h(x)=\frac{h(mx)}{m^2}=\cdots=\frac{h(m^nx)}{m^{2n}}$$

for all $x \in X$. Hence

$$\begin{aligned}
\|g(x)-h(x)\| &\le \left\|\frac{f(m^nx)}{m^{2n}}-\frac{g(m^nx)}{m^{2n}}\right\|+\left\|\frac{f(m^nx)}{m^{2n}}-\frac{h(m^nx)}{m^{2n}}\right\| \\
&\le \frac{2}{m^{2n}}\Big\{\frac{1}{m}\Phi(a_1m^nx,a_2m^nx)+\Phi(m^nx,0) \\
&\qquad +\frac{m^2+1}{m^2-1}\|f(0)\|\Big\} \\
&= \frac{2}{m^{2n}}\Big\{\sum_{i=1}^{\infty}\frac{1}{m^{2i}}\varphi(a_1m^{n+i-1}x,a_2m^{n+i-1}x) \\
&\qquad +\sum_{i=1}^{\infty}\frac{1}{m^{2i-1}}\varphi(m^{n+i-1}x,0)+\frac{m^2+1}{m^2-1}\|f(0)\|\Big\} \\
&= 2\sum_{k=n+1}^{\infty}\frac{1}{m^{2k}}\varphi(a_1m^{k-1}x,a_2m^{k-1}x) \\
&\quad +2\sum_{k=n+1}^{\infty}\frac{1}{m^{2k-1}}\varphi(m^{k-1}x,0)+\frac{2}{m^{2n}}\Big(\frac{m^2+1}{m^2-1}\|f(0)\|\Big)
\end{aligned}$$

for all $n \in N$ and $x \in X$. Therefore we can conclude that $g(x) = h(x)$ for all $x \in X$. Thus we complete the proof. □

COROLLARY 2. *Let $0 < p < 2$ and $\theta > 0$ be real numbers. If a function $f : X \longrightarrow Y$ satisfies*

$$\begin{aligned}
&\|f(a_1x+a_2y)+f(a_2x-a_1y)-(a_1^2+a_2^2)(f(x)+f(y))\| \\
&\le \theta(\|x\|^p+\|y\|^p)
\end{aligned}$$

for every $x, y \in X$ and fixed $a_1, a_2 \in R$ with $a_1^2 + a_2^2 > 1$, then there exists a unique function $g : X \longrightarrow Y$ such that

$$g(a_1x+a_2y)+g(a_2x-a_1y)=(a_1^2+a_2^2)(g(x)+g(y))$$

for all $x \in X$ and

$$\|f(x)-g(x)\| \le \frac{\theta\|x\|^p(|a_1|^p+|a_2|^p+m)}{m^2-m^p}$$

for all $x \in X$, where $m = a_1^2 + a_2^2$.

PROOF. Let $\varphi(x,y) = \theta(\|x\|^p + \|y\|^p)$ for all $x \in X$. Then

$$\begin{aligned}\frac{1}{m}\Phi(a_1x, a_2x) &= \sum_{i=1}^{\infty} \frac{1}{m^{2i}} \varphi(a_1 m^{i-1}x, a_2 m^{i-1}x) \\ &= \frac{\theta(|a_1|^p + |a_2|^p)\,\|x\|^p}{m^p} \sum_{i=1}^{\infty} \frac{1}{m^{(2-p)i}} \\ &= \frac{\theta(|a_1|^p + |a_2|^p)\,\|x\|^p}{m^2 - m^p}, \\ \Phi(x,0) &= \sum_{i=1}^{\infty} \frac{1}{m^{2i-1}} \varphi(m^{i-1}x, 0) \\ &= \frac{\theta\,\|x\|^p \cdot m}{m^p} \sum_{i=1}^{\infty} \frac{1}{m^{(2-p)i}} \\ &= \frac{\theta\,\|x\|^p \cdot m}{m^2 - m^p}\end{aligned}$$

for all $x \in X$. Replacing x, y by 0 in assumption, we have

$$\|f(0)\| \le \frac{1}{2(m-1)}\varphi(0,0) = \frac{0}{2(m-1)} = 0.$$

Hence we complete the proof. □

COROLLARY 3. *If a function $f : X \longrightarrow Y$ satisfies*

$$\left\|f(a_1x + a_2y) + f(a_2x - a_1y) - (a_1^2 + a_2^2)(f(x) + f(y))\right\| \le \delta$$

for all $x, y \in X$ fixed $a_1, a_2 \in R$ with $m = a_1^2 + a_2^2 > 1$, and $\delta > 0$, then there exists a unique function $g : X \longrightarrow Y$ such that g satisfies the Eq. (1) and

$$\|f(x) - g(x)\| \le \frac{(3m^2 - 1)\delta}{2(m-1)(m^2-1)}$$

for all $x \in X$.

PROOF. Let $\varphi(x,y) = \delta$. Then we have

$$\frac{1}{m}\Phi(a_1x, a_2x) = \frac{\delta}{m^2 - 1}$$

and

$$\Phi(x,0) = \frac{m\delta}{m^2-1}$$

for all $x \in X$. Hence

$$\frac{1}{m}\Phi(a_1x, a_2x) + \Phi(x,0) + \frac{m^2+1}{m^2-1} \cdot \frac{\delta}{2(m-1)}$$
$$= \frac{(3m^2-1)\delta}{2(m-1)(m^2-1)}$$

for all $x \in X$.

Let $\Psi : X \longrightarrow [0,\infty)$ be a mapping such that

$$\Psi(x,y) = \sum_{i=1}^{\infty} m^{2i-1}\psi\Big(\frac{x}{m^i}, \frac{y}{m^i}\Big) < \infty$$

for all $x, y \in X$. Note that

$$\sum_{i=1}^{\infty} m^{2(i-1)}\psi\Big(\frac{a_1x}{m^i}, \frac{a_2y}{m^i}\Big) = \frac{1}{m}\Psi(a_1x, a_2y) < \infty$$

for all $x, y \in X$. □

THEOREM 4. *Assume that $f : X \longrightarrow Y$ is a function for which*

$$\begin{aligned}(14)\qquad &\big\|f(a_1x + a_2y) + f(a_2x - a_1y) - (a_1^2 + a_2^2)(f(x) + f(y))\big\| \\ &\le \psi(x,y)\end{aligned}$$

hold for all $x, y \in X$ and any fixed real numbers a_1, a_2 such that $0 < m = a_1^2 + a_2^2 < 1$. Then there exists a unique function $g : X \longrightarrow Y$ such that

$$g(a_1x + a_2y) + g(a_2x - a_1y) = (a_1^2 + a_2^2)(g(x) + g(y))$$

for all $x \in X$ and

$$\|f(x) - g(x)\| \le \frac{1}{m}\Psi(a_1x, a_2x) + \Psi(x,0) + \frac{m^2+1}{1-m^2}\,\|f(0)\|$$

for all $x \in X$. The function g is given by

$$g(x) = \lim_{n\to\infty}(a_1^2 + a_2^2)^{2n} f\Big(\frac{x}{m^n}\Big)$$

for all $x \in X$.

PROOF. Putting $y = 0$ in (14), we get

$$\|f(a_1x) + f(a_2x) - mf(x)\| \le \psi(x,0) + m\|f(0)\| \tag{15}$$

for all $x \in X$. Replacing x by a_1x and y by a_2x in (14), we obtain

$$\|f(mx) - m(f(a_1x) + f(a_2x))\| \le \psi(a_1x, a_2x) + \|f(0)\| \tag{16}$$

for all $x \in X$. Multiplying by m in (15) and using (16), we have

$$\left\|f(mx) - m^2 f(x)\right\| \le \psi(a_1x, a_2x) + m\psi(x,0) + (m^2+1)\|f(0)\|$$

for all $x \in X$. Replacing x by $\frac{x}{m}$, we get

$$\left\|f(x) - m^2 f(\frac{x}{m})\right\| \le \psi\Big(\frac{a_1x}{m}, \frac{a_2x}{m}\Big) + m\psi\Big(\frac{x}{m}, 0\Big) + (m^2+1)\|f(0)\| \tag{17}$$

for all $x \in X$. Replacing x by $\frac{x}{m^{n-1}}$ and multiplying by $m^{2(n-1)}$ in (17), we have

$$\begin{aligned}&\left\|m^{2(n-1)} f\Big(\frac{x}{m^{n-1}}\Big) - m^{2n} f\Big(\frac{x}{m^n}\Big)\right\| \\ &\le m^{2(n-1)}\psi\Big(\frac{a_1x}{m^n}, \frac{a_2x}{m^n}\Big) + m^{2n-1}\psi\Big(\frac{x}{m^n}, 0\Big) + m^{2(n-1)}(m^2+1)\|f(0)\|\end{aligned}$$

for all $x \in X$. Induction argument implies

$$\begin{aligned}\left\|f(x) - m^{2n} f\Big(\frac{x}{m^n}\Big)\right\| \le \sum_{i=1}^{\infty} m^{2(i-1)}\psi\Big(\frac{a_1x}{m^i}, \frac{a_2x}{m^i}\Big) + \sum_{i=1}^{\infty} m^{2i-1}\psi\Big(\frac{x}{m^i}, 0\Big) \\ + \sum_{i=1}^{\infty} m^{2(i-1)}(m^2+1)\|f(0)\|\end{aligned} \tag{18}$$

for all $x \in X$ and $n \in N$. Hence

$$\begin{aligned}\left\|m^{2n} f\Big(\frac{x}{m^n}\Big) - m^{2k} f\Big(\frac{x}{m^k}\Big)\right\| \le & \sum_{i=k+1}^{n} m^{2(i-1)}\psi\Big(\frac{a_1x}{m^i}, \frac{a_2x}{m^i}\Big) \\ & + \sum_{i=k+1}^{n} m^{2i-1}\psi\Big(\frac{x}{m^i}, 0\Big) \\ & + \sum_{i=k+1}^{n} m^{2(i-1)}(m^2+1)\|f(0)\|\end{aligned}$$

for all $x \in X$ and $n, k \in N$ with $n > k$.

This shows that $\{m^{2n} f(\frac{x}{m^n})\}$ is a Cauchy sequence for all $x \in X$ and thus converges. Therefore we can define a function $g : X \longrightarrow Y$ by

$$g(x) := \lim_{n\to\infty} m^{2n} f(\frac{x}{m^n}) \qquad \text{for all } x \in X.$$

By (14) we have

$$g(a_1x + a_2y) + g(a_2x - a_1y) = (a_1^2 + a_2^2)(g(x) + g(y))$$

for all $x \in X$. Taking limit in (18) as n to ∞, we obtain

$$\|f(x) - g(x)\| \le \frac{1}{m}\Psi(a_1x, a_2x) + \Psi(x, 0) + \frac{m^2+1}{1-m}\|f(0)\|$$

for all $x \in X$. Now we prove the uniqueness of g. Let h be an another function satisfying

$$h(a_1x + a_2y) + h(a_2x - a_1y) = (a_1^2 + a_2^2)(h(x) + h(y)) \tag{19}$$

and

$$\|f(x) - h(x)\| \le \frac{1}{m}\Psi(a_1x, a_2x) + \Psi(x, 0) + \frac{m^2+1}{1-m}\|f(0)\|$$

for all $x \in X$. Then $h(0) = 0$. Replacing y by 0 in (19), we have

$$h(a_1x) + h(a_2x) = mh(x) \tag{20}$$

for all $x \in X$. Replacing x by $\frac{x}{m}$ and multiplying by m in (20), we get

$$h(\frac{a_1x}{m}) + h(\frac{a_2x}{m}) = mh(\frac{x}{m}) \tag{21}$$

for all $x \in X$. Replacing x by $\frac{a_1x}{m}$ and y by $\frac{a_2x}{m}$ in (19), we have

$$h(x) = m(h(\frac{a_1x}{m}) + h(\frac{a_2x}{m})) \tag{22}$$

for all $x \in X$. By (21) and (22),

$$h(x) = m^2h(\frac{x}{m}) = \cdots = m^{2n}h(\frac{x}{m^n})$$

for all $x \in X$ and for all $n \in N$. Hence

$$\begin{aligned}
&\|g(x) - h(x)\| \\
&\leq \left\|m^{2n} g(\frac{x}{m^n}) - m^{2n} f(\frac{x}{m^n})\right\| + \left\|m^{2n} f(\frac{x}{m^n}) - m^n h(\frac{x}{m^n})\right\| \\
&\leq 2m^{2n}[\frac{1}{m}\Psi(\frac{a_1 x}{m^n}, \frac{a_2 x}{m^n}) + \Psi(\frac{x}{m^n}, 0) + \frac{m^2+1}{1-m}\|f(0)\|] \\
&\longrightarrow 0
\end{aligned}$$

as $n \to \infty$ for all $x \in X$. Therefore we can conclude that $g(x) = h(x)$ for all $x \in X$. Thus g is unique. This completes the proof. □

COROLLARY 5. *Let $p > 2$ and $\theta > 0$ be real numbers. If a function $f : X \longrightarrow Y$ satisfies*

$$\begin{aligned}
&\left\|f(a_1 x + a_2 y) + f(a_2 x - a_1 y) - (a_1^2 + a_2^2)(f(x) + f(y))\right\| \\
&\leq \theta(\|x\|^p + \|y\|^p)
\end{aligned}$$

for every $x, y \in X$ and fixed $a_1, a_2 \in R$ with $m = a_1^2 + a_2^2 < 1$, then there exists a unique function $g : X \longrightarrow Y$ such that

$$g(a_1 x + a_2 y) + g(a_2 x - a_1 y) = (a_1^2 + a_2^2)(g(x) + g(y))$$

for all $x \in X$ and

$$\|f(x) - g(x)\| \leq \frac{\theta\,\|x\|^p\,(|a_1|^p + |a_2|^p + m)}{(m^p - m^2)}$$

for all $x \in X$.

PROOF. Let $\varphi(x,y) = \theta(\|x\|^p + \|y\|^p)$ for all $x \in X$. Then

$$\frac{1}{m}\Psi(a_1 x, a_2 x) = \theta(|a_1|^p + |a_2|^p)\,\|x\|^p\,\frac{1}{(m^p - m^2)}$$

and

$$\Psi(x, 0) = \theta\,\|x\|^p \cdot \frac{m}{(m^p - m^2)}$$

for all $x \in X$. □

COROLLARY 6. *If a function $f : X \to Y$ satisfies*

$$f(a_1x + a_2y) + f(a_2x - a_1y) - (a_1^2 + a_2^2)(f(x) + f(y)) \leq \delta$$

for all $x, y \in X$, fixed $a_1, a_2 \in R$ with $m = a_1^2 + a_2^2 < 1$, and $\delta > 0$, then there exists a unique function $g : X \longrightarrow Y$ such that g satisfies the Eq. (1) *and for all $x \in X$*

$$\|f(x) - g(x)\| \leq \frac{(3 - m^2)\delta}{2(1 - m)(1 - m^2)}.$$

PROOF. Let $\varphi(x, y) = \delta$. Note that $\|f(0)\| \leq \frac{\delta}{2(1-m)}$. Thus we have

$$\frac{1}{m}\Psi(a_1x, a_2x) + \Psi(x, 0) + \frac{m^2 + 1}{1 - m^2}\|f(0)\|$$
$$= \frac{(3 - m^2)\delta}{2(1 - m)(1 - m^2)}.$$

□

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