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INTEGRABILITY AS VALUES OF CUSP FORMS IN IMAGINARY QUADRATIC

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ABSTRACT. Let $\mathfrak{h}$ be the complex upper half plane, let $h(\tau)$ be a cusp form, and let τ be an imaginary quadratic in $\mathfrak{h}$. If $h(\tau) \in \Omega\ (g_2(\tau)^m g_3(\tau)^l)$ with Ω the field of algebraic numbers and m, l positive integers, then we show that $h(\tau)$ is integral over the ring $\mathbb{Q}[h(\frac{\tau}{n}) \cdots h(\frac{\tau+n-1}{n})]$.

0. Introduction

Let $\mathfrak{M}_k$ be the space of cusp forms with weight k, where k is an even integer and $k \geq 4$. It is well-known that $\mathfrak{M}_k$ has finite dimension([5]).

Let $\Delta(\tau)$ denote the modular discriminant on the upper half plane $\mathfrak{h}$ and let K be an imaginary quadratic field. In this work we study integrability as values of cusp forms in imaginary quadratic. The basic argument in the proof of theorems is the following: For any $N \geq 1$, the modular function $\Delta(Nz)/\Delta(z)$ is, when suitably normalized, integral over $\mathbb{Z}[j]$([4]). This fact leads to many interesting results in number theory and geometry. In Section 1, we consider the integrability of $\Delta(\tau)$ in imaginary quadratic. In Section 2, we consider the integrability of $f(\tau) \in \mathfrak{M}_k$ in imaginary quadratic, where the coefficients of $f(\tau)$ are algebraic numbers.

1. Infinite product formula and algebraic integer

Let $\mathfrak{h}$ be the complex upper half plane, let $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$ ($\tau \in \mathfrak{h}$) be a lattice, and let $p = e^{\pi i \tau}$. The *Eisenstein series of weight* $2k$ *(for* Λ_τ

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and $k > 1$) is the series $G_{2k}(\Lambda_\tau) = \sum_{\substack{\omega\in\Lambda_\tau \\ \omega\neq 0}} \omega^{-2k}$, and the *Weierstrass* $\wp$*-function* (*relative to* Λ_τ) is defined by the series

$$\wp(z;\Lambda_\tau) = \frac{1}{z^2} + \sum_{\substack{\omega\in\Lambda_\tau \\ \omega\neq 0}} \left\{ \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right\}.$$

We shall use the notations $\wp(z)$ instead of $\wp(z;\Lambda_\tau)$, when the lattice Λ_τ has been fixed.

As is customary, by setting

$$g_2(\tau) = g_2(\Lambda_\tau) = 60G_4(\Lambda_\tau) \quad \text{and} \quad g_3(\tau) = g_3(\Lambda_\tau) = 140G_6(\Lambda_\tau),$$

the algebraic relation between $\wp(z)$ and $\wp'(z)$ becomes

$$\begin{aligned} \wp'(z)^2 &= 4\wp(z)^3 - g_2(\tau)\wp(z) - g_3(\tau) \\ &= 4\Big(\wp(z) - \wp(\tfrac{1}{2})\Big)\Big(\wp(z) - \wp(\tfrac{\tau}{2})\Big)\Big(\wp(z) - \wp(\tfrac{\tau+1}{2})\Big). \end{aligned} \tag{1.0}$$

PROPOSITION 1.0. ([4, p. 251]) *Let* $p = e^{\pi i\tau}$.

(1) $\wp\left(\frac{\tau}{2}\right) - \wp\left(\frac{1}{2}\right) = -\pi^2\prod_{n=1}^{\infty}(1-p^{2n})^4(1+p^{2n-1})^8.$

(2) $\wp\left(\frac{\tau+1}{2}\right) - \wp\left(\frac{1}{2}\right) = -\pi^2\prod_{n=1}^{\infty}(1-p^{2n})^4(1-p^{2n-1})^8.$

(3) $\wp\left(\frac{\tau+1}{2}\right) - \wp\left(\frac{\tau}{2}\right) = 16\pi^2 p\prod_{n=1}^{\infty}(1-p^{2n})^4(1+p^{2n})^8.$

In [2] and [3], we derive that

$$\wp\left(\frac{\tau}{2}\right) = -\frac{\pi^2}{3}\prod_{n=1}^{\infty}(1-p^{2n})^4\left(\prod_{n=1}^{\infty}(1+p^{2n-1})^8 + 16p\prod_{n=1}^{\infty}(1+p^{2n})^8\right),$$

$$\wp\left(\frac{\tau+1}{2}\right) = -\frac{\pi^2}{3}\prod_{n=1}^{\infty}(1-p^{2n})^4\left(\prod_{n=1}^{\infty}(1+p^{2n-1})^8 - 32p\prod_{n=1}^{\infty}(1+p^{2n})^8\right),$$

$$\wp\left(\frac{1}{2}\right) = \frac{\pi^2}{3}\prod_{n=1}^{\infty}(1-p^{2n})^4\left(2\prod_{n=1}^{\infty}(1+p^{2n-1})^8 - 16p\prod_{n=1}^{\infty}(1+p^{2n-1})^8\right).$$

By above equations of $\wp(\frac{\tau}{2})$, $\wp(\frac{\tau+1}{2})$, $\wp(\frac{1}{2})$ and (1.0), we give the equations of $g_2(\tau)$ and $g_3(\tau)$,

$$\begin{aligned} g_2(\tau) = \frac{4\pi^4}{3}\prod_{n=1}^{\infty}(1-p^{2n})^8\Big[\prod_{n=1}^{\infty}(1+p^{2n-1})^{16} - 16p\prod_{n=1}^{\infty}(1+p^n)^8 \\ + 256p^2\prod_{n=1}^{\infty}(1+p^{2n})^{16}\Big], \end{aligned} \tag{1.1}$$

$$\begin{aligned} g_3(\tau) = \frac{8\pi^6}{27}\prod_{n=1}^{\infty}(1-p^{2n})^{12}\Big(\prod_{n=1}^{\infty}(1+p^{2n-1})^{24} \\ - 24p\prod_{n=1}^{\infty}(1+p^{2n-1})^{16}(1+p^{2n})^8 \\ - 384p^2\prod_{n=1}^{\infty}(1+p^{2n-1})^8(1+p^{2n})^{16} \\ + 4096p^3\prod_{n=1}^{\infty}(1+p^{2n})^{24}\Big). \end{aligned} \tag{1.2}$$

We consider the formula for modular discriminant $\Delta(\tau) = (2\pi)^{12}\eta(\tau)^{24} = g_2(\tau)^3 - 27g_3(\tau)^2$, where the Dedekind η-function is given by the infinite product $\eta(\tau) = p^{\frac{1}{12}}\prod_{n=1}^{\infty}(1-p^{2n})$.

Let $\alpha = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with b mod d and $|\alpha|$ the determinant of α, and let

$$\phi_\alpha(\tau) := |\alpha|^{12}\frac{\Delta\left(\alpha\binom{\tau}{1}\right)}{\Delta\left(\binom{\tau}{1}\right)} = |\alpha|^{12}d^{-12}\frac{\Delta(\alpha\tau)}{\Delta(\tau)}.$$

We begin with an important proposition which tells us when the value $\phi_\alpha(\tau)$ is an algebraic integer.

PROPOSITION 1.1. ([4, p. 164]) *For any $z \in K \cap \mathfrak{h}$, the value $\phi_\alpha(z)$ is an algebraic integer, which divide $|\alpha|^{12}$.*

Let n be any positive integer, and let $\alpha_j = \begin{pmatrix} 1 & j \\ 0 & n \end{pmatrix}$ with $j = 0, 1, \cdots, n-1$. Then,

$$\phi_{\alpha_j}(\tau) = \frac{1}{n^{12}}n^{12}\frac{\Delta(\frac{\tau+j}{n})}{\Delta(\tau)} = \frac{\eta(\frac{\tau+j}{n})^{24}}{\eta(\tau)^{24}} \tag{1.3}$$

is an algebraic integer for all j, which divides n^{12}. Thus,

$$\phi_{\alpha_0}(\tau)\cdots\phi_{\alpha_{n-1}}(\tau) = \frac{\Delta(\frac{\tau}{n})\Delta(\frac{\tau+1}{n})\cdots\Delta(\frac{\tau+n-1}{n})}{\Delta(\tau)^n}$$

is an algebraic integer dividing n^{12n}. So, there exists

$$F(x) = x^m + a_{m-1}x^{m-1} + \cdots + a_0 \in \mathbb{Z}[x]$$

satisfying

$$F\left(\frac{\Delta(\frac{\tau}{n})\Delta(\frac{\tau+1}{n})\cdots\Delta(\frac{\tau+n-1}{n})}{\Delta(\tau)^n}\right) = 0.$$

Thus $a_0\Delta(\tau)^{mn} + \cdots + \Delta(\frac{\tau}{n})^m\Delta(\frac{\tau+1}{n})^m\cdots\Delta(\frac{\tau+n-1}{n})^m = 0$. From this, we have the following:

THEOREM 1.2. *Let n be any positive integer, and let $\tau \in \mathfrak{h} \cap K$. Then $\Delta(\tau)$ is integral over $\mathbb{Q}$ $[\Delta(\frac{\tau}{n}), \Delta(\frac{\tau+1}{n}), \cdots, \Delta(\frac{\tau+n-1}{n})]$.*

First, we consider

$$\frac{\Delta(2\tau)}{\Delta(\tau)} = \frac{(2\pi)^{12}p^4\prod_{n=1}^{\infty}(1-p^{4n})^{24}}{(2\pi)^{12}p^2\prod_{n=1}^{\infty}(1-p^{2n})^{24}} = p^2\prod_{n=1}^{\infty}(1+p^{2n})^{24}$$

and

$$\frac{\Delta(\tau)}{\Delta(2\tau)} = \frac{(2\pi)^{12}p^2\prod_{n=1}^{\infty}(1-p^{2n})^{24}}{(2\pi)^{12}p^4\prod_{n=1}^{\infty}(1-p^{4n})^{24}} = p^{-2}\frac{1}{\prod_{n=1}^{\infty}(1+p^{2n})^{24}}.$$

Let $\beta_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $\beta_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

By (1.3), we derive that

$$\phi_{\beta_1}(\tau) = 2^{12}\frac{\Delta(2\tau)}{\Delta(\tau)} = 2^{12}\frac{\eta(2\tau)^{24}}{\eta(\tau)^{24}},$$

$$\phi_{\beta_2}(2\tau) = 2^{12}\frac{1}{2^{12}}\frac{\Delta(\tau)}{\Delta(2\tau)} = \frac{\eta(\tau)^{24}}{\eta(2\tau)^{24}},$$

and thus

$$\text{(1.4)} \qquad \sqrt{2}p^{\frac{1}{24}}\prod_{n=1}^{\infty}(1+p^n) \quad \text{and} \quad p^{-\frac{1}{24}}\frac{1}{\prod_{n=1}^{\infty}(1+p^n)}$$

are algebraic integers. By (1.1), (1.2), (1.3), and (1.4), we get the following:

PROPOSITION 1.3. ([2]) *Let $\tau \in K \cap \mathfrak{h}$. Then,*

(a) $\sqrt{2}p^{\frac{1}{24}}\prod_{n=1}^{\infty}(1+p^n)$, $p^{-\frac{1}{24}}\prod_{n=1}^{\infty}(1+p^{2n-1})$, $\sqrt{2}\prod_{n=1}^{\infty}(1+p^n)(1+p^{2n-1})$, *and* $p^{-\frac{1}{24}}\prod_{n=1}^{\infty}(1-p^{2n-1})$ *are algebraic integers.*

(b) $\frac{3}{\pi^2}\frac{\wp(\frac{\tau}{2})}{\eta(\tau)^4}$, $\frac{3}{4\pi^4}\frac{g_2(\tau)}{\eta(\tau)^8}$, *and* $\frac{27}{\pi^6}\frac{g_3(\tau)}{\eta(\tau)^{12}}$ *are algebraic integers.*

It is well-known that the natural logarithm $\log\beta$ is transcendental for any algebraic $\beta \neq 0, 1$ ([1]). Thus by Proposition 1.3(a), we get the following:

COROLLARY 1.4. *Let $\tau \in K \cap \mathfrak{h}$. Then,* $\frac{1}{24}\pi i\tau + \sum_{n=1}^{\infty}\log(1+e^{n\pi i\tau})$, $-\frac{1}{24}\pi i\tau + \sum_{n=1}^{\infty}\log(1+e^{(2n-1)\pi i\tau})$ *and* $-\frac{1}{24}\pi i\tau + \sum_{n=1}^{\infty}\log(1-e^{(2n-1)\pi i\tau})$ *are transcendental numbers.*

The Weber functions are defined by

$$\begin{aligned} h_1(z) &= -\frac{2^7 3^5 g_2(\tau) g_3(\tau)}{\Delta(\tau)}\wp(z), \\ h_2(z) &= \frac{2^8 3^4 g_2^2(\tau)}{\Delta(\tau)}\wp(z)^2, \\ h_3(z) &= -\frac{2^9 3^6 g_3(\tau)}{\Delta(\tau)}\wp(z)^3. \end{aligned}$$

Taking $z = \frac{\tau}{2}$, we obtain that

$$\begin{aligned} h_1(\frac{\tau}{2}) &= -\frac{1}{8}\cdot\frac{3}{4\pi^4}\frac{g_2(\tau)}{\eta(\tau)^8}\cdot\frac{3^3 g_3(\tau)}{\pi^6\eta(\tau)^{12}}\cdot\frac{3\wp(\frac{\tau}{2})}{\pi^2\eta(\tau)^4}, \\ h_2(\frac{\tau}{2}) &= \frac{3^2}{2^4\pi^8}\frac{g_2^2(\tau)}{\eta(\tau)^{16}}\cdot\frac{3^2\wp(\frac{\tau}{2})^2}{\pi^4\eta(\tau)^8}, \\ h_3(\frac{\tau}{2}) &= -\frac{1}{8}\cdot\frac{3^3}{\pi^6}\frac{g_3(\tau)}{\eta(\tau)^{12}}\cdot\frac{3^3\wp(\frac{\tau}{2})^3}{\pi^6\eta(\tau)^{12}}. \end{aligned}$$

Then by what we have got just above and Proposition 1.3, we get the following:

COROLLARY 1.5. $8h_1(\frac{\tau}{2})$, $h_2(\frac{\tau}{2})$ *and* $8h_3(\frac{\tau}{2})$ *are algebraic integers.*

2. $g_2(\tau)$ and $g_3(\tau)$

Let n be any positive integer. By Proposition 1.3, we come up with

$$\frac{g_2(\frac{\tau}{n})}{\pi^4\eta(\frac{\tau}{n})^8} \quad \text{and} \quad \frac{\pi^4\eta(\tau)^8}{g_2(\tau)}$$

are algebraic numbers.

Thus we get $\frac{g_2(\frac{\tau}{n})}{g_2(\tau)} \cdot \frac{\eta(\tau)^8}{\eta(\frac{\tau}{n})^8}$ is an algebraic number. Also, $\frac{g_2(\frac{\tau}{n})}{g_2(\tau)}$ is an algebraic number, since $\frac{\eta(\tau)}{\eta(\frac{\tau}{n})}$ is an algebraic number.

Similarly, we get

$$\frac{g_2(\frac{\tau+j}{n})}{g_2(\tau)} \tag{2.1}$$

is an algebraic number with $j = 1, \cdots, n-1$.

Thus, we deduce from (2.1) that

$$\frac{g_2(\frac{\tau}{n})g_2(\frac{\tau+1}{n})\cdots g_2(\frac{\tau+n-1}{n})}{g_2(\tau)^n}$$

is an algebraic number.

This implies that there exists $F(x) = x^m + b_{m-1}x^{m-1} + \cdots + b_0$ in $\mathbb{Q}[x]$ satisfying

$$\begin{aligned} &F\Big(\frac{g_2(\frac{\tau}{n})g_2(\frac{\tau+1}{n})\cdots g_2(\frac{\tau+n-1}{n})}{g_2(\tau)^n}\Big) \\ &= \Big(\frac{g_2(\frac{\tau}{n})g_2(\frac{\tau+1}{n})\cdots g_2(\frac{\tau+n-1}{n})}{g_2(\tau)^n}\Big)^m + \cdots + b_0 \\ &= 0. \end{aligned}$$

Therefore, we get an equation

$$b_0 g_2(\tau)^{mn} + \cdots + g_2(\frac{\tau}{n})^m g_2(\frac{\tau+1}{n})^m \cdots g_2(\frac{\tau+n-1}{n})^m = 0.$$

In a similar way, we are working with the matrices $\begin{pmatrix} 1 & j \\ 0 & n \end{pmatrix}$ $(0 \le j \le n-1)$, we derive that

$$\frac{g_3(\frac{\tau+j}{n})}{g_3(\tau)} \tag{2.2}$$

are algebraic numbers with $j = 1, \cdots, n-1$; hence

$$\frac{g_3(\frac{\tau}{n}) \cdots g_3(\frac{\tau+n-1}{n})}{g_3(\tau)^n}$$

is an algebraic number.

Thus, we get the following:

THEOREM 2.1. *Let n be any positive integer, let $\tau \in K \cap \mathfrak{h}$. Then $g_2(\tau)$ (respectively, $g_3(\tau)$) is integral over $\mathbb{Q}[\, g_2(\frac{\tau}{n})\, g_2(\frac{\tau+1}{n}) \cdots g_2\,(\frac{\tau+n-1}{n})]$ (respectively, $\mathbb{Q}\,[g_3\,(\frac{\tau}{n}) \cdots g_3\,(\frac{\tau+n-1}{n})]$).*

We shall generalize Theorem 2.1. Let

$$f(\tau) := \sum_i^{\text{finite}} e_i g_2(\tau)^{a_i} g_3(\tau)^{b_i} \Delta(\tau)^{c_i}$$

where all e_i are algebraic numbers, $4a_i + 6b_i + 12c_i = k$ for all i. In fact, since $\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2$, we shall take $f(\tau) \in \Omega[g_2(\tau), g_3(\tau)]$ with Ω the field of algebraic numbers.

Then we get

$$\begin{aligned}
\frac{f(\frac{\tau}{n})}{f(\tau)} &= \frac{\sum_i^{\text{finite}} e_i g_2(\frac{\tau}{n})^{a_i} g_3(\frac{\tau}{n})^{b_i} \Delta(\frac{\tau}{n})^{c_i}}{\sum_i^{\text{finite}} e_i g_2(\tau)^{a_i} g_3(\tau)^{b_i} \Delta(\tau)^{c_i}} \\
&= \frac{\sum_i^{\text{finite}} \frac{e_i g_2(\frac{\tau}{n})^{a_i} g_3(\frac{\tau}{n})^{b_i} \Delta(\frac{\tau}{n})^{c_i}}{\pi^k \eta(\frac{\tau}{n})^k}}{\sum_i^{\text{finite}} \frac{e_i g_2(\tau)^{a_i} g_3(\tau)^{b_i} \Delta(\tau)^{c_i}}{\pi^k \eta(\tau)^k}} \frac{\eta(\frac{\tau}{n})^k}{\eta(\tau)^k} \\
&= \frac{\sum_i^{\text{finite}} e_i \cdot \frac{g_2(\frac{\tau}{n})^{a_i}}{\pi^{4a_i}\eta(\frac{\tau}{n})^{4a_i}} \cdot \frac{g_3(\frac{\tau}{n})^{6b_i}}{\pi^{6b_i}\eta(\frac{\tau}{n})^{6b_i}} \cdot \frac{\Delta(\frac{\tau}{n})^{c_i}}{\pi^{12i}\eta(\frac{\tau}{n})^{12i}}}{\sum_i^{\text{finite}} e_i \cdot \frac{g_2(\tau)^{a_i}}{\pi^{4a_i}\eta(\tau)^{4a_i}} \cdot \frac{g_3(\tau)^{6b_i}}{\pi^{6b_i}\eta(\tau)^{6b_i}} \cdot \frac{\Delta(\tau)^{c_i}}{\pi^{12i}\eta(\tau)^{12i}}} \cdot \frac{\eta(\frac{\tau}{n})^k}{\eta(\tau)^k}.
\end{aligned}$$

Since

$$e_i, \ \frac{g_2(\frac{\tau}{n})^{a_i}}{\pi^{4a_i}\eta(\frac{\tau}{n})^{4a_i}}, \ \frac{g_3(\frac{\tau}{n})^{b_i}}{\pi^{6b_i}\eta(\tau)^{6b_i}}, \ \frac{\Delta(\frac{\tau}{n})^{c_i}}{\pi^{12c_i}\eta(\frac{\tau}{n})^{12c_i}}, \ \text{and} \ \frac{\eta(\frac{\tau}{n})}{\eta(\tau)}$$

are algebraic numbers, $\dfrac{f(\frac{\tau}{n})}{f(\tau)}$ is an algebraic number. Similarly, $\dfrac{f(\frac{\tau+i}{n})}{f(\tau)}$ is an algebraic number for $i = 1, \cdots, n-1$. Thus

$$\prod_{i=0}^{n-1} \frac{f(\frac{\tau+i}{n})}{f(\tau)^n}$$

is an algebraic number.

And let

$$g(\tau) = \sum_{j}^{\text{finite}} h_j g_2(\tau)^{a'_j} g_3(\tau)^{b'_j} \Delta(\tau)^{c'_j},$$

where h_j are all algebraic numbers, $4a'_j + 6b'_j + 12c'_j = k'$ for all j. Similarly, $\dfrac{g(\tau)}{g(\frac{\tau+j}{n})}$ is an algebraic number, for all $j = 0, 1, \cdots, n-1$. And let $h(\tau) = \frac{f(\tau)}{g(\tau)}$. By the same method as above, we get

$$\frac{h(\frac{\tau}{n})}{h(\tau)} = \frac{\sum_i^{\text{finite}} e_i g_2(\frac{\tau}{n})^{a_i} g_3(\frac{\tau}{n})^{b_i} \Delta(\frac{\tau}{n})^{c_i}}{\sum_j^{\text{finite}} h_j g_2(\frac{\tau}{n})^{a'_j} g_3(\frac{\tau}{n})^{b'_j} \Delta(\frac{\tau}{n})^{c'_j}} \cdot \frac{\sum_j^{\text{finite}} h_j g_2(\tau)^{a'_j} g_3(\tau)^{b'_j} \Delta(\tau)^{c'_j}}{\sum_i^{\text{finite}} e_i g_2(\tau)^{a_i} g_3(\tau)^{b_i} \Delta(\tau)^{c_i}}.$$

In other words, we get

$$\frac{\frac{\sum_i^{\text{finite}} e_i g_2(\frac{\tau}{n})^{a_i} g_3(\frac{\tau}{n})^{b_i} \Delta(\frac{\tau}{n})^{c_i}}{\eta(\frac{\tau}{n})^k}}{\frac{\sum_j^{\text{finite}} h_j g_2(\frac{\tau}{n})^{a'_j} g_3(\frac{\tau}{n})^{b'_j} \Delta(\frac{\tau}{n})^{c'_j}}{\eta(\frac{\tau}{n})^{k'}}} \cdot \frac{\frac{\sum_j^{\text{finite}} h_j g_2(\tau)^{a'_j} g_3(\tau)^{b'_j} \Delta(\tau)^{c'_j}}{\eta(\tau)^{k'}}}{\frac{\sum_i^{\text{finite}} e_i g_2(\tau)^{a_i} g_3(\tau)^{b_i} \Delta(\tau)^{c_i}}{\eta(\tau)^k}} \cdot \frac{\eta(\tau)^{k'} \eta(\frac{\tau}{n})^k}{\eta(\frac{\tau}{n})^{k'} \eta(\tau)^k}.$$

Since each term is an algebraic number, so is $\dfrac{h(\frac{\tau}{n})}{h(\tau)}$. Similarly, $\dfrac{h(\frac{\tau+j}{n})}{h(\tau)}$ is also an algebraic number with j any integer.

Therefore,

$$\prod_{i=0}^{n-1} \left(\frac{f(\frac{\tau+i}{n})}{g(\frac{\tau+i}{n})} \middle/ \frac{f(\tau)}{g(\tau)} \right)$$

is an algebraic number. Consequently, there exists $F(x) = x^d + \cdots + c_d \in \mathbb{Q}[x]$ satisfying $F\left(\prod_{i=0}^{n-1} \left(\frac{f(\frac{\tau+i}{n})}{g(\frac{\tau+i}{n})} \middle/ \frac{f(\tau)}{g(\tau)} \right) \right) = 0$.

So,

$$c_d \left(\frac{f(\tau)}{g(\tau)} \right)^{nd} + \cdots + \left(\frac{f(\frac{\tau}{n})}{g(\frac{\tau}{n})} \right)^d \cdots \left(\frac{f(\frac{\tau+n-1}{n})}{g(\frac{\tau+n-1}{n})} \right)^d = 0.$$

Thus we get the following:

THEOREM 2.2. *Let n be any positive integer, let $\tau \in K \cap \mathfrak{h}$, and let $f(\tau),\ g(\tau) \in \Omega(g_2(\tau), g_3(\tau))$ with homogeneous degree k and k', where Ω is the field of algebraic numbers. Then $h(\tau) = \frac{f(\tau)}{g(\tau)}$ is integral over $\mathbb{Q}$ $[h(\frac{\tau}{n}) \cdots h(\frac{\tau+n-1}{n})]$. Also $h(\tau)$ is integral over $\mathbb{Q}$ $[h(\frac{\tau+i_1}{n}) \cdots h(\frac{\tau+i_n}{n})]$ with $i_1, \cdots, i_n$ integers.*

If $\dim\mathfrak{M}_k = 1$ and $f(\tau) \in \mathfrak{M}_k$, then $f(\tau) = rg_2(\tau)^a g_3(\tau)^b \Delta(\tau)^c$ with $4a + 6b + 12c = k$ and $r \in \mathbb{C}$. And we get

$$\frac{f(\frac{\tau}{n})}{f(\tau)} = \frac{rg_2(\frac{\tau}{n})^a g_3(\frac{\tau}{n})^b \Delta(\frac{\tau}{n})^c}{rg_2(\tau)^a g_3(\tau)^b \Delta(\tau)^c} = \left(\frac{g_2(\frac{\tau}{n})}{g_2(\tau)}\right)^a \left(\frac{g_3(\frac{\tau}{n})}{g_3(\tau)}\right)^b \left(\frac{\Delta(\frac{\tau}{n})}{\Delta(\tau)}\right)^c.$$

By (1.3), (2.1) and (2.2), we get $\dfrac{f(\frac{\tau+i}{n})}{f(\tau)}$ is an algebraic number with $i \in \mathbb{Z}$. Thus, there exists $F(x) = a_0x^m + \cdots + a_m \in \mathbb{Q}[x]$ such that $F\left(\prod_{i=0}^{n-1} \dfrac{f(\frac{\tau+i}{n})}{f(\tau)^n}\right) = 0$. Thus we have the following:

COROLLARY 2.3. *Let n be any positive integer, let $\tau \in K \cap \mathfrak{h}$, and let $f(\tau) \in \mathbb{C}(g_2(\tau), g_3(\tau))$ be a polynomial with homogeneous degree in $\mathfrak{M}_k$, and $\dim\mathfrak{M}_k = 1$. Then $f(\tau)$ is integral over $\mathbb{Q}$ $[f(\frac{\tau}{n}) \cdots f(\frac{\tau+n-1}{n})]$. Also $f(\tau)$ is integral over $\mathbb{Q}$ $[f(\frac{\tau+i_1}{n}) \cdots f(\frac{\tau+i_n}{n})]$ with $i_1, \cdots, i_n$ integers.*

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