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IDEALS AND QUOTIENTS OF INCLINE ALGEBRAS

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ABSTRACT. In this paper we introduce the notion of quotient incline and obtain the structure of incline algebra. Moreover, we also introduce the notion of prime and maximal ideal in incline, and study some relations between them in incline algebra.

0. Introduction

Z. Q. Cao, K. H. Kim, and F. W. Roush [3] introduced the notion of incline algebras in their book, *Incline algebra and applications*, and was studied by some authors [1, 2, 6, 7]. Inclines are a generalization of both Boolean and fuzzy algebras, and a special type of a semiring, and they give a way to combine algebras with ordered structures to express the degree of intensity of binary relations.

An *incline* is a structure which has an associative, commutative addition, and a distributive multiplication such that $x+x=x$, $x+xy=x$ for all x, y. It has both a semiring structure and a poset structure. Inclines can also be used to represent automa and other mathematical systems, in optimization theory, to study inequalities for nonnegative matrices of polynomials. The present authors [4] considered the fuzzification of subinclines (ideals) in inclines, and also stated the product and projections of fuzzy subinclines (ideals). They discussed fuzzy relations, fuzzy characteristic subinclines (ideals) and fuzzy k-ideals. In this paper we introduce the notion of quotient incline and obtain the structure of incline algebras. Moreover, we also introduce the notion of prime and maximal ideals in an incline, and study some relations between them in incline algebras.

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1. Preliminaries

DEFINITION 1.1 ([3]). An *incline* (*algebra*) is a set $\mathcal{K}$ with two binary operations denoted by " $+$ " and " $*$ " satisfying the following axioms: for all $x, y, z \in \mathcal{K}$,

(i) $x + y = y + x$,
(ii) $x + (y + z) = (x + y) + z$,
(iii) $x * (y * z) = (x * y) * z$,
(iv) $x * (y + z) = (x * y) + (x * z)$,
(v) $(y + z) * x = (y * x) + (z * x)$,
(vi) $x + x = x$,
(vii) $x + (x * y) = x$,
(viii) $y + (x * y) = y$.

Furthermore, an incline algebra $\mathcal{K}$ is said to be *commutative* if $x*y = y*x$ for all $x, y \in \mathcal{K}$.

For convenience, we pronounce "+" (resp. "*") as *addition* (resp. *multiplication*). Every distributive lattice is an incline. An incline is a distributive lattice (as a semiring) if and only if $x * x = x$ for all $x \in \mathcal{K}$ ([3, Proposition (1.1.1)]). Note that $x \leq y \Longleftrightarrow x+y = y$ for all $x, y \in \mathcal{K}$. A *subincline* of an incline $\mathcal{K}$ is a non-empty subset M of $\mathcal{K}$ which is closed under addition and multiplication. A subincline M is said to be an *ideal* of an incline $\mathcal{K}$ if $x \in M$ and $y \leq x$ then $y \in M$. An element 0 in an incline algebra $\mathcal{K}$ is a *zero element* if $x + 0 = x = 0 + x$ and $x * 0 = 0 * x = 0$, for any $x \in \mathcal{K}$. By a *homomorphism* of inclines we shall mean a mapping f from an incline $\mathcal{K}$ into an incline $\mathcal{L}$ such that $f(x + y) = f(x) + f(y)$ and $f(x * y) = f(x) * f(y)$ for all $x, y \in \mathcal{K}$.

2. Quotient inclines

In this section, we discuss the quotient incline and investigate their properties.

The present authors [4] introduced the notion of the k-ideal, i.e., a subincline I of an incline $\mathcal{K}$ is said to be *k-ideal* if $x + y \in I$, $y \in I$, then $x \in I$. We show this notion is another equivalent definition of the ideal of an incline.

PROPOSITION 2.1. *Let I be a subincline of an incline $\mathcal{K}$. Then I is an ideal of $\mathcal{K}$ if and only if I is a k-ideal of $\mathcal{K}$.*

PROOF. Let I be an ideal of $\mathcal{K}$, and let $x \in \mathcal{K}$ and $y, z \in I$ such that $x + y = z$. Since $x + x = x$, $x + z = x + (x + y) = x + y = z$ and hence $x \leq z$. Hence $x \in I$, i.e., I is a k-ideal of $\mathcal{K}$. Conversely, assume that $y \in \mathcal{K}$ and $x \in I$ with $y \leq x$. Then $y + x = x$. Since I is a k-ideal of $\mathcal{K}$, $y \in I$, proving that I is an ideal of $\mathcal{K}$. □

Suppose that I is an ideal of an incline $\mathcal{K}$ with zero element 0. For any $x, y \in \mathcal{K}$, we define a relation $\sim$ on $\mathcal{K}$ by $x \sim y$ if and only if there exist $i_1, i_2 \in I$ such that $x + i_1 = y + i_2$. Now we prove that $\sim$ is an equivalence relation on $\mathcal{K}$. Since $x + 0 = x$ for any $x \in I$, $0 \leq x$. Since I is an ideal, we obtain $0 \in I$. Since $x + 0 = x + 0$, $x \sim x$ for any $x \in \mathcal{K}$. This means that $\sim$ is reflexive. By definition of the relation, $\sim$ is symmetric.

If $x \sim y$ and $y \sim z$, then there are i_1, i_2, i_3, and $i_4 \in I$ such that $x + i_1 = y + i_2$ and $y + i_3 = z + i_4$. Hence $(x + i_1) + i_3 = (y + i_2) + i_3 = y + (i_2 + i_3) = y + (i_3 + i_2) = (y + i_3) + i_2 = (z + i_4) + i_2$ and so $x + (i_1 + i_3) = z + (i_4 + i_2)$. Thus $x \sim z$. This shows that $\sim$ is transitive. Therefore $\sim$ is an equivalence relation on $\mathcal{K}$.

Futhermore we have the following lemma:

LEMMA 2.2. *If $a \sim b$, then $a * x \sim b * x$ and $a + x \sim b + x$ for all $x \in \mathcal{K}$, i.e., $\sim$ is a congruence relation on $\mathcal{K}$.*

PROOF. Since $a \sim b$, there exist $i_1, i_2 \in I$ such that

$$\text{(a)} \qquad a + i_1 = b + i_2.$$

Multiplying x on the right side of (a), $(a + i_1) * x = (b + i_2) * x$ and so $a * x + i_1 * x = b * x + i_2 * x$. By Definition 1.1-(vii), we have $i_1 + i_1 * x = i_1$ and so $i_1 * x \leq i_1 \in I$. Since I is an ideal of $\mathcal{K}$, $i_1 * x \in I$. Similarly, $i_2 * x \in I$ and hence $a * x \sim b * x$. Adding x on both side of (a), $(a + i_1) + x = (b + i_2) + x$. This means that $(a + x) + i_1 = (b + x) + i_2$. Hence $a + x \sim b + x$, completing the proof. □

We denote by $[x]_I := \{y \in \mathcal{K} | x \sim y\}$ the equivalence class of x determined by an ideal I. In fact, if $x \in I$, then $x * 0 = 0$ and $x + 0 = x$. Since $0, x \in I$, $x + 0 = 0 + x$, i.e., $x \sim 0$. Thus $x \in [0]_I$. Conversely, if $x \in [0]_I$, then $0 \sim x$ and so there are $i_1, i_2 \in I$ such that $0 + i_1 = x + i_2$. Hence $i_1 = x + i_2$. Since I is an ideal of $\mathcal{K}$, $x \in I$. Therefore $[0]_I = I$.

Denote by $\mathcal{K}/I = \{[x]_I | x \in \mathcal{K}\}$ the set of all equivalence classes $[x]_I$ determined by an ideal I, and we define two operations on $\mathcal{K}/I$ by

$$[a]_I + [b]_I := [a+b]_I$$

and

$$[a]_I * [b]_I := [a * b]_I.$$

Since $\sim$ is a congruence relation on $\mathcal{K}$, the operations "+" and "$*$" are well-defined. It is easy to show that $(\mathcal{K}/I, +, *)$ is an incline (algebra). Furthermore, if $\mathcal{K}$ is a commutative incline, then

$$[a]_I * [b]_I = [a * b]_I = [b * a]_I = [b]_I * [a]_I,$$

and hence $\mathcal{K}/I$ is a commutative incline. Summarizing the above facts we have:

THEOREM 2.3. *Let $\mathcal{K}$ be an (commutative) incline with zero element 0 and let I be an ideal of $\mathcal{K}$. Then $(\mathcal{K}/I; +, *)$ is also an (commutative) incline with zero element $[0]_I = I$.*

The incline $\mathcal{K}/I$ described in Theorem 2.3 is called a (*commutative*) *factor incline* or (*commutative*) *quotient incline* of $\mathcal{K}$ via an ideal I.

There are some close relations of ideals between $\mathcal{K}$ and $\mathcal{K}/I$.

THEOREM 2.4. *If I and J are any ideals of $\mathcal{K}$ and $I \subseteq J$, then*
(a) *I is also an ideal of the subincline J,*
(b) *$J/I := \{[x]_I | x \in J\}$ is an ideal of the quotient incline $\mathcal{K}/I$.*

PROOF. (a) is immediately follows from the definition of an ideal of incline.

To show (b), first we have to show that each element of J/I is an also element of $\mathcal{K}/I$. To avoid the ambiguity, we denote the element of J/I containing x by $[x]_I^J$. Let $x \in J$. If $y \in [x]_I$, then $y \in X$ and $y \sim x$ with respect to I. It follows that there exist $i_1, i_2 \in I$ such that $x + i_1 = y + i_2$. Since $I \subseteq J$ and $x \in J$, we obtain $y + i_2 \in J$. Since J is a k-ideal, it follows that $y \in J$. Hence $y \in [x]_I^J$, i.e., $[x]_I \subseteq [x]_I^J$. Obviously, $[x]_I^J \subseteq [x]_I$. This means that each element of J/I is also an element of $\mathcal{K}/I$.

Next we prove that J/I is an ideal of $\mathcal{K}/I$. Clearly, J/I is a subincline of $\mathcal{K}/I$. Since $I \subseteq J$, $[0]_I = I \in J/I$. If $[b]_I \in J/I$ and $[a]_I \leq [b]_I$ where

$[a]_I \in \mathcal{K}/I$, then $[a]_I + [b]_I = [b]_I$ and hence $[a+b]_I = [b]_I$. This means that $a + b \sim b$ with respect to I, i.e., there exist $i_1, i_2 \in I$ such that $a+b+i_1 = b+i_2$. Since J is an ideal of $\mathcal{K}$, $b \in J$ implies that $a \in J$ and hence $[a]_I \in J/I$. This means that J/I is an ideal of $\mathcal{K}/I$. It is easy to show that J/I is an ideal of $\mathcal{K}/I$, and we omit the proof. This completes the proof. □

THEOREM 2.5. *If J^* is an ideal of $\mathcal{K}/I$, then*

$$J := \cup\{\ [x]_I \mid [x]_I \in J^*\}$$

is an ideal of $\mathcal{K}$ and $I \subseteq J$.

PROOF. Since $I = [0]_I \in J^*$, $[0]_I \subseteq J$ and so $I \subseteq J$. If $x \leq y$ and $y \in J$, then $[x]_I \leq [y]_I \in J^*$. Since J^* is an ideal of $\mathcal{K}$, $[x]_I \in J^*$ and $x \in J$. This shows that J is an ideal of $\mathcal{K}$. □

The set of all ideals on $\mathcal{K}$ is denoted by $\mathcal{I}(\mathcal{K})$, and the set of all ideals containing I on $\mathcal{K}$ is denoted by $\mathcal{I}(\mathcal{K}, I)$. The mapping f from $\mathcal{I}(\mathcal{K}, I)$ to $\mathcal{I}(\mathcal{K}/I)$ is defined by, for any $J \in \mathcal{I}(K, I)$, $f(J) := J/I$. It follows from Theorems 2.4 and 2.5 that f is onto. We claim that f is one to one. In fact, let $A, B \in \mathcal{I}(\mathcal{K}, I)$ and $A \neq B$. Without loss of generality, we suppose that there is $x \in B - A$. If $f(A) = f(B)$, then $[x]_I \in f(B)$ and $[x]_I \in f(A)$. Then there exists $y \in A$ such that $[x]_I = [y]_I$, so $x \sim y$ with respect to I, i.e., there are $i_1, i_2 \in I$ such that $x + i_1 = y + i_2$. Since $i_1, y + i_2 \in A$ and A is an ideal of $\mathcal{K}$, $x \in A$, which is a contradiction to $x \notin A$. Summarizing the above facts we obtain:

THEOREM 2.6. *If I is an ideal of an incline $\mathcal{K}$ with zero element, then there is a bijection from $\mathcal{I}(\mathcal{K}, I)$ to $\mathcal{I}(\mathcal{K}/I)$.*

Suppose I is an ideal of $\mathcal{K}$. The mapping ν from $\mathcal{K}$ to $\mathcal{K}/I$ which is defined by $\nu(x) = [x]_I$ for $x \in \mathcal{K}$ satisfies $\nu(x+y) = \nu(x) + \nu(y)$ and $\nu(x * y) = \nu(x) * \nu(y)$. This means that ν is a homomorphism, called the *natural homomorphism*. By means of this terminology, Theorem 2.6 can be reformed as follows:

THEOREM 2.7. *Let I be an ideal of an incline $\mathcal{K}$. If A is an ideal of a quotient incline $\mathcal{K}/I$, then $\nu^{-1}(A)$ is an ideal of $\mathcal{K}$ containing I.*

3. Prime and maximal ideals in inclines

In this section, we define prime and maximal ideals in incline algebras and investigate their properties.

DEFINITION 3.1. Let $\mathcal{K}$ be an incline. A proper ideal P of $\mathcal{K}$ is said to be *prime* if for all $a, b \in \mathcal{K}, a * b \in P$ implies either $a \in P$ or $b \in P$. An ideal M in $\mathcal{K}$ is called a *maximal ideal* of $\mathcal{K}$ if $M \neq \mathcal{K}$ and for every ideal N with $M \subseteq N \subseteq \mathcal{K}$, either $N = M$ or $N = \mathcal{K}$.

THEOREM 3.2 ([5]). *All two-sided ideals of a ring (or semigroup) form an incline under lattice sum and product.*

EXAMPLE 3.3. The ring $(Z_6, +, \cdot)$ has 4 ideals as follows:

$$I_1 =< 0 >, \quad I_2 =< 1 >, \quad I_3 =< 2 >, \quad I_4 =< 3 > .$$

We define sum " + " and product " $*$ " on $\mathcal{I} := \{I_i \,|\, i = 1, 2, 3, 4\}$ as follows:

$+$	I_1	I_2	I_3	I_4
I_1	I_1	I_2	I_3	I_4
I_2	I_2	I_2	I_2	I_2
I_3	I_3	I_2	I_3	I_2
I_4	I_4	I_2	I_2	I_4

Table 1

$*$	I_1	I_2	I_3	I_4
I_1	I_1	I_1	I_1	I_1
I_2	I_1	I_2	I_3	I_4
I_3	I_1	I_3	I_3	I_1
I_4	I_1	I_4	I_1	I_4

Table 2

Then $(\mathcal{I}, +, *)$ is an incline algebra by Theorem 3.2 and I_1 is the zero element of $\mathcal{I}$. If we define the sets

$$L_1 := \{I_i \in \mathcal{I} | I_i \leq I_1\} = \{I_1\},$$

$$L_2 := \{I_i \in \mathcal{I} | I_i \leq I_2\} = \{I_1, I_2, I_3, I_4\},$$

$$L_3 := \{I_i \in \mathcal{I} | I_i \leq I_3\} = \{I_1, I_3\},$$

and

$$L_4 := \{I_i \in \mathcal{I} | I_i \le I_4\} = \{I_1, I_4\},$$

then all L_i are ideals of $\mathcal{I}$ and especially L_3, L_4 are both prime ideals and maximal ideals of $\mathcal{I}$.

DEFINITION 3.4. An element $1_{\mathcal{K}}$ ($\neq$ zero element) in an incline algebra $\mathcal{K}$ is called a *mutiplicative identity* if for any $x \in \mathcal{K}$, $x * 1_{\mathcal{K}} = 1_{\mathcal{K}} * x = x$.

EXAMPLE 3.5. Let $\mathcal{K}_1 := ([0,1]; \max, \min)$, $\mathcal{K}_2 := ([0,1]; +, *)$ where $x + y := \min\{x, y\}$ and $x * y := \min\{x + y, 1\}$ and $K_3 := ([0,1]; +, *)$ where $x + y := \max\{x, y\}$ and $x * y := xy$ (ordinary multiplication). Then $\mathcal{K}_i$ $(i = 1, 2, 3)$ are incline algebras. In $\mathcal{K}_3$, 1 is a (multiplicative) identity and 0 is a zero element.

DEFINITION 3.6. A non-zero element a in an incline algebra $\mathcal{K}$ with zero element is said to be a *left* (resp. *right*) *zero divisor* if there exists a non-zero $b \in \mathcal{K}$ such that $a * b = 0$ (resp. $b * a = 0$). A *zero divisor* is an element of $\mathcal{K}$ which is both a left zero divisor and a right zero divisor.

EXAMPLE 3.7. In Example 3.3, the ideals I_3, I_4 are zero divisors of $\mathcal{I}$, since $I_3 * I_4 = I_4 * I_3 = I_1$.

LEMMA 3.8. *If an incline algebra $\mathcal{K}$ with zero element satisfies the cancellation laws, i.e., for all $a, b, c \in \mathcal{K}$ with $a \neq 0$, $a * b = a * c$ or $b * a = c * a$ implies $b = c$, then it has no zero divisor.*

PROOF. Let $\mathcal{K}$ be an incline in which the cancellation laws hold, and suppose $a * b = 0$ for some $a, b \in \mathcal{K}$. We must show that either a or b is 0. If $a \neq 0$, then $a * b = 0 = a * 0$ implies that $b = 0$, by cancellation laws. Similarly, $b \neq 0$ implies that $a = 0$. □

DEFINITION 3.9. An incline $\mathcal{K}$ with multiplicative identity $1_{\mathcal{K}} \neq 0$ and zero element 0 is called an *integral incline* if it has no zero divisors. An incline $\mathcal{K}$ with multiplicative identity $1_{\mathcal{K}} \neq 0$ and zero element 0 is called a *pre-integral incline* if it has both right and left cancellation laws in $\mathcal{K}$. An element $u \in \mathcal{K}$ is called a *unit* if it has a multiplicative invertible element. An incline with $1_{\mathcal{K}} \neq 0$ and zero element in which every non-zero element is a unit is called a *field incline.*

THEOREM 3.10. *A pre-integral incline is an integral incline.*

PROOF. It follows from Lemma 3.8. □

THEOREM 3.11. *In an incline $\mathcal{K}$ with identity $1_{\mathcal{K}}$ and zero element 0, an ideal P is prime if and only if the quotient incline $\mathcal{K}/P$ is an integral incline.*

PROOF. We can easily see that $\mathcal{K}/P$ is an incline with identity $[1_{\mathcal{K}}]_P$ and zero element $[0]_P = P$. If P is prime, then $[1_{\mathcal{K}}]_P \neq P$ since $P \neq \mathcal{K}$. Futhermore, $\mathcal{K}/P$ has no zero divisors, since

$$\begin{aligned}[a]_P * [b]_P = P &\Longrightarrow [a*b]_P = P \Longrightarrow a*b \in P \\ &\Longrightarrow a \in P \text{ or } b \in P \Longrightarrow [a]_P = P \text{ or } [b]_P = P.\end{aligned}$$

Therefore, $\mathcal{K}/P$ is an integral incline.

Conversely, if $\mathcal{K}/P$ is an integral incline, then $[1_{\mathcal{K}}]_P \neq [0]_P$, $1_{\mathcal{K}} \notin P$ and so $P \neq \mathcal{K}$. Since $\mathcal{K}/P$ is an integral incline, $\mathcal{K}/P$ has no zero divisors. Suppose $a * b \in P$. Then $[a*b]_P = P$ and so $[a]_P * [b]_P = P$. Hence either $[a]_P = P$ or $[b]_P = P$, i.e., $a \in P$ or $b \in P$. Thus P is prime. □

Every incline $\mathcal{K}$ with zero element has two ideals, the improper ideal $\mathcal{K}$ and the trivial ideal $\{0\}$. For these ideals, the factor inclines are $\mathcal{K}/\mathcal{K}$, which has only one element, and $\mathcal{K}/\{0\}$, which is isomorphic to $\mathcal{K}$. A proper non-trivial ideal of an incline $\mathcal{K}$ is an ideal N of $\mathcal{K}$ such that $N \neq \mathcal{K}$ and $N \neq \{0\}$.

THEOREM 3.12. *Any commutative finite pre-integral incline $\mathcal{K}$ is a field incline.*

PROOF. For a fixed non-zero element a and $a_1, \cdots, a_n$ in $\mathcal{K}$, we consider the n products

$$a * a_1,\ a * a_2,\ \cdots,\ a * a_n.$$

These products are all distinct, since if $a * a_i = a * a_j$, by cancellation laws we obtain $a_i = a_j$. It follows that each element of $\mathcal{K}$ must be of the form $a * a_i$ for some choice of i. In particular, there exists some a_i' such that $a * a_i' = 1_{\mathcal{K}}$. From the commutativity of multiplication, we claim that

$$a^{-1} = a_i',$$

whence every non-zero element of $\mathcal{K}$ possesses a multiplicative inverse.□

COROLLARY 3.13. *Any commutative finite integral incline is a field incline.*

THEOREM 3.14. *Every field incline $\mathcal{K}$ is an integral incline.*

PROOF. Since every field incline is a commutative incline with identity $1_{\mathcal{K}}$ and zero element, we need only to prove that $\mathcal{K}$ contains no zero divisors. Assume that $a, b \in \mathcal{K}$ with $a * b = 0$. If $a \neq 0$, then it must possess a multiplicative inverse $a^{-1} \in \mathcal{K}$. Thus $a * b = 0$ yields

$$0 = a^{-1} * 0 = a^{-1} * (a * b) = (a^{-1} * a) * b = 1_{\mathcal{K}} * b = b.$$

This proves the theorem. □

THEOREM 3.15. *Every field incline $\mathcal{K}$ is a pre-integral incline.*

PROOF. It is sufficient to show that $\mathcal{K}$ satisfies cancellation laws. Suppose that $a * b = a * c$ and $a \neq 0$. Then

$$a^{-1} * (a * b) = a^{-1} * (a * c),$$

$$(a^{-1} * a) * b = (a^{-1} * a) * c,$$

$$1_{\mathcal{K}} * b = 1_{\mathcal{K}} * c.$$

Hence $b = c$. This completes the proof. □

DEFINITION 3.16. A proper ideal I of an incline $\mathcal{K}$ is said to be *irreducible* if $I = A \cap B$ implies $I = A$ or $I = B$ for any ideals A, B of $\mathcal{K}$.

EXAMPLE 3.17. In Example 3.3, the ideal L_3 is irreducible, since $L_3 = L_3 \cap L_2$.

THEOREM 3.18. *In an incline $\mathcal{K}$, the following are equivalent:*
(a) *I is an irreducible ideal,*
(b) *I is a prime ideal,*
(c) *the ideal I satisfies that, for any elements A, B in the set of all ideals on $\mathcal{K}$, $A \subseteq I$ or $B \subseteq I$ whenever $A \cap B \subseteq I$.*

PROOF. (a)$\Longrightarrow$ (b). Assume that I is not prime. Then $x * y \in I$ for some $x, y \in \mathcal{K} - I$. It is easy to show that $I \subset I_1 := I \cup \{x\}$ and $I \subset I_2 := I \cup \{y\}$ and $I = I_1 \cap I_2$, i.e., I is not irreducible, a contradiction.

(b)$\Longrightarrow$ (c). If (c) does not hold, then for some A, B in the set of all ideals on $\mathcal{K}$, $A \cap B \subseteq I$, $A \not\subseteq I$ and $B \not\subseteq I$. Then there exist a, b such that $a \in A - I$ and $b \in B - I$. It follows from $a * b \leq a$ and $a * b \leq b$ that $a * b \in A$ and $a * b \in B$, since A, B are ideals of $\mathcal{K}$. Hence $a * b \in A \cap B \subseteq I$, which contradicts to that I is prime. Therefore (c) holds.

(c) $\Longrightarrow$ (a). Suppose that there exist ideals A, B of $\mathcal{K}$ such that $I = A \cap B$. By assumption either $A \subseteq I$ or $B \subseteq I$, i.e., either $I = A$ or $I = B$. Thus I is irreducible. This completes the proof. □

DEFINITION 3.19. An incline $\mathcal{K}$ is said to be *simple* if it has no proper non-zero ideals.

THEOREM 3.20. *In an incline $\mathcal{K}$ with identity $1_{\mathcal{K}} \neq 0$ and zero element 0, an ideal M is maximal if and only if the quotient incline $\mathcal{K}/M$ is simple.*

PROOF. Let ν be the natural homomorphism from $\mathcal{K}$ to $\mathcal{K}/M$. Let M be a maximal ideal of $\mathcal{K}$. Suppose that there exists a proper ideal B of $\mathcal{K}/M$. Then, by Theorem 2.7, $\nu^{-1}(B)$ is a proper ideal of $\mathcal{K}$ properly containing M. This contradicts to the maximality of M.

Conversely, suppose that $\mathcal{K}/M$ is simple. If M is not maximal, then there is a proper ideal A of $\mathcal{K}$ properly containing M. Hence A/M is a proper non-zero ideal of $\mathcal{K}/M$ by Theorem 2.4. This contradicts to the simplicity of $\mathcal{K}/M$. The proof is complete. □

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